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Ostrowski's inequalities for functions whose first derivatives are s-logarithmically preinvex in the second sense

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Abstract. In this paper, some Ostrowski's inequalities for functions whose first derivatives are s-logarithmically preinvex in the second sense are established.

1. Introduction

In 1938, A. M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

Theorem 1.1 ([9]). *Let $I \subseteq \mathbb{R}$ be an interval. Let $f : I \to \mathbb{R}$, be a differentiable mapping in the interior I° of I, and $a, b \in I^\circ$ with $a < b$. If $|f'| \leq M$ for all $x \in [a,b]$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}\,t \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad x \in [a,b].$$

This is well-known Ostrowski's inequality. In recent years, a number of authors have written about generalizations, extensions and variants of such inequalities one can see [4, 5, 6, 7, 8, 15] and the reference cited therein.

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson in [3], introduced a new class of generalized convex functions, called invex functions. In [2], the authors gave the concept of preinvex functions which is special case of invexity. Pini [13], Noor [10, 11], Yang and Li [18] and Weir [17], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

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Meftah [8] established the following Ostrowski's inequality for functions whose derivative are log-preinvex.

Theorem 1.2. *Let $K \subseteq [0,\infty)$ be an invex subset with respect to $\eta : K \times K \to \mathbb{R}$, $a, b \in K^\circ$ (K° interior of K) with $\eta(b,a) > 0$ and $[a, a+\eta(b,a)] \subset K$. Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a differentiable function such that $f' \in L([a, a+\eta(b,a)])$ and $f'(a) \neq 0$. If $|f'|$ is logarithmically preinvex function, then the following inequality holds:*

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right| \leq \frac{\eta(b,a)\,|f'(a)|}{2}$$

$$\times \begin{cases} \left(\left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1 - \frac{x-a}{\eta(b,a)}\right)^2 \right), & \text{if } A = 1, \\ 2\left[\left(2\frac{x-a}{\eta(b,a)} - 1 \right) \frac{A^{\frac{x-a}{\eta(b,a)}}}{\ln A} + \frac{1 - 2A^{\frac{x-a}{\eta(b,a)}} + A}{\ln^2 A} \right], & \text{if } A \neq 1, \end{cases}$$

for all $x \in [a, a+\eta(b,a)]$, where $A = \frac{|f'(b)|}{|f'(a)|}$.

Theorem 1.3. *Let $K \subseteq [0,\infty)$ be an invex subset with respect to $\eta : K \times K \to \mathbb{R}$, $a, b \in K^\circ$ (K° interior of K) with $\eta(b,a) > 0$ and $[a, a+\eta(b,a)] \subset K$. Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a differentiable function such that $f' \in L([a, a+\eta(b,a)])$ and $f'(a) \neq 0$, let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f'|^q$ is a logarithmically preinvex function, then the following inequality holds:*

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right| \leq \frac{\eta(b,a)\,|f'(a)|}{(p+1)^{\frac{1}{p}}}$$

$$\times \begin{cases} \left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1 - \frac{x-a}{\eta(b,a)}\right)^2, \ \text{if } A = 1, \\ \left(\frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}} \left(\frac{A^{q\frac{x-a}{\eta(b,a)}} - 1}{q \ln A} \right)^{\frac{1}{q}} + \left(1 - \frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}} \left(\frac{A^q - A^{q\frac{x-a}{\eta(b,a)}}}{q \ln A} \right)^{\frac{1}{q}}, \\ \text{if } A \neq 1, \end{cases}$$

for all $x \in [a, a+\eta(b,a)]$, where $A = \frac{|f'(b)|}{|f'(a)|}$.

Theorem 1.4. *Let $K \subseteq [0,\infty)$ be an invex subset with respect to $\eta : K \times K \to \mathbb{R}$, $a, b \in K^\circ$ (K° interior of K) with $\eta(b,a) > 0$ and $[a, a+\eta(b,a)] \subset K$. Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a differentiable function such that $f' \in L([a, a+\eta(b,a)])$ and $f'(a) \neq 0$, let $q > 1$. If $|f'|^q$ is a logarithmically preinvex function, then the following inequality holds:*

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right| \leq \frac{\eta(b,a)}{2^{1-\frac{1}{q}}} |f'(a)|$$

$$\times\begin{cases}\frac{1}{2^{\frac{1}{q}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^2+\left(1-\frac{x-a}{\eta(b,a)}\right)^2\right), \ \textit{if } A=1,\\ \left(\left(\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}}\left(\frac{x-a}{\eta(b,a)}\frac{A^{q\frac{x-a}{\eta(b,a)}}}{\ln A}+\frac{1-A^{q\frac{x-a}{\eta(b,a)}}}{\ln^2 A}\right)^{\frac{1}{q}}+\left(1-\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}}\right.\\ \left.\times\left(\frac{A^q-A^{q\frac{x-a}{\eta(b,a)}}}{\ln^2 A}-\left(1-\frac{x-a}{\eta(b,a)}\right)\frac{A^{q\frac{x-a}{\eta(b,a)}}}{\ln A}\right)^{\frac{1}{q}}\right), \ \textit{if } A\neq 1,\end{cases}$$

for all $x\in[a,a+\eta(b,a)]$, *where* $A=\frac{|f'(b)|}{|f'(a)|}$.

