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On $\mathcal{I}$-Fréchet-Urysohn spaces and sequential $\mathcal{I}$-convergence groups

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ABSTRACT. In this paper, we introduce the concept of sequential $\mathcal{I}$-convergence spaces and $\mathcal{I}$-Fréchet-Urysohn space and study their properties. We give a sufficient condition for the product of two sequential $\mathcal{I}$-convergence spaces to be a sequential $\mathcal{I}$-convergence space.

Finally, we introduce sequential $\mathcal{I}$-convergence groups and obtain an $\mathcal{I}$-completion of these groups satisfying certain conditions.

1. INTRODUCTION

In [4], Hong introduced the notion of Fréchet spaces and sequential convergence groups. It has been discussed and developed by many authors [3, 5]. We try to extend this concept on ideal topological spaces. A non-empty collection $\mathcal{I}$ of subsets of a set X is said to be an *ideal* on X [7] if it satisfies the following two conditions:

(i) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$.

(ii) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.

A non-trivial ideal $\mathcal{I}$ is called *admissible* [2] if and only if $\mathcal{I} \supset \{\{x\} \mid x \in X\}$. Several examples of nontrivial admissible ideals may be seen in [6]. Let (X, τ) be a topological space. A sequence (x_n) in X is said to be *$\mathcal{I}$-convergent* to $x_0 \in X$ [7] if for any non-empty open set U containing x_0, $\{n \in \mathbb{N} \mid x_n \notin U\} \in \mathcal{I}$. It is denoted by $(x_n) \xrightarrow{\mathcal{I}} x_0$ and x_0 is called an *$\mathcal{I}$-limit* of the sequence (x_n). A topological space (X, τ) is $\mathcal{I}$-Fréchet or *$\mathcal{I}$-Fréchet-Urysohn space* [9] if every point in the closure of a subset A of X is a $\mathcal{I}$-limit of a sequence of A. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be *pseudo open* [2] if whenever $f^{-1}(y) \subset U$ with U open in X, $y \in int(f(U))$.

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Let $A \subset \mathbb{N}$. Put $A(n) = \{k \mid k \leq n, k \in A\}$. Then we call $\underline{\delta}(A) = \lim_{n\to\infty} \inf \frac{|A(n)|}{n}$ and $\bar{\delta}(A) = \lim_{n\to\infty} \sup \frac{|A(n)|}{n}$, the lower and upper asymptotic density [8] of A, respectively. If $\underline{\delta}(A) = \bar{\delta}(A)$, then $d(A) = \delta(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$ is called the asymptotic density (or natural density) of A.

This paper consists of four sections with new results. In Section 2, we introduce the concept of sequential $\mathcal{I}$-convergence spaces, $\mathcal{I}$-Fréchet-Urysohn spaces and study their properties. Recall that $\mathcal{I}$-Fréchet and I-sequential spaces are generalizations of statistical versions of Fréchet-Urysohn spaces and sequential spaces considered in [1] and [11]. In Section 3, we give a sufficient condition for the product of two sequential $\mathcal{I}$-convergence spaces to be a sequential $\mathcal{I}$-convergence space. In Section 4, we introduce sequential $\mathcal{I}$-convergence groups and an $\mathcal{I}$-completion of these groups satisfying given condition $(**)$. Throughout this paper, we consider only an admissible ideal.

The following lemma will be useful in the sequel.

Lemma 1.1. [10] *Let $f : X \to Y$ be a quotient mapping and X be an $\mathcal{I}$-Fréchet-Urysohn space. Then Y is an $\mathcal{I}$-Fréchet-Urysohn space if and only if f is pseudo open.*

2. Sequential $\mathcal{I}$-convergence spaces

Let X be a non-empty set and $S[X]$ be the set of all sequences in X. We use the notation $x_n \xrightarrow{\mathcal{I}} x$ for $((x_n), x) \in L_{\mathcal{I}}$. A non-empty subfamily $L_{\mathcal{I}}$ of $S[X] \times X$ is called a *sequential $\mathcal{I}$-convergence structure* (S$\mathcal{I}C$) on X if it satisfies the following properties:

(S$\mathcal{I}$C1) For each $x \in X$, $((x), x) \in L_{\mathcal{I}}$, where (x) is the constant sequence whose n-th term is x for all indices $n \in \mathbb{N}$,

(S$\mathcal{I}$C2) If $((x_n), x) \in L_{\mathcal{I}}$, then $((x_{n_i}), x) \in L_{\mathcal{I}}$ for each subsequence (x_{n_i}) of (x_n).

(S$\mathcal{I}$C3) Let $x \in X$ and $A \subset X$. If $((x_n), x) \notin L_{\mathcal{I}}$ for each $(x_n) \in S[A]$, then $((y_n), y) \notin L_{\mathcal{I}}$ for each $(y_n) \in S[\{y \in X | ((x_n), y) \in L_{\mathcal{I}}$ for some $(x_n) \in S[A]\}]$.

If a sequential $\mathcal{I}$-convergence structure $L_{\mathcal{I}}$ on X is given, the pair $(X, L_{\mathcal{I}})$ is called a *sequential $\mathcal{I}$-convergence space*. Hereafter, we use the notation SC$\mathcal{I}[X]$ for the set of all sequential $\mathcal{I}$-convergence structures on X.

Let $(X, \tau_{\mathcal{I}})$ be a $\mathcal{I}$-Fréchet-Urysohn space and let $L_{\tau_{\mathcal{I}}}$ denote the set of all pairs $((x_n), x) \in S[X] \times X$ such that $(x_n) \xrightarrow{\mathcal{I}} x$ in $(X, \tau_{\mathcal{I}})$.

(i) Suppose for each $x \in X$, then $((x), x) \in L_{\tau_{\mathcal{I}}}$, where (x) is constant sequence.

(ii) Suppose $((x_n),x)\in L_{\tau_{\mathcal{I}}}$ then there exists a subsequence (x_{n_k}) of (x_n) such that $((x_{n_k}),x)\in L_{\tau_{\mathcal{I}}}$.

(iii) Let $x\in X$ and $A\subset X$. Suppose that $((x_n),x)\notin L_{\tau_{\mathcal{I}}}$ for each $(x_n)\in S[A]$, then $((y_n),y)\notin L_{\tau_{\mathcal{I}}}$ for each

$$(y_n)\in S[\{y\in X|((x_n),y)\in L_{\tau_{\mathcal{I}}} \text{ for some } (x_n)\in S[A]\}].$$

Then, it is clear that $L_{\tau_{\mathcal{I}}}\in S\mathcal{I}C[X]$ and two topological spaces $(X,\tau_{\mathcal{I}})$ and $(X,L_{\tau_{\mathcal{I}}})$ are precisely same, since $(X,\tau_{\mathcal{I}})$ is a $\mathcal{I}$-Fréchet-Urysohn space. Hence every $\mathcal{I}$-Fréchet-Urysohn space is a sequential $\mathcal{I}$-convergence space. And, for each $L_{\mathcal{I}}\in S\mathcal{I}C[X]$, define a mapping $c_{L_{\mathcal{I}}}$ of the power set $\mathcal{P}(X)$ of X into itself as follows:

$$c_{L_{\mathcal{I}}}(A)=\{x\in X|((x_n),x)\in L_{\mathcal{I}} \text{ for some } (x_n)\in S[A]\}$$

The following Lemma 2.1 gives the properties of the operator $c_{L_{\mathcal{I}}}$.

