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Transcendental Picard-Mann hybrid Julia and Mandelbrot sets

Mahesh Chandra Joshi, Mamta Rani, Naveen Chandra*

Abstract. In this paper, the fascinating Julia and Mandelbrot sets for the complex-valued transcendental functions $z \to \sin(z^m) + c$, $(m \geq 2) \in \mathbb{N}$ have been obtained in Picard, Ishikawa and Noor orbits. The purpose of the paper is to visualize transcendental Julia and Mandelbrot sets in Picard-Mann hybrid orbit.

1. Introduction and Preliminaries

The science of fractal graphics has spread in almost all branches of science and engineering. Fractals are either natural or generated using a mathematical recipe [1]. Gaston Julia iterated complex polynomials and introduced Julia set as a classical example of fractals. Benoit B. Mandelbrot (1977) extended the work of Gaston Julia and created Mandelbrot set using computer graphics. Since then, the study of Julia and Mandelbrot sets has become an area of intense research. Julia and Mandelbrot sets have been studied for quadratic, cubic and higher degree polynomials in Picard orbit (see [7] and references therein). Generally, Mandelbrot and Julia sets lie in a complex plane. Rochon [17] studied a more generalized form of Mandelbrot sets in bi-complex plane (see also [12, 19, 20] and references therein). For a detailed study of Julia and Mandelbrot sets, one may refer to [2]. Transcendental function forms a rich dynamics for well-known Julia and Mandelbrot sets. The study of dynamical behaviour of the transcendental functions was initiated by Fatou [6]. For a transcendental function, points with unbounded orbits are not in Fatou sets but they must lay in Julia sets. For the entire transcendental functions, the point 1 is an essential singularity. Erneko (see [12] and references therein) studied that for every transcendental function,

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the set of escaping points is always non-empty. For the special cases of exponential functions, every escaping point may be connected to 1 along with unique curve running entirely through the escaping points. The process of generating fractals from $z \to \sin(z^m) + c$ and self-squared function is similar [11]. In 2004, Rani and Kumar [16] introduced superior iterates (Mann iterates) in the study of fractal theory, and created superior Julia and Mandelbrot sets. Later on, in a series of papers Rani jointly with other researchers generated and analyzed superior Julia and superior Mandelbrot sets for quadratic, cubic and higher degree complex polynomials (see [13, 15, 16] and references therein). Per-turbed superior Julia and superior Mandelbrot sets due to noise have also been studied by researchers (see [14] and references therein). For a complete literature review of superior fractals until 2010, one may refer to Singh, Mishra and Sinkala [18]. In 2010, Chauhan, Rana and Negi [5] obtained relative Julia and Mandelbrot sets via Ishikawa iterates. Ashish, Rani and Chugh [4] visualized new Julia sets and Mandelbrot sets in Noor orbit [10]. Further, the cubic Julia sets and antifractals [3] were studied in the Noor orbit. Julia and Mandelbrot sets have been studied in Jungck orbit also (see [9] and references therein).

In the series of Julia and Mandelbrot sets via various iterates, Picard-Mann hybrid iterates was introduced by Rani and others (see [13, 14] and references therein). She obtained not only hybrid Julia and Mandelbrot sets in Picard-Mann hybrid orbit but used iterates in chaos theory also. The purpose of this paper is to obtain transcendental Picard-Mann hybrid Julia and Mandelbrot sets.

Definition 1.1 (**Picard iterates).** Let Z be a non-empty subset of real or complex numbers and $f : Z \to Z$. We consider a sequence $\{z_n\}$ of iterates for an initial point $z_0 \in Z$ such that $O(f, z_0) = \{z_n : z_n = f(z_{n-1}), n \in \mathbb{N}\}$.

The sequence constructed above is called as Picard orbit, abbreviated as O [16].

Definition 1.2 (**Picard-Mann hybrid iterates).** Let Z be a non-empty subset of real or complex numbers and $f : Z \to Z$. We consider a sequence $\{z_n\}$ of iterates for an initial point $z_0 \in Z$ such that $(f, z_0, \beta_n) = \big\{z_{n+1} : z_{n+1} = fy_n;\ y_n = (1-\beta_n)z_n + \beta_n f z_n,\ n = 0, 1, 2, \dots\big\}$, where $\{\beta_n\}$ is the sequence in $[0, 1]$. We call P-MO as Picard-Mann hybrid orbit [8].