Sarikaya et al. [14] established the following midpoint inequalities for differentiable log-preinvex functions.

Theorem 1.5. *Let* $K\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta : K\times K\to\mathbb{R}$. *Suppose that* $f:K\to\mathbb{R}$ *is a differentiable function. If* $|f'|$ *is* log-*preinvex on* K *then, for every* $a,b\in K$ *the following inequality holds:*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u-f\left(\frac{2a+\eta(b,a)}{2}\right)\right|\leq\eta(b,a)\left(\frac{|f'(b)|^{\frac{1}{2}}-|f'(a)|^{\frac{1}{2}}}{\log|f'(a)|-\log|f'(a)|}\right)^2.$$

Theorem 1.6. *Let* $K\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta : K\times K\to\mathbb{R}$. *Suppose that* $f:K\to\mathbb{R}$ *is a differentiable function. Assume* $q\in\mathbb{R}$ *with* $q\geq 1$. *If* $|f'|^q$ *is* log-*preinvex on* K *then, for every* $a,b\in K$ *the following inequality holds:*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u-f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq\ \eta(b,a)\frac{|f'(a)|^{\frac{1}{2}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}q^{\frac{1}{q}}}\left(\frac{|f'(b)|^{\frac{q}{2}}-|f'(a)|^{\frac{q}{2}}}{\log|f'(a)|-\log|f'(a)|}\right)^{\frac{1}{q}}.$$

Motivated by the above results, in this paper we establish some new Ostrowski type inequalities for functions whose first derivatives are logarithmically s-preinvex in the second sense.

2. Preliminaries

In this section we recall some concepts of convexity that are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 2.1 ([12])**.** A positive function $f:I\to\mathbb{R}$ is said to be logarithmically convex, if

$$f(tx+(1-t)y)\ \leq[f(x)]^t\,[f(y)]^{1-t}$$

holds for all $x,y\in I$ and $t\in[0,1]$.

Definition 2.2. A positive function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-logarithmically convex function in the second sense on I, if the following inequality

$$f(tx + (1-t)y) \ \leq [f(x)]^{t^s} \, [f(y)]^{(1-t)^s}$$

holds for some $s \in (0, 1]$, all $x, y \in I$ and $t \in [0, 1]$.

Definition 2.3 ([17])**.** A set K is said to be invex at x with respect to η, if

$$x + t\eta\,(y, x) \in K$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 2.4 ([10])**.** A positive function f on the invex set K is said to be logarithmically preinvex function with respect to η, if

$$f\,(x + t\eta\,(y, x)) \leq [f\,(x)]^{1-t} \, [f(y)]^{t}$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.5 ([16])**.** A positive function f on the invex set $K \subseteq [0, \infty)$ is said to be s-logarithmically preinvex function in the second sense with respect to η, if

$$f\,(x + t\eta\,(y, x)) \leq [f\,(x)]^{(1-t)^s} \, [f(y)]^{t^s}$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

The following lemmas are essential to establishing our main results.

Lemma 2.1 ([1])**.** *Let $0 < \phi \leq 1 \leq \psi$ and $t, s \in (0, 1]$, then*

$$\begin{aligned} \phi^{t^s} &\leq \phi^{st}, \\ \psi^{t^s} &\leq \psi^{st+1-s}. \end{aligned}$$

Lemma 2.2 ([4])**.** *Let $A \subset \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \ \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a + \eta(b, a)]$, then the following equality*

$$f(x) - \frac{1}{\eta\,(b, a)} \int\limits_{a}^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u$$

$$= \eta\,(b, a) \left(\int\limits_{0}^{\frac{x-a}{\eta(b,a)}} t f'(a + t\eta\,(b, a))\,\mathrm{d}\,t + \int\limits_{\frac{x-a}{\eta(b,a)}}^{1} (t-1)\, f'(a + t\eta\,(b, a))\,\mathrm{d}\,t \right)$$

holds for all $x \in [a, a + \eta(b, a)]$.

3. Main Results

In what follows we assume that $K \subseteq [0,\infty)$ be an invex subset with respect to the bifunction $\eta : K\times K \to \mathbb{R}$ and $a,b\in K^{\circ}$interior of K with $a<a+\eta(b,a)$ such that $[a,a+\eta(b,a)]\subset K$.