Lemma 2.1. *Let (X,τ) be a topological space and $A,B\subset X$. Then the following hold.*

(a) $c_{L_{\mathcal{I}}}(\emptyset)=\emptyset$.
(b) $A\subset c_{L_{\mathcal{I}}}(A)$.
(c) $c_{L_{\mathcal{I}}}(c_{L_{\mathcal{I}}}(A))\subset c_{L_{\mathcal{I}}}(A)$.
(d) $A\subset B\Rightarrow c_{L_{\mathcal{I}}}(A)\subset c_{L_{\mathcal{I}}}(B)$.
(e) $c_{L_{\mathcal{I}}}(A\cup B)=c_{L_{\mathcal{I}}}(A)\cup c_{L_{\mathcal{I}}}(B)$.

Proof. (a) $c_{L_{\mathcal{I}}}(\emptyset)=\emptyset$ is clear.

(b) Suppose $x\in A$, and consider $(x_n)=(x,x,...,x)$. Then $(x_n)\in S[A]$ and $((x_n),x)\in L_{\mathcal{I}}$. Therefore, $x\in c_{L_{\mathcal{I}}}(A)$.

(c) Suppose $x\in c_{L_{\mathcal{I}}}(c_{L_{\mathcal{I}}}(A))$, then there exists a sequence $(x_n)\in S[c_{L_{\mathcal{I}}}(A)]$ such that $((x_n),x)\in L_{\mathcal{I}}$. Suppose that $x\notin c_{L_{\mathcal{I}}}(A)$. Then for each $(y_n)\in S[A]$, $((y_n),x)\notin L_{\mathcal{I}}$. By $(S\mathcal{I}C3)$, $((x_n),x)\notin L_{\mathcal{I}}$, which is a contradiction. Therefore, $x\in c_{L_{\mathcal{I}}}(A)$. Hence $c_{L_{\mathcal{I}}}(c_{L_{\mathcal{I}}}(A))\subset c_{L_{\mathcal{I}}}(A)$.

(d) Suppose $x\in c_{L_{\mathcal{I}}}(A)$, then $((x_n),x)\in L_{\mathcal{I}}$ for some $(x_n)\in S[A]$. Since $A\subset B$, $(x_n)\in B$ and so $x\in c_{L_{\mathcal{I}}}(B)$. Therefore, $c_{L_{\mathcal{I}}}(A)\subset c_{L_{\mathcal{I}}}(B)$.

(e) We have $c_{L_{\mathcal{I}}}(A)\cup c_{L_{\mathcal{I}}}(B)\subset c_{L_{\mathcal{I}}}(A\cup B)$, by (d). Let $x\in c_{L_{\mathcal{I}}}(A\cup B)$. Then $((x_n),x)\in L_{\mathcal{I}}$ for some $(x_n)\in A\cup B$. Note that either A or B contains infinitely many terms of (x_n). If A contains infinitely many terms of (x_n), then there exists a subsequence (x_{nm}) of (x_n) in A with $((x_{nm}),x)\in L_{\mathcal{I}}$, by (S$\mathcal{I}$C2). Therefore, $x\in c_{L_{\mathcal{I}}}(A)$. Similarly, $x\in c_{L_{\mathcal{I}}}(B)$. Hence $x\in c_{L_{\mathcal{I}}}(A)\cup c_{L_{\mathcal{I}}}(B)$. Thus, $c_{L_{\mathcal{I}}}(A\cup B)=c_{L_{\mathcal{I}}}(A)\cup c_{L_{\mathcal{I}}}(B)$. □

Thus, $c_{L_{\mathcal{I}}}$ is a Kuratowski closure operator on X and $(X,c_{L_{\mathcal{I}}})$ is a $\mathcal{I}$-Fréchet-Urysohn space as it satisfies the above properties. Let $\mathcal{L}(c_{L_{\mathcal{I}}})$ denote the set of all pairs $((x_n),x)\in S[X]\times X$ such that $(x_n)\xrightarrow{\mathcal{I}}x$ in $(X,c_{L_{\mathcal{I}}})$. By the following Example 2.1, $L_{\mathcal{I}}\subsetneqq\mathcal{L}(c_{L_{\mathcal{I}}})$, in general. Hence not every sequential $\mathcal{I}$-convergence space $(X,L_{\mathcal{I}})$ need be a $\mathcal{I}$-Fréchet-Urysohn space even if $(X,L_{\mathcal{I}})$ determines a $\mathcal{I}$-Fréchet-Urysohn space $(X,c_{L_{\mathcal{I}}})$ as above.

Example 2.1. Let $\mathbb{Q}$ be the set of all rational numbers and $L_{\mathcal{I}} = \{((x), x) | x \in \mathbb{Q}\} \cup \{((x_n), x) \in S[\mathbb{Q}] \times \mathbb{Q} | (x_n) \xrightarrow{\mathcal{I}} x \in \mathbb{Q}\}$ with the usual topology and (x_n) is either increasing or decreasing }.

Let $(x_n) = (\frac{1}{n})$ and $(x) = (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, ...)$. Then there is an open set U of 0 such that $\{n | (x_n) = \frac{1}{n} \notin (U \cap \mathbb{Q})\} \in \mathcal{I}$ where $\mathcal{I} = \{J \subset \mathbb{N} \mid d(J) = 0\}$. Therefore, $((x), \frac{1}{2}) \in L_{\mathcal{I}}$ satisfy (S$\mathcal{I}$C1). Choose $(x_{n_i}) = (\frac{1}{2}, \frac{1}{4}, ...,)$ of (x_n). Then $((x_{n_i}, 0) \in L_{\mathcal{I}}$ satisfy (S$\mathcal{I}$C2). Suppose $(x_n) = (-1, 1, -1, ...,)$ does not $\mathcal{I}$-converge in $L_{\mathcal{I}}$ but $(y_n) = \{1 \mid (1, 1, ...,) \xrightarrow{\mathcal{I}} 1$ for some $(x_n) \in S[\mathbb{Q}]\}$. Therefore, (S$\mathcal{I}C$3) is satisfied. Thus, $L_{\mathcal{I}} \in S\mathcal{I}C[\mathbb{Q}]$ but $L_{\mathcal{I}} \subsetneqq \mathcal{L}(c_{L_{\mathcal{I}}}) = \{((x_n), 0) \in [S\mathbb{Q}] \times \mathbb{Q} \mid (x_n) \xrightarrow{\mathcal{I}} 0 \textit{ in } \mathbb{Q}$ with the usual topology$\}$.