Notice that when $\beta_n = 0$, then P-MO reduces to O. Also, at $\beta_n = 1$, P-MO may be considered as double Picard iteration method. Khan [8] claimed that the P-MO converges faster than Ishikawa orbit. Rani [12] too showed superiority of P-MO over O by showing some examples. In fact, she gave two examples in which it was shown that diverging and oscillatory sequences in O are convergent in P-MO.

Definition 1.3 (**Picard-Mann hybrid Julia (P-MHJ) set** [12]**).** The set of complex points FJ whose orbits $\{z_n\}$ constructed above for suitable choices of z_0 are bounded under Picard-Mann hybrid iterates of a complex function $Q(z)$ is called the filled Picard-Mann hybrid Julia set. Picard-Mann hybrid Julia set of $Q(z)$ is the boundary of filled Picard-Mann hybrid Julia set FJ.

Definition 1.4 (**Picard-Mann hybrid Mandelbrot (P-MHM) set** [12]**).** Picard-Mann hybrid Mandelbrot set for the functions of the form $z^m + c$, $(m \geq 2) \in \mathbb{N}$, is defined as the collection of $c \in Z$ for which the orbit of the point 0 is bounded, i.e., $P\text{-}MHM = \Big\{c \in Z : \{Q_c^k(0) : k = 0, 1, 2, \ldots\} \text{ is bounded}\Big\}$.

2. TRANSCENDENTAL PICARD-MANN HYBRID JULIA AND MANDELBROT SETS

The transcendental function of the form $\sin(z^m) + c$, $(m \geq 2) \in \mathbb{N}$ is iterated in $P\text{-}MO$ for some initial choice z_0 as follows to obtain the sequence $\{z_n\}$:

$$
\begin{aligned}
z_1 &= f(y_0) = f((1-\beta_0)z_0 + \beta_0 f(z_0)) = f((1-\beta_0)z_0 + \beta_0(\sin(z_0^m) + c)) \\
&= \sin((1-\beta_0)z_0 + \beta_0(\sin(z_0^m) + c))^m + c, \\
z_2 &= \sin((1-\beta_1)z_1 + \beta_1(\sin(z_1^m) + c))^m + c, \\
z_3 &= \sin((1-\beta_2)z_2 + \beta_2(\sin(z_2^m) + c))^m + c, \\
&\vdots \\
z_n &= \sin((1-\beta_{n-1})z_{n-1} + \beta_{n-1}(\sin(z_{n-1}^m) + c))^m + c.
\end{aligned}
$$

Now, we define transcendental Picard-Mann hybrid Julia and Mandelbrot sets.

Definition 2.1 (**Transcendental Picard-Mann hybrid Julia set**)**.** The set of complex points FJ whose orbits $\{z_n\}$ constructed above for suitable choices of z_0 are bounded under Picard-Mann hybrid iterates of a transcendental function $Q(z)$ is called the filled transcendental Picard-Mann hybrid Julia set. Transcendental Picard-Mann hybrid Julia set, abbreviated as **transcendental P-MHJ set**, of $Q(z)$ is the boundary of filled transcendental Picard-Mann hybrid Julia set FJ.

Definition 2.2 (**Transcendental Picard-Mann hybrid Mandelbrot set**)**.** Transcendental Picard-Mann hybrid Mandelbrot set, abbreviated as **transcendental P-MHM set**, for the transcendental functions $\sin(z^m)+c$, $(m \geq 2) \in \mathbb{N}$ is defined as the collection of $c \in Z$ for which the orbit of the point 0 is bounded, i.e., transcendental $P\text{-}MHM = \Big\{c \in Z : \{Q_c^k(0) : k = 0, 1, 2, \ldots\} \text{ is bounded}\Big\}$.

3. Generation of Transcendental P-MHJ and P-MHM sets

The escape criterion for the complex polynomial $z^m + c$, $(m \geq 2) \in \mathbb{N}$ to obtain Julia and Mandelbrot sets in superior orbit or (Mann orbit) is $\max\left\{c, (\frac{2}{\beta})^{\frac{1}{n-1}}\right\}$ (see [16] and references therein). The same escape criterion has been used obtain Julia and Mandelbrot sets in *P-MO* [12]. The same escape criterion may be used in the iteration of the transcendental function $z \to \sin(z^m) + c$. We generate transcendental Julia and Mandelbrot sets for $m \geq 2$ at $\beta_n = \beta$.