Lemma 2.2. *Let $L_{\mathcal{I}} \in S\mathcal{I}C[X]$ and $x \in A \subset X$. If A is a neighbourhood of x in $(X, c_{L_{\mathcal{I}}})$, then for each $((x_n), x) \in L_{\mathcal{I}}$, there is an open set U in $(X, c_{L_{\mathcal{I}}})$ containing x such that $\{n \mid x_n \notin U\} \in \mathcal{I}$.*

Proof. Let A be a neighbourhood of x in $(X, c_{L_{\mathcal{I}}})$ and $((x_n), x) \in L_{\mathcal{I}}$. Then there exists an open set U in $(X, c_{L_{\mathcal{I}}})$ such that $x \in U \subset A$. It follows that $c_{L_{\mathcal{I}}}(X \backslash U) = X \backslash U$, and there does not exist (y_n) in $X \backslash U$ such that $((y_n), x) \in L_{\mathcal{I}}$, by the definition of $c_{L_{\mathcal{I}}}$. Now we show that $\{k \in \mathbb{N} \mid x_k \in X \backslash U\} \in \mathcal{I}$. If $\{k \in \mathbb{N} \mid x_k \in X \backslash U\} \notin \mathcal{I}$, then there exists a subsequence (x_{n_i}) of (x_n) in $X \backslash U$. Since $((x_n), x) \in L_{\mathcal{I}}$, $((x_{n_i}), x) \in L_{\mathcal{I}}$, which is a contradiction. Therefore, $\{k \in \mathbb{N} \mid x_k \in X \backslash U\} \in \mathcal{I}$. □

Theorem 2.1. *If $L_{\mathcal{I}} \in S\mathcal{I}C[X]$, then the following hold:*

(a) $L_{\mathcal{I}} \subset \mathcal{L}(c_{L_{\mathcal{I}}})$.
(b) $c_{L_{\mathcal{I}}} = c_{\mathcal{L}(c_{L_{\mathcal{I}}})}$.

Proof. (a) Let $((x_n), x) \in L_{\mathcal{I}}$. Then by Lemma 2.2, for each neighborhood A of x in $(X, c_{L_{\mathcal{I}}})$, there is an open set U in $(X, c_{L_{\mathcal{I}}})$ such that $\{n \mid x_n \notin U\} \in \mathcal{I}$. Therefore, $((x_n), x) \in \mathcal{L}(c_{L_{\mathcal{I}}})$.

(b) Let A be a non-empty subset of X. Then by Lemma 2.1(b) and (d), $c_{L_{\mathcal{I}}}(A) \subset c_{\mathcal{L}(c_{L_{\mathcal{I}}})}(A)$. Conversely, let $x \in c_{\mathcal{L}(c_{L_{\mathcal{I}}})}(A)$. Then $((x_n), x) \in \mathcal{L}_{(c_{L_{\mathcal{I}}})}$ for some $(x_n) \in S[A]$. By the definition of $\mathcal{L}_{(c_{L_{\mathcal{I}}})}$, $(x_n) \xrightarrow{\mathcal{I}} x$ in $(X, c_{L_{\mathcal{I}}})$. Therefore, $x \in c_{L_{\mathcal{I}}}(A)$. □

Theorem 2.2. *For each $\mathcal{I}$-Fréchet-Urysohn topology $\tau_{\mathcal{I}}$ on X, $L_{\tau_{\mathcal{I}}} = \mathcal{L}(c_{L_{\tau_{\mathcal{I}}}}) \in S\mathcal{I}C[X]$, where $L_{\tau_{\mathcal{I}}} = \{((x_n), x) \in S[X] \times X \mid (x_n) \xrightarrow{\mathcal{I}} x \textit{ in } (X, \tau_{\mathcal{I}})\}$.*

Proof. Note that $c_{L_{\tau_{\mathcal{I}}}}$ is the closure operator for $(X, \tau_{\mathcal{I}})$. Since $\tau_{\mathcal{I}}$ is an $\mathcal{I}$-Fréchet-Urysohn topology and $L_{\tau_{\mathcal{I}}} \in S\mathcal{I}C[X]$, $L_{\tau_{\mathcal{I}}} \subset \mathcal{L}(c_{L_{\tau_{\mathcal{I}}}})$, by Theorem 2.1. □

Corollary 2.1. *Let $F_{\mathcal{I}}[X]$ denote the set of all $\mathcal{I}$-Fréchet-Urysohn topologies on X and $TS\mathcal{I}[X] = \{\mathcal{L}(c_{L_{\mathcal{I}}}) \mid L_{\mathcal{I}} \in S\mathcal{I}C[X]\}$. Then partially ordered sets*

$F_{\mathcal{I}}[X]$ and $TS\mathcal{I}C[X]$ endowed with the set inclusion order are dual isomorphic under the correspondence $\tau_{\mathcal{I}} \to L_{\tau_{\mathcal{I}}}$.

Proof. Since $c_{L_{\tau_{\mathcal{I}}}}$ is the closure operator for $(X, \tau_{\mathcal{I}})$, $L_{\tau_{\mathcal{I}_1}} = L_{\tau_{\mathcal{I}_2}}$ implies $\tau_{\mathcal{I}_1} = \tau_{\mathcal{I}_2}$. Hence this correspondence is one-to-one. Take any $L_{\mathcal{I}}$ in S$\mathcal{I}C[X]$ and let $\tau_{c_{L_{\mathcal{I}}}}$ be the $\mathcal{I}$-Fréchet topology on X with the closure operator $c_{L_{\mathcal{I}}}$. Then $L_{\tau_{c_{L_{\mathcal{I}}}}} = \mathcal{L}(c_{L_{\mathcal{I}}})$. Hence this correspondence is onto. □

Theorem 2.3. *There exists an one-to-one correspondence between the set of all $\mathcal{I}$-Fréchet-Urysohn topologies on a set X and $\{c_{L_{\mathcal{I}}} \mid L_{\mathcal{I}} \in S\mathcal{I}C[X]\}$.*

Proof. Follows from Corollary 2.1. □

3. Product of sequential $\mathcal{I}$-convergence spaces

In general, the product of two sequential $\mathcal{I}$-convergence spaces need not be a sequential $\mathcal{I}$-convergence space, but we give a sufficient condition for the product of two sequential $\mathcal{I}$-convergence spaces to be a sequential $\mathcal{I}$-convergence space. The following Example 3.1 shows that the product of two sequential $\mathcal{I}$-convergence spaces is not a sequential $\mathcal{I}$-convergence space.

Example 3.1. Let $X = \mathbb{R}/\mathbb{Z}$, $\mathbb{R}$ is the real line (equipped with the usual topology) with the integers identified and let $I = [0, 1]$ be the closed unit interval in the real line. Since $\mathcal{I} = \{J \subset \mathbb{N} \mid d(J) = 0\}$ and every point in the closure of I is an $\mathcal{I}$-limit of a sequence of points of I, I is $\mathcal{I}$-Fréchet-Urysohn space. A quotient map $\phi : \mathbb{R} \to X$ is pseudo-open. Hence by Lemma 1.1, X is an $\mathcal{I}$-Fréchet-Urysohn space. For each $n \in \mathbb{N}$, let $A_n = \{(n - \frac{1}{k}, \frac{1}{n}) \mid k \in \mathbb{N}\}$ and let $A = \bigcup\{A_n \mod n \in \mathbb{N}\}$. Then $(0, 0) \in \bar{A}$, but no sequence in A $\mathcal{I}$-converging to $(0, 0)$. Hence $X \times I$ is not $\mathcal{I}$-Fréchet-Urysohn. Next we show that $X \times I$ is not a sequential $\mathcal{I}$-convergence space. For each $n, k \in \mathbb{N}$, let $z_{n_k} = (n - \frac{1}{k+1}, \frac{1}{n})$ and let $A = \{z_{n_k} \mid n, k \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$, $(z_{n_k}) \xrightarrow{\mathcal{I}} (0, \frac{1}{n})$ in $X \times I$ and the sequence $((0, \frac{1}{n})) \xrightarrow{\mathcal{I}} (0, 0)$ in $X \times I$. But there does not exist a (z_{n_k}) in A such that $(z_{n_k}) \xrightarrow{\mathcal{I}} (0, 0)$. Therefore, $X \times I$ is not a sequential $\mathcal{I}$-convergence space.

The following condition $(*)$ is sufficient for the product of two sequential $\mathcal{I}$-convergence spaces to be a sequential $\mathcal{I}$-convergence space.

$(*)$ Let $((x_n), x) \in L_{\mathcal{I}}$ and let $((x_{n_m}), x_n) \in L_{\mathcal{I}}$ for each $n \in \mathbb{N}$. It is possible to choose a cross-sequence $(x_{n_{m(n)}})$ in the double sequence (x_{n_m}) such that (i) $((x_{n_{m(n)}}), x) \in L_{\mathcal{I}}$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $((x_{n_{k(n)}}), x) \in L_{\mathcal{I}}$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$.

$(*)$ implies (S$\mathcal{I}$C3) Let $((x_n), x) \notin L_{\mathcal{I}}$ for all $(x_n) \in A$ and $(y_n) \in S[B]$ where $B = \{y \in X | ((X_n), y) \in L_{\mathcal{I}}$ for some (x_n) in $A\}$. Then $y_i \in B$ for all $i \in \mathbb{N}$ and for each i, there exists $(x_{n_i}) \in S[A]$ such that $((x_{n_i}), y_i) \in L_{\mathcal{I}}$. By hypothesis, $((x_{n_i}), x) \notin L_{\mathcal{I}}$ for all i. Suppose $((y_n), y) \in L_{\mathcal{I}}$, then $(i)((x_{n_{i(n)}}), x) \in L_{\mathcal{I}}$ $(ii) i(n) \geq n$ for all n and (iii) $((x_{n_{k(n)}}), x) \in L_{\mathcal{I}}$ if

$k(n) \geq m(n)$ for all $n \in \mathbb{N}$, which is a contradiction to $((x_{n_i}), x) \notin L_{\mathcal{I}}$ for all i. Therefore, $((y_n), y) \notin L_{\mathcal{I}}$.

Example 3.2. Let $L_{\mathcal{I}_{\mathbb{Q}}} = \{((x_n), x) \in S[\mathbb{Q}] \times \mathbb{R} \mid (x_n)\ \mathcal{I}-$converges to x in the real line $\mathbb{R}\}$. By Example 2.1, $(\mathbb{R}, L_{\mathcal{I}_{\mathbb{Q}}})$ is a sequential $\mathcal{I}$-convergence space satisfying $(*)$, but not an $\mathcal{I}$-Fréchet-Urysohn space

Theorem 3.1. *Let $(X, L_{\mathcal{I}_X})$ and $(Y, L_{\mathcal{I}_Y})$ be any two sequential $\mathcal{I}$-convergence spaces satisfying $(*)$ and let*

$$L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y} = \{((x_n, y_n), (x, y)) \mid ((x_n), x) \in L_{\mathcal{I}_X} \textit{ and } ((y_n), y) \in L_{\mathcal{I}_Y}\}$$

Then $(X \times Y, L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y})$ is a sequential $\mathcal{I}$-convergence space satisfying $(*)$.

Proof. Suppose that $(x_n) = (x)$, $(y_n) = (y)$. Then $(X \times Y, L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y})$ satisfies (S$\mathcal{I}C$1). Choose the subsequences (x_{n_i}) and (y_{n_i}) of (x_n) and (y_n), $L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y} = \{((x_{n_i}, y_{n_i}), (x, y)) \mid ((x_{n_i}), x) \in L_{\mathcal{I}_X}$ and $((y_{n_i}), y) \in L_{\mathcal{I}_Y}\}$. Thus, $L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$ satisfies (S$\mathcal{I}$C2). Since $(*)$ implies (S$\mathcal{I}C$3), it is enough to show that $L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$ satisfies $(*)$.

Let $((x_n, y_n), (x, y)) \in L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$ and let $((x_{n_m}, y_{n_l}), (x_n, y_n)) \in L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$ for each $n \in \mathbb{N}$. Then, by the definition of $L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$, $((x_n), x) \in L_{\mathcal{I}_X}, ((y_n), y) \in L_{\mathcal{I}_Y}$, $((x_{n_m}), x_n) \in L_{\mathcal{I}_X}$ for each $n \in \mathbb{N}$ and $((y_{n_l}), y_n) \in L_{\mathcal{I}_Y}$ for each $n \in \mathbb{N}$. Since $(X, L_{\mathcal{I}_X})$ and $(Y, L_{\mathcal{I}_Y})$ satisfy $(*)$, there are two cross-sequence $x_{n_{m(n)}}$ and $y_{n_{l(n)}}$ in the double sequences (x_{n_m}) and (y_{n_l}), respectively, such that (i) $((x_{n_{m(n)}}), x) \in L_{\mathcal{I}_X}$ and $((y_{n_{l(n)}}), y) \in L_{\mathcal{I}_Y}$ and those cross-sequences also satisfy the properties (ii) and (iii), respectively.