3.1. Transcendental P-MHJ sets for $\sin(\mathbf{z^m}) + \mathbf{c}$.

Quadratic transcendental *P-MHJ* sets have been obtained at $c = 0.275$ for $\beta = 1$ and 0.1 in Fig. 1. The quadratic transcendental *P-MHJ* sets hold a perfect symmetry about both the axes, and have 4 sticks around them. It can be observed that the set for $\beta = 0.1$ is fattier than the set at $\beta = 1$.

Fig. 2 shows cubic transcendental *P-MHJ* sets at $c = 0.5\breve{\ }0.1i$ for $\beta = 1$ and 0.1. Cubic transcendental *P-MHJ* sets are symmetrical about both the axes, and have 6 sticks. It is observed that the set for $\beta = 0.1$ is fatter than the set at $\beta = 1$.

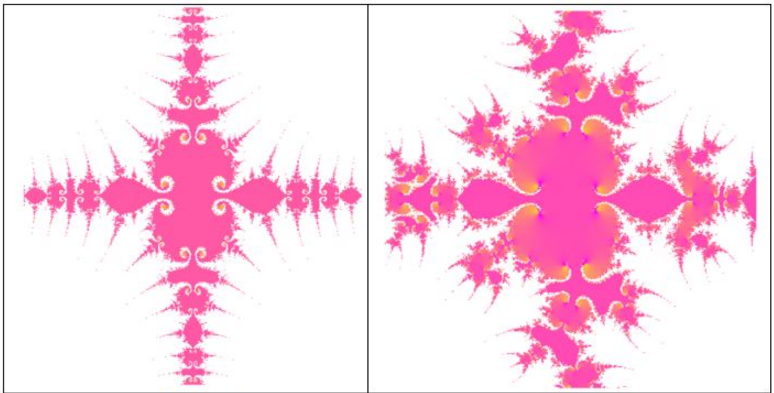
Various biquadratic transcendental *P-MHJ* sets have been shown in Fig. 3. Above observations are true in this case also. Finally, transcendental *P-MHJ* sets have been generated for $\sin(z^m) + c$. See transcendental *P-MHJ* sets in Fig. 4 at $\beta = 0.5$ and $c = 0$, for $m = 3, 4, 5, 6, 25$ and 100. It is observed that all the transcendental *P-MHJ* sets in Fig. 4 have $2m$ sticks and symmetrical about both the axes.

3.2. Transcendental P-MHM sets for $\sin(\mathbf{z^m}) + \mathbf{c}$.

A few quadratic transcendental *P-MHM* sets have been obtained in Fig. 5 at $\beta = 1, 0.5, 0.3, 0.1$. Fig. 6 shows some of the cubic transcendental *P-MHM* sets at $\beta = 0.8, 0.5, 0.3, 0.1$. Some example of biquadratic transcendental *P-MHM* sets at $\beta = 1, 0.8, 0.5, 0.1$ are in Fig. 7. See two transcendental *P-MHM* sets for $\sin(z^5) + c$ at $\beta = 1$ and 0.5 in Fig. 8. Transcendental *P-MHM* sets for higher degree of polynomial are shown in Fig. 9. In all the transcendental *P-MHM* sets, it can be observed that the number of bulbs are $(m-1)$, and for higher values of m, the sets become circular saw.