Let $p(n) =\ max\ \{m(n), l(n)\}$ for each $n \in \mathbb{N}$. Then $((x_{n_{p(n)}}), x) \in L_{\mathcal{I}_X}$ and $((y_{n_{p(n)}}), y) \in L_{\mathcal{I}_Y}$, and we obtain a cross-sequence $(x_{n_{p(n)}}, y_{n_{p(n)}})$ in the double sequence (x_{n_m}, y_{n_l}) such that (i) $((x_{n_{p(n)}} y_{n_{p(n)}}), (x, y)) \in L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$; (ii) $p(n) \geq n$ for all $n \in \mathbb{N}$; (iii) $((x_{n_{q(n)}}, y_{n_{q(n)}}), (x, y))$, for all $n \in \mathbb{N}$. □

4. Sequential $\mathcal{I}$-convergence groups

Definition 4.1. A sequential $\mathcal{I}$-convergence space $(X, L_{\mathcal{I}})$ is called Hausdorff if $L_{\mathcal{I}}$ satisfies the following property:

If $((x_n), x) \in L_{\mathcal{I}}$ and $((x_n), y) \in L_{\mathcal{I}}$, then $x = y$.

Definition 4.2. Let $(X, L_{\mathcal{I}})$ be a Hausdorff sequential $\mathcal{I}$-convergence space satisfying $(*)$ and let $\cdot$ be a commutative group operator on X. The triple $(X, \cdot, L_{\mathcal{I}})$ is called a sequential $\mathcal{I}$-convergence group if it satisfies the following property:

($S\mathcal{I}G$) For each $((x_n), x) \in L_{\mathcal{I}}$ and $((y_n), y) \in L_{\mathcal{I}}$, $((x_n y_n^{-1}), xy^{-1}) \in L_{\mathcal{I}}$.

Remark 4.1. Define a mapping ϕ of $X \times X$ onto $(X, \cdot, L_{\mathcal{I}})$ by taking $\phi(x, y) = xy^{-1}$ for each $(x, y) \in X \times X$. Then (S$\mathcal{I}$G) is equivalent to the following:

($S\mathcal{I}G'$) For each $((x_n, y_n), (x, y)) \in L_{\mathcal{I}} \times L_{\mathcal{I}}$, $(\phi(x_n, y_n), \phi(x, y)) \in L_{\mathcal{I}}$.

Let $(X, \cdot, L_{\mathcal{I}})$ be a sequential $\mathcal{I}$-convergence group. A sequence $(x_n) \in S[X]$ is called $\mathcal{I}$-Cauchy if for each subsequences (x_{n_i}) and (x_{n_j}) of (x_n), $((x_{n_i}x_{n_j}^{-1}), e) \in L_{\mathcal{I}}$ where e is the identity element of the group $(X, \cdot)$. Let $C_{\mathcal{I}}[X]$ denote the set of all $\mathcal{I}$-Cauchy sequences in $(X, \cdot, L_{\mathcal{I}})$. A sequential $\mathcal{I}$-convergence group $(X, \cdot, L_{\mathcal{I}})$ is called $\mathcal{I}$-complete if for each $(x_n) \in C_{\mathcal{I}}[X]$, $((x_n), x) \in L_{\mathcal{I}}$ for some $x \in X$. Let $(X, \cdot, L_{\mathcal{I}})$ be a sequential $\mathcal{I}$-convergence group and let $\sim$ be an equivalence relation on $C_{\mathcal{I}}[X]$ defined by $(x_n) \sim (y_n)$ if and only if $((x_{n_i}y_{n_j}^{-1}), e) \in L_{\mathcal{I}}$ for each subsequence (x_{n_i}) of (x_n) and each subsequence (y_{n_j}) of (y_n). We denote the class of all $\mathcal{I}$-Cauchy sequences which are equivalent to an $(x_n) \in C_{\mathcal{I}}[X]$ by $[(x_n)]$.

In particular, for each constant sequence (x), x will be used for the equivalence class $[(x)]$. Let $X^* = \{[(x_n)] \mid (x_n) \in C_{\mathcal{I}}[X]\}$ and $\phi : X \to X^*$ defined by $\phi(x) = x$ for all $x \in X$. Then ϕ is injective and if $(X, \cdot, L_{\mathcal{I}})$ is $\mathcal{I}$-complete, then $\phi(x) = x$ for each $(x_n) \in C_{\mathcal{I}}[X]$. Thus, ϕ is bijective. Define an (group) operator $*$ on X^* by $[(x_n)] * [(y_n)] = [(x_ny_n)] = [(y_nx_n)] = [(y_n)] * [(x_n)]$. It follows that $(X^*, *)$ is a commutative group.

Lemma 4.1. *Let $(X, \cdot, L_{\mathcal{I}})$ be a sequential $\mathcal{I}$-convergence group. Then the following hold:*

(a) $((x_n), x) \in L_{\mathcal{I}}$ *if and only if* $(x_n) \in C_{\mathcal{I}}[X]$ *and* $(x_n) \in x$.
(b) *If* $(x_n) \in C_{\mathcal{I}}[X]$ *and* $(y_n) \in C_{\mathcal{I}}[X]$, *then* $(x_ny_n) \in C_{\mathcal{I}}[X]$
(c) *If* $(x_n) \in C_{\mathcal{I}}[X]$ *and* $(y_n) \in S[X]$ *with* $((x_ny_n^{-1}), e) \in L_{\mathcal{I}}$, *then* $(y_n) \in C_{\mathcal{I}}[X]$ *and* $(y_n) \in [(x_n)]$.

Proof. (a) Let $((x_n), x) \in L_{\mathcal{I}}$ and $(x_{n_i}), (x_{n_j})$ be subsequences of (x_n). Then by (S$\mathcal{I}$2), $((x_{n_i}), x) \in L_{\mathcal{I}}$ and $((x_{n_j}), x) \in L_{\mathcal{I}}$. Hence $((x_{n_i}x_{n_j}^{-1}),\ xx^{-1}) \in L_{\mathcal{I}}$, by (S$\mathcal{I}G$) and so $((x_{n_i}x_{n_j}^{-1}), e) \in L_{\mathcal{I}}$. Therefore, $(x_n) \in C_{\mathcal{I}}[X]$. Since $((x_n), x) \in L_{\mathcal{I}}$, $(x, x) \in L_{\mathcal{I}}, ((x_nx^{-1}), xx^{-1}) \in L_{\mathcal{I}}$. Therefore, $((x_nx^{-1}), e) \in L_{\mathcal{I}}$. Hence $(x_n) \in x$.

Conversely, let $(x_n) \in C_{\mathcal{I}}[X]$ with $(x_n) \in x$. Then for each subsequence (x_{n_i}) of (x_n), $((xx_{n_i}^{-1}), x) \in L_{\mathcal{I}}$, since $(x_n) \in x$. Hence $((xx_n^{-1}), e) \in L_{\mathcal{I}}$. It follows that $((x_n), x) \in L_{\mathcal{I}}$, by (S$\mathcal{I}C$1).

(b) Let $(x_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in C_{\mathcal{I}}[X]$. Then for each subsequences (x_{n_i}) and (x_{n_j}) of (x_n) and subsequences (y_{n_i}) and (y_{n_j}) of (y_n), $((x_{n_i}x_{n_j}^{-1}), e) \in L_{\mathcal{I}}$ and $((y_{n_i}y_{n_j}^{-1}), e) \in L_{\mathcal{I}}$. Hence $(((x_{n_i}x_{n_j}^{-1})(y_{n_i}y_{n_j}^{-1})),$
$e) = (((x_{n_i}y_{n_i})(x_{n_j}y_{n_j})^{-1})), e) \in L_{\mathcal{I}}$. Therefore, $(x_ny_n) \in C_{\mathcal{I}}[X]$.