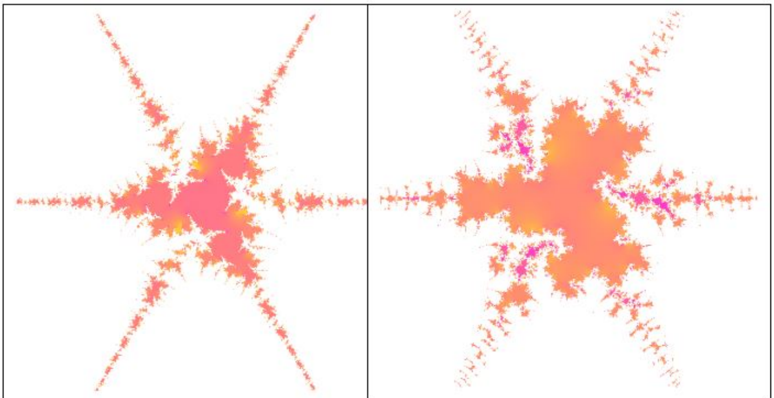
4. Conclusion

In this paper, we have obtained Julia and Mandelbrot sets for the transcendental function $\sin(z^m)+c$, where $(m \geq 2) \in \mathbb{N}$, via Picard-Mann hybrid iterates. It is concluded that in Picard-Mann hybrid orbit, the number of bulbs in a transcendental Mandelbrot set is $(m-1)$. Further, transcendental Julia and Mandelbrot sets both become circular saw for higher values of m.



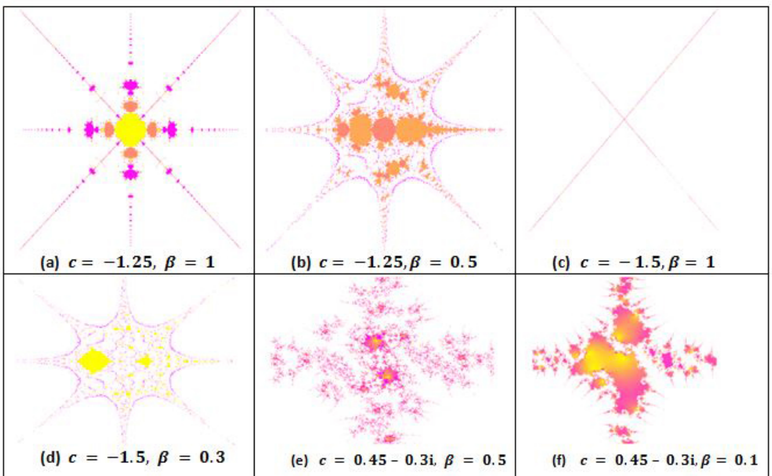
(a) $\beta = 1$ | *(b)* $\beta = 0.1$

Fig. 1: Quadratic transcendental *P-MHJ* sets at $c = 0.275$



(a) $\beta = 1$ | *(b)* $\beta = 0.1$

Fig. 2: Cubic transcendental *P-MHJ* sets at $c = 0.5 - 0.1\,i$



(a) $c = -1.25,\ \beta = 1$ | (b) $c = -1.25, \beta = 0.5$ | (c) $c = -1.5, \beta = 1$

(d) $c = -1.5,\ \beta = 0.3$ | (e) $c = 0.45 - 0.3i,\ \beta = 0.5$ | (f) $c = 0.45 - 0.3i, \beta = 0.1$

Fig. 3: Bi-quadratic transcendental *P-MHJ* sets

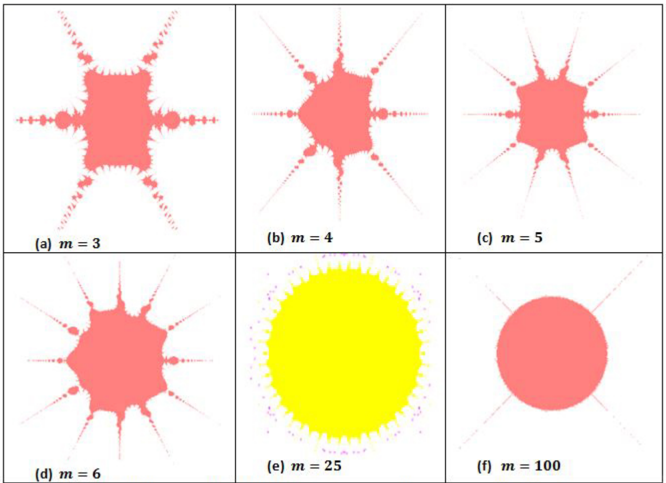


Fig. 4: Transcendental *P-MHJ* sets for $sin(z^m) + c$ at $(c, \beta) = (0, 0.5)$