(c) Suppose that $(x_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in S[X]$ with $((x_ny_n^{-1}), e) \in L_{\mathcal{I}}$. Then by (S$\mathcal{I}C$2), for each subsequences of $(x_{n_i}y_{n_i}^{-1})$ and $(x_{n_j}y_{n_j}^{-1})$ of $(x_ny_n^{-1})$ such that $((x_{n_i}y_{n_i}^{-1}), e) \in L_{\mathcal{I}}$ and $((x_{n_j}y_{n_j}^{-1}), e) \in L_{\mathcal{I}}$. Since $y_{n_i}y_{n_j}^{-1} = (x_{n_i}^{-1}y_{n_i})(x_{n_j}y_{n_j}^{-1})(x_{n_i}x_{n_j}^{-1})$ and $x_{n_i}y_{n_j}^{-1} = (x_{n_i}x_{n_j}^{-1})(x_{n_j}y_{n_j}^{-1})$ and $(x_n) \in C_{\mathcal{I}}[X]$. Hence $(x_{n_j}y_{n_j}^{-1}, e) \in L_{\mathcal{I}}$ and $((y_{n_i}y_{n_j}^{-1}), e) \in L_{\mathcal{I}}$. Therefore, $(y_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in [(x_n)]$. □

Theorem 4.1. *Let $(X, \cdot, L_{\mathcal{I}})$ be a sequential $\mathcal{I}$-convergence group. Then $(X^*, *)$ is a commutative group containing $(X, \cdot)$ as a subgroup.*

Proof. First we shall show that $(X^*, *)$ is a commutative group.

(1) Since X^* is non-empty, there exists $[(x_n)] \in X^*$. If we take $a = [(x_n)]$ and $b = [(x_n)]$, then $a * b^{-1} = [(x_n)] * [(x_n)]^{-1} = e$ where e is the identity element. Therefore, $e \in X^*$.

(2) Let $[(x_n)] \in X^*$. If $a = e$ and $b = [(x_n)]$, it follows that $a * b^{-1} = e * [(x_n)]^{-1} = [(x_n)]^{-1} \in X^*$.

(3) Let $[(x_n)], [(y_n)] \in X^*$. Then $[(y_n)]^{-1} \in X^*$. Hence $[(x_n)]*[[(y_n)]^{-1}]^{-1} = [(x_n)] * [(y_n)]$ in X^*. So X^* is closed under the operation $*$.

(4) Let $[(x_n)], [(y_n)]$ and $[(z_n)]$ in X^*. Then $[(x_n)]*([(y_n)]*[(z_n)]) = [(x_n)]*([(y_n * z_n)]) = [(x_n(y_n z_n))] = [(x_n y_n)(z_n))] = [(x_n y_n)] * [(z_n)]$. Therefore, X^* satisfies associative property.

(5) If $[(x_n)], [(y_n)] \in X^*$, then $[(x_n)] * [(y_n)] = [(x_n y_n)] = [(y_n x_n)] = [(y_n)] * [(x_n)]$. Therefore, $(X^*, *)$ is a commutative group. Hence $(X^*, *)$ is a commutative group containing $(X, \cdot)$, since $(X, \cdot)$ is a subgroup of $(X^*, *)$.

Now we will construct a sequential $\mathcal{I}$-convergence structure $L^*_{\mathcal{I}}$ on X^*. Let $L^*_{\mathcal{I}}$ be the set of all pairs $((\alpha_n), \alpha) \in S[X^*] \times X^*$ satisfying the condition that there exists $(x_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m x_m^{-1} = \alpha[(x_n)]^{-1}$ for each $m \in \mathbb{N}$. □

Lemma 4.2. *If $L^*_{\mathcal{I}}$ is a sequential $\mathcal{I}$-convergence structure on X^*, then the following hold:*

(a) *For each $((x_n), x) \in L_{\mathcal{I}}$, $((x_n), x) \in L^*_{\mathcal{I}}$*
(b) *For each $(x_n) \in C_{\mathcal{I}}[X]$, $((x_n), [(x_n)]) \in L^*_{\mathcal{I}}$*
(c) *$L^*_{\mathcal{I}}$ satisfies (S$\mathcal{I}$C1) and (S$\mathcal{I}$C2).*
(d) *If $((\alpha_n), \alpha) \in L^*_{\mathcal{I}}$, then $\alpha_n \alpha_m^{-1} \in X$ for each $n, m \in \mathbb{N}$.*
(e) *If $((\alpha_n), \alpha) \in L^*_{\mathcal{I}}$ and $((\beta_n), \beta) \in L^*_{\mathcal{I}}$, then $((\alpha_n \beta_n^{-1}), \alpha\beta^{-1})$, that is, L^* satisfied (S$\mathcal{I}$G) .*

Proof. (a) Suppose $((x_n), x) \in L_{\mathcal{I}}$. By Lemma 4.1(a), $(x_n) \in C_{\mathcal{I}}[X]$ and $(x_n) \in x$. Therefore, for each subsequence (x_{n_i}) of (x_n), $((x x_{n_i}^{-1}), e) \in L_{\mathcal{I}}$. Therefore, $x \in [(x_n)]$ implies $x[(x_n)]^{-1} = e$. Thus, $((x_n), x) \in L^*_{\mathcal{I}}$

(b) For each $(x_n) \in C_{\mathcal{I}}[X]$, $[(x_n)] \in X^*$. Therefore, $((x_n), [(x_n)]) \in L^*_{\mathcal{I}}$, since $x_m x_m^{-1} = [(x_n)][(x_n)]^{-1}$

(c) Since $(x) \in C_{\mathcal{I}}[X]$, $((x), x) \in L^*_{\mathcal{I}}$ for each $x \in X$. If $((x_n), x) \in L^*_{\mathcal{I}}$, then there exists $(y_n) \in C_{\mathcal{I}}[X]$ such that $x_m y_m^{-1} = x[(y_n)]^{-1}$ for each $m \in \mathbb{N}$. Let (x_{n_i}) be a subsequence of (x_n). Then $x_{n_i} y_{m_i}^{-1} = x[(y_{n_i})]^{-1}$ for each $m_i \in \mathbb{N}$. Therefore, $((x_{n_i}), x) \in L^*_{\mathcal{I}}$.

(d) Since $((\alpha_n), \alpha) \in L^*_{\mathcal{I}}$, there exists $(x_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m x_m^{-1} = \alpha[(x_n)]^{-1}$ for each $m \in \mathbb{N}$. Hence $(\alpha_n x_n^{-1})(\alpha_m x_m^{-1})^{-1} = e$ for each $n, m \in \mathbb{N}$. Thus, $\alpha_n \alpha_m^{-1} = x_n x_m^{-1} \in X$ for each $n, m \in \mathbb{N}$.