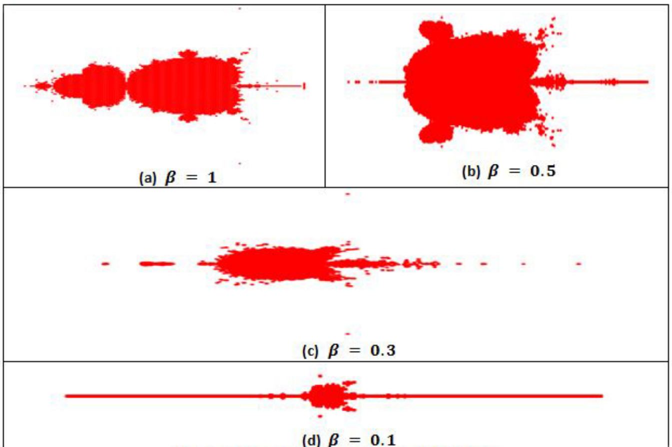


Fig. 5: Transcendental quadratic *P-MHM* sets

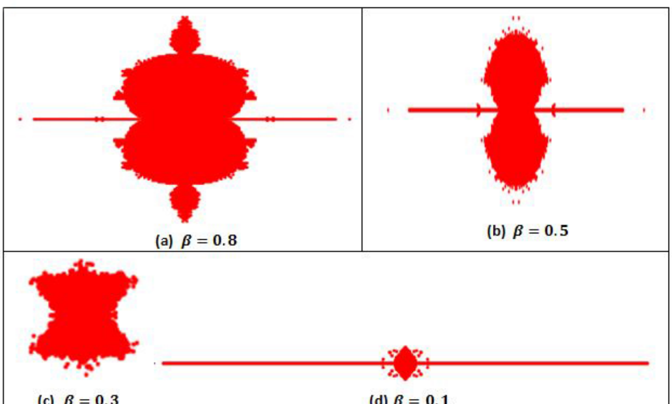


Fig. 6: Transcendental cubic *P-MHM* sets

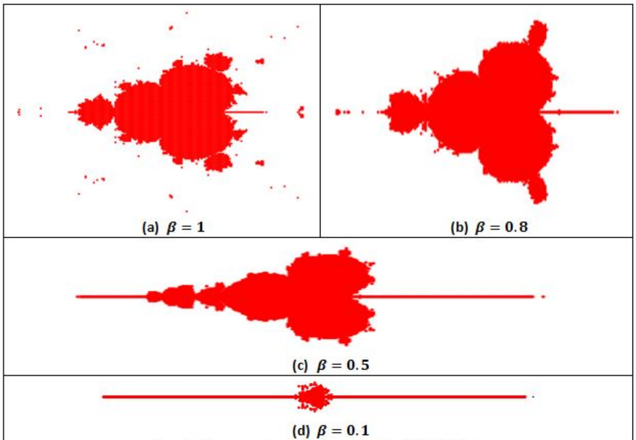


Fig. 7: Transcendental bi-quadratic *P-MHM* sets

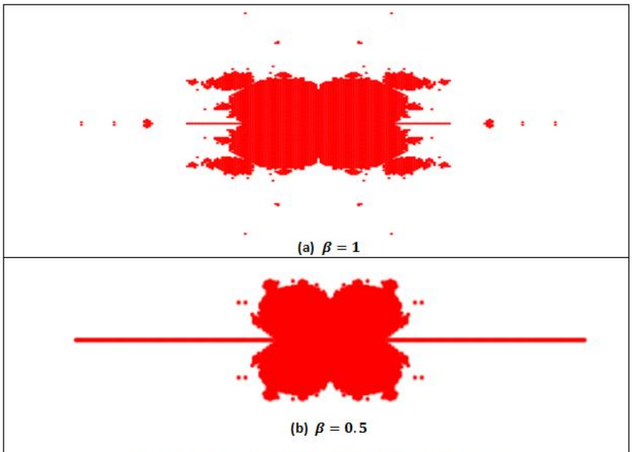


Fig. 8: Transcendental *P-MHM* sets for $sin(z^5) + c$

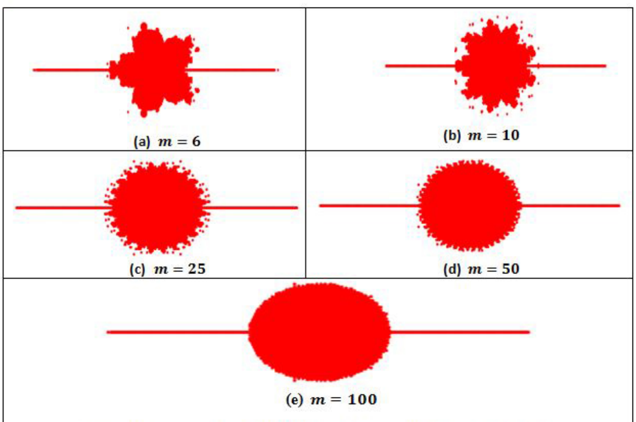


Fig. 9: Transcendental *P-MHM* sets for $sin(z^m) + c$ at $\beta = 0.5$

REFERENCES

[1] P. S. Addison, *Fractals and Chaos an Illustrated Course*, Overseas Press, New Delhi, 2005.

[2] I. Andreadis, T. E. Karakasidis, *On a topological closeness of per-turbed Julia sets*, Appl. Math. Comput., 217 (6) (2010), 2883–2890.

[3] Ashish, M. Rani, R. Chugh, *Dynamics of antifractals in Noor orbit*, Int. J. Comput. Appl., 57 (4) (2012), 11–15.

[4] Ashish, M. Rani, R. Chugh, *Julia sets and Mandelbrot sets in Noor Orbit*, Appl. Math. Comput., 228 (2014), 615–631.

[5] Y. S. Chauhan, R. Rana, A. Negi, *Complex dynamics of Ishikawa iterates for non integer values*, Int. J. Comput. Appl., 9 (2) (2010), 9–16.

[6] P. Fatou, *Sur iteration des functions transcendantes enteries*, Acta. Math., 47 (1926), 337–378.

[7] M. Frame, J. Robertson, *A generalized Mandelbrot set and the role of critical points*, Computer and Graphics, 16 (1) (1992), 35–40.

[8] S. H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory Appl., 69 (1) (2013), 10 pages.

[9] W. Nazeer, S. M. Kang, M. Tanveer, A. A. Shahid, *Fixed point results in the generation of Julia and Mandelbrot sets*, J. Inequal. Appl., (2015) 2015:298, 16 pages.

[10] M. A Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., 251 (2000), 217–229.

[11] H. O. Peitgen, P. H. Richter, *The Beauty of Fractals*, Springer, Berlin, 1986.

[12] M. Rani, *Theoretical Framework for Fractal Models under Two-Step Feedback Process*, D. Sc. Thesis, Kumaun University, Nainital (India), 2016.

[13] M. Rani, *Superior antifractals*, in: IEEE Proc. ICCAE, 1 (2010), 798–802.

[14] M. Rani, Rashi Agarwal, *Effect of noise on Julia sets generated by logistic map*, in: Proc. 2nd IEEE International Conference on Computer and Automation Engineering (ICCAE), 2 (2010), 55–59.

[15] M. Rani, R. Agarwal, *Generation of fractals from complex logistic map*, Chaos Solit. Fractals, 42 (1) (2009), 447–452.

[16] M. Rani, V. Kumar, *Superior Mandelbrot set*, J. Korean Soc. Math. Edu. Res. Ser. D, 8 (4) (2004), 279–291.

[17] D. Rochon, *A generalized Mandelbrot set for bicomplex numbers*, Fractals, 8 (4) (2000), 355–368.

[18] S. L. Singh, S. N. Mishra, W. Sinkala, *A new iterative approach to fractal models*, Commun. Nonlinear Sci. Numer. Simul., 17 (2) (2011), 521–529.

[19] Y. Sun, X. Wang, *Quaternion M set with none zero critical points*, Fractals, 17 (4) (2009), 427–439.

[20] X. Wang, Y. Sun, *The general quaternionic M-J sets on the mapping* $z^\alpha + c\,(\alpha \in \mathbb{N})$, Comput. Math. Appl., 53 (2007), 1718–1732.

Mahesh Chandra Joshi
Department of Mathematics
D.S.B. Campus, Kumaun University
Nainital-263002
India
E-mail address: mcjoshi69@gmail.com

Mamta Rani
Department of Computer Science
Central University of Rajasthan
Rajasthan-305801
India
E-mail address: mamtarsingh@gmail.com

Naveen Chandra
Department of Mathematics
D.S.B. Campus, Kumaun University
Nainital-263002
India
E-mail address: cnaveen329@gmail.com