(e) Since $((\alpha_n), \alpha) \in L^*_{\mathcal{I}}$ and $((\beta_n), \beta) \in L^*_{\mathcal{I}}$, there are $(x_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m x_m^{-1} = \alpha[(x_n)]^{-1}$ for each $m \in \mathbb{N}$ and $\beta_k y_k^{-1} = \beta[(y_n)]^{-1}$ for each $k \in \mathbb{N}$. Hence $(\alpha_m \beta_k^{-1})(x_m y_k^{-1})^{-1} = (\alpha\beta)^{-1}[(x_n y_n^{-1})]^{-1}$

for each $m, k \in \mathbb{N}$. Thus, $((\alpha_n \beta_n^{-1}), \alpha\beta^{-1}) \in L^*_{\mathcal{I}}$, by (S$\mathcal{I}G$) and Lemma 4.1(b). □

Lemma 4.3. *Let $((\alpha_n), \alpha) \in L^*_{\mathcal{I}}$ and let (x_{n_m}) be a double sequence in X with $[(x_{n_m})] = \alpha_n$ for each $n \in \mathbb{N}$. Then there exists a cross-sequence $(x_{n_{m(n)}})$ in (x_{n_m}) such that (i) $[(x_{n_{m(n)}})] = \alpha$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $[(x_{n_{k(n)}})] = \alpha$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$.*

Proof. We show that in particular case $\alpha_n = \alpha$ for each $n \in \mathbb{N}$. By Lemma 4.2(e), we have $(x_{n_m} x_{1_m}^{-1})$ with $((x_{n_m} x_{1_m}^{-1}), e) \in L^*_{\mathcal{I}}$ for each $n \in \mathbb{N}$. First we prove that $L_{\mathcal{I}}$ satisfy $(*)$. (S$\mathcal{I}$C1) obviously. Let $((x_n), x) \in L_{\mathcal{I}}, ((x_{n_i}), x) \in L_{\mathcal{I}}$ and $((x_{n_m},)x_n) \in L_{\mathcal{I}}$ for each $n \in \mathbb{N}$. Choose a cross-sequence $(x_{n_{m(ni)}})$ in the double sequence (x_{n_m}) such that (i) $((x_{n_{m(ni)}}), x) \in L_{\mathcal{I}}$, (ii) $m_{ni} \geq ni$ for all $ni \in \mathbb{N}$ and (iii) $((x_{n_{k(ni)}}), x) \in L_{\mathcal{I}}$ if $k(ni) \geq m(ni)$ for all $n \in \mathbb{N}$, (S$\mathcal{I}C$3) already satisfied $(*)$. Thus, $L_{\mathcal{I}}$ satisfy $(*)$. Therefore, there exists a cross-sequence $(x_{n_{m(n)}} x_{l_{m(n)}}^{-1}) \in C_{\mathcal{I}}[X]$ in $(x_{n_m} x_{l_m}^{-1})$ such that (1) $((x_{n_{m(n)}} x_{l_{m(n)}}^{-1}), e) \in L_{\mathcal{I}}$, (2) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (3) $((x_{n_{k(n)}} x_{l_{k(n)}}^{-1}), e) \in L_{\mathcal{I}}$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$. Since $[(x_{l_m})] = \alpha$ and $L^*_{\mathcal{I}}$ satisfies (S$\mathcal{I}$2), $[(x_{l_{m(n)}})] = \alpha$ so $[(x_{n_{m(n)}})] = \alpha$, by Lemma 4.2(d). Thus, we have a cross-sequence $(x_{n_{m(n)}})$ in x_{n_m} such that (i) $[(x_{n_{m(n)}})] = \alpha$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $[(x_{n_{k(n)}})] = \alpha$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$.

Now we prove this lemma in general case. Since $((\alpha_n), \alpha) \in L^*_{\mathcal{I}}$ and by definition of $L^*_{\mathcal{I}}$, there exists $(y_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m y_m^{-1} = \alpha[(y_n)]^{-1}$ for each $m \in \mathbb{N}$. Hence a double sequence $(x_{n_m} y_n^{-1})$ in X with $[(x_{p_m} y_p^{-1})] = \alpha[(y_n)]^{-1}$ for each $p \in \mathbb{N}$. By the above particular case, there exists a cross-sequence $(x_{n(m)} y_n^{-1}) \in C_{\mathcal{I}}[X]$ such that (i) $[(x_{n_{m(n)}})] = \alpha[(y_n)]^{-1}$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $[(x_{n_{k(n)}})] = \alpha[(y_n)]^{-1}$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$. Hence by Lemma 4.2(d), a cross-sequence $(x_{n_{m(n)}})$ in (x_{n_m}) is such that (i) $\alpha = \alpha[(y_n)]^{-1}[(y_n)]$, where $\alpha[(y_n)]^{-1} = [(x_{n_{m(n)}} y_n^{-1})]$. Therefore, $\alpha = [(x_{n_{m(n)}} y_n^{-1})][(y_n)] = [(x_{n_{m(n)}} y_n^{-1}) y_n] = [(x_{n_{m(n)}})]$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $[(x_{n_{k(n)}})] = \alpha$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$. □

Theorem 4.2. *Assume that $L^*_{\mathcal{I}}$ satisfies the following condition:*

$(**)$ *Let $\alpha \in X^*$ and (α_{nm}) be a double sequence in X^* with $((\alpha_{nm}), \alpha) \in L^*_{\mathcal{I}}$ for each $n \in \mathbb{N}$. It is possible to choose $(x_n) \in C_{\mathcal{I}}[X]$ satisfying the property that for each $p \in \mathbb{N}$, there exists a sequence $(x_{n_{p(m)}})$ of (x_n) such that $\alpha_{pm} x_{n_{p(m)}}^{-1} = \alpha[(x_n)]^{-1}$ for all $m \in \mathbb{N}$.*

Then $(X^*, *, L^*_{\mathcal{I}})$ is an $\mathcal{I}$-complete sequential $\mathcal{I}$-convergence group containing $(X, \cdot, L_{\mathcal{I}})$.

Proof. First, we prove that $L^*_{\mathcal{I}}$ satisfies $(*)$. Let $((\alpha_n), \alpha) \in L^*_{\mathcal{I}}$ and let (α_{nm}) be a double sequence in X^* with $((\alpha_{nm}), \alpha_n) \in L^*_{\mathcal{I}}$ for each $n \in \mathbb{N}$. Since $((\alpha_n), \alpha) \in L^*_{\mathcal{I}}$, there exists $(x_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m x_m^{-1} = \alpha[(x_n)]^{-1}$ for each $m \in \mathbb{N}$.

Let $\alpha[(x_n)]^{-1} = \beta$. Then by $(**)$, there exists $(y_n) \in C_{\mathcal{I}}[X]$ satisfying the property that for each $p \in \mathbb{N}$, there exists a subsequence $(y_{n_{p(m)}})$ of (y_n) such that $\alpha_{p_m} x_p^{-1} y_{n_{p(m)}}^{-1} = \beta[(y_n)]^{-1}$ for all $m \in \mathbb{N}$. Since $(y_n) \in C_{\mathcal{I}}[X]$ and by Lemma 4.2(b) and (c), $((y_n), [(y_n)]) \in L_{\mathcal{I}}^*$ and so $[(y_{n_{p(m)}})] = [(y_n)]$ for each $p \in \mathbb{N}$ and hence by Lemma 4.3, there exists a cross-sequence $(y_{n_m(im)}) \in C_{\mathcal{I}}[X]$ in the double sequence $(y_{n_{p(m)}})$ such that (1) $[(y_{n_m(i(m))})] = [(y_n)]$, (2) $i(m) \geq m$ for all $m \in \mathbb{N}$ and (3) $[(y_{n_m(j(m))})] = [(y_n)]$ if $j(m) \geq i(m)$ for all $m \in \mathbb{N}$.

Thus, $((\alpha_{m_{i(m)}} x_m^{-1} y_{n_m(i(m))}^{-1}), \beta[(y_n)]^{-1}) \in L_{\mathcal{I}}^*$.

That is, $\alpha_{1_{i}(1)} x_1^{-1} y_{n_1(i(1))}^{-1}, \alpha_{2_{i}(2)} x_2^{-1} y_{n_2(i(2))}^{-1}, ..., \xrightarrow{\mathcal{I}} \beta[(y_n)]^{-1}$. Hence by Lemma 4.2(e), $((\alpha_{n_{i(n)}}), \alpha) \in L_{\mathcal{I}}^*$. Since the cross-sequence $y_{n_m(i(m))}$ satisfies the above three properties, $i(n) \geq n$ for all $n \in \mathbb{N}$ and $((\alpha_{n_{j(n)}}), \alpha) \in L_{\mathcal{I}}^*$ if $j(n) \geq i(n)$ for all $n \in \mathbb{N}$. Therefore, $L_{\mathcal{I}}^*$ satisfies $(*)$.

Therefore, by Theorem 4.1 and Lemma 4.2, $(X^*, *, L_{\mathcal{I}}^*)$ is a sequential $\mathcal{I}$-convergence group containing $(X, \cdot, L_{\mathcal{I}})$.

Next we show that $(X^*, *, L_{\mathcal{I}}^*)$ is $\mathcal{I}$-complete. Let (α_n) be an $\mathcal{I}$-Cauchy sequence in X^*. Then, by the definition of $\mathcal{I}$-Cauchy, $((\alpha_{i_n} \alpha_{j_n}), e) \in L_{\mathcal{I}}^*$ for each subsequences (α_{i_n}) and (α_{j_n}) of (α_n). The proof will be divided into two cases.

Case 1. There exists a subsequence (α_{i_n}) of (α_n) such that $\alpha_{i_n} \alpha_{i_m}^{-1}$ for each $n, m \in \mathbb{N}$. Then by Lemma 4.2(b), $(\alpha_{i_n} \alpha_{i_1})^{-1} \in C_{\mathcal{I}}[X]$ and so

$$((\alpha_{i_n} \alpha_{i_1}^{-1}), [(\alpha_{i_n} \alpha_{i_1}^{-1})]) \in L_{\mathcal{I}}^*.$$

Therefore, $((\alpha_n), [(\alpha_{i_n})]) \in L_{\mathcal{I}}^*$, by Lemma 4.2(e).

Case 2. There does not exist a subsequence (α_{n_j}) of (α_n) such that $\alpha_{i_n} \alpha_{i_m}^{-1}$ for each $n, m \in \mathbb{N}$. Without loss of generality, we assume that $\alpha_n \alpha_m^{-1} \notin X$ for each $n \neq m \in \mathbb{N}$. Now, we determine a subsequence (α_{i_n}) of (α_n). Let $\alpha_{i_1} = \alpha_2$, then we can choose α_{i_2} in $\{\alpha_3, \alpha_4\}$. Satisfying $(\alpha_1 \alpha_{i_1}^{-1})(\alpha_2 \alpha_{i_2}^{-1})^{-1} \notin X$.

Suppose we choose $k-1$ natural numbers i_m such that $2^{m-1} \lneqq i_m \leq 2^m$ and $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \notin X$ for each $n \neq m \in \{1, 2, ..., k-1\}$.

Now, we show that there exists $\alpha_{i_k} \in \{\alpha_{2^{k-1}+1}, \alpha_{2^{k-1}+2}, ..., \alpha_{2^k}\}$ such that $(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_{i_k}^{-1})^{-1} \notin X$ for each $n \in \{1, 2, ..., k-1\}$. Assume that there does not exists such α_{i_k}. Then $(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_p^{-1})^{-1} \in X$ for all $p \in \{2^{k-1}+1, 2^{k-1}+2, ..., 2^k\}$ and for all $n \in \{1, 2, ..., k-1\}$. Therefore, $\{(\alpha_n \alpha_{n_i}^{-1})(\alpha_k \alpha_p^{-1})^{-1}\}\{(\alpha_n \alpha_{n_i}^{-1})(\alpha_k \alpha_q^{-1})^{-1}\}^{-1} = \alpha_p \alpha_q^{-1} \in X$ which is a contradiction to $\alpha_n \alpha_m^{-1} \notin X$. Thus, by induction, we have a subsequence (α_{i_n}) of (α_n) such that $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \notin X$ for all $n \neq m \in \mathbb{N}$.

Since (α_n) is $\mathcal{I}$-Cauchy sequence in X^*, $((\alpha_n \alpha_{i_n})^{-1}, e) \in L_{\mathcal{I}}^*$. Therefore, by Lemma 4.2(d), $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \in X$ for each $n, m \in \mathbb{N}$, which is a contradiction to $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \notin X$. It follows that the case 2 can not occur. Hence $(X^*, *, L_{\mathcal{I}}^*)$ is an $\mathcal{I}$-complete. □

Example 4.1. Let $L_{\mathcal{I}} = \{((x_n), x) \in S[\mathbb{R}] \times \mathbb{R} \mid (x_n)\ \mathcal{I}-$converges to $x \in \mathbb{R}$ with the usual topology $\}$, where $\mathcal{I} = \{J \subset \mathbb{N} \mid d(J) = 0\}$. Let $\cdot$ be the usual multiplication on $\mathbb{R}$. Suppose $(x_n) = (\frac{1}{n})$ and $(y_n) = (\frac{1}{n+1})$. Then $(\mathbb{R}, \cdot, L_{\mathcal{I}})$ is a sequential $\mathcal{I}$-convergence group and $L^*_{\mathcal{I}}$ satisfies $(**)$. Hence $(\mathbb{R}, \cdot, L_{\mathcal{I}})$ is $\mathcal{I}$-complete.

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