Theorem 3.1. *Let $f:[a,a+\eta(b,a)]\to(0,\infty)$ be a differentiable function such that $f'\in L([a,a+\eta(b,a)])$. If $|f'|$ is s-logarithmically preinvex function in the second sense for some fixed $s\in(0,1]$ with $|f'(a)|\neq 0$, then for all $x\in[a,a+\eta(b,a)]$ we have the following inequality*

$$\left|f(x)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)\,\mathrm{d}u\right|\leq\eta(b,a)$$
$$\times\begin{cases} N_{(s,\lambda)}\left(\frac{\left(2\frac{x-a}{\eta(b,a)}-1\right)\lambda^{s\frac{x-a}{\eta(b,a)}}}{s\ln\lambda}+\frac{1+\lambda^s-2\lambda^{s\frac{x-a}{\eta(b,a)}}}{(s\ln\lambda)^2}\right), & if\ \lambda\neq 1;\\ \frac{|f'(a)|^s}{2}\left(\left(\frac{x-a}{\eta(b,a)}\right)^2+\left(1-\frac{x-a}{\eta(b,a)}\right)^2\right), & if\ |f'(a)|<1=\lambda;\\ \frac{|f'(a)||f'(b)|^{1-s}}{2}\left(\left(\frac{x-a}{\eta(b,a)}\right)^2+\left(1-\frac{x-a}{\eta(b,a)}\right)^2\right), & if\ |f'(a)|>1=\lambda;\end{cases}$$

where

$$\lambda=\frac{|f'(b)|}{|f'(a)|}, \tag{1}$$

and

$$N_{(s,\lambda)}=\begin{cases}|f'(a)|^s, & if\ |f'(a)|,|f'(b)|<1;\\ |f'(a)|^s|f'(b)|^{1-s}, & if\ |f'(a)|\leq 1\leq|f'(b)|;\\ |f'(a)|, & if\ |f'(b)|\leq 1\leq |f'(a)|;\\ |f'(a)||f'(b)|^{1-s} & if\ |f'(a)|,|f'(b)|>1.\end{cases} \tag{2}$$

Proof. From Lemma 2.2, and property of modulus, we have

$$\left|f(x)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)\,\mathrm{d}u\right|$$
$$\leq\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}}t\left|f'(a+t\eta(b,a))\right|\mathrm{d}t+\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)\left|f'(a+t\eta(b,a))\right|\mathrm{d}t\right).$$

Since $|f'|$ is s-logarithmically preinvex function, we deduce

$$\begin{aligned} &\left| f(x) - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right| \\ \le\ & \eta(b,a) \left(\int_0^{\frac{x-a}{\eta(b,a)}} t\left|f'(a)\right|^{(1-t)^s} \left|f'(b)\right|^{t^s} \mathrm{d}\,t \right. \\ &\left. + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)\left|f'(a)\right|^{(1-t)^s} \left|f'(b)\right|^{t^s} \mathrm{d}\,t \right). \end{aligned} \tag{3}$$

From Lemma 2.1, we have

$$\left|f'(a)\right|^{(1-t)^s} \left|f'(b)\right|^{t^s} \le N_{(s,\lambda)} \times \lambda^{st}, \tag{4}$$

where λ and $N_{(s,\lambda)}$ are defined by (1) and (2) respectively.

Substituting (4) into (3), we obtain

$$\begin{aligned} &\left| f(x) - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right| \le \eta(b,a) \times N_{(s,\lambda)} \\ &\times \left(\int_0^{\frac{x-a}{\eta(b,a)}} t\lambda^{st} dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)\lambda^{st}\,\mathrm{d}\,t \right). \end{aligned} \tag{5}$$

Clearly, in the case where $\lambda \neq 1$, we have

$$\int_0^{\frac{x-a}{\eta(b,a)}} t\lambda^{st} = \frac{\frac{x-a}{\eta(b,a)} \lambda^{s\frac{x-a}{\eta(b,a)}}}{s\ln\lambda} + \frac{1-\lambda^{s\frac{x-a}{\eta(b,a)}}}{(s\ln\lambda)^2}, \tag{6}$$

and

$$\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)\lambda^{st}\,\mathrm{d}\,t = \frac{\lambda^s - \lambda^{s\frac{x-a}{\eta(b,a)}}}{(s\ln\lambda)^2} - \frac{(1-\frac{x-a}{\eta(b,a)})\lambda^{s\frac{x-a}{\eta(b,a)}}}{s\ln\lambda}. \tag{7}$$

Substituting (6) and (7) into (5), we obtain

$$\left| f(x) - \frac{1}{\eta(b,a)} \int\limits_{a}^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right| \leq \eta(b,a)$$

$$\times N_{(s,\lambda)} \left(\frac{\left(2\frac{x-a}{\eta(b,a)}-1\right)\lambda^{s\frac{x-a}{\eta(b,a)}}}{s\ln\lambda} + \frac{1+\lambda^s-2\lambda^{s\frac{x-a}{\eta(b,a)}}}{(s\ln\lambda)^2} \right). \tag{8}$$

Now, we assume that $\lambda = 1$, then (5) gives

$$\left| f(x) - \frac{1}{\eta(b,a)} \int\limits_{a}^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right| \leq \eta(b,a)$$

$$\times \begin{cases} \frac{|f'(a)|^s}{2} \left(\left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1 - \frac{x-a}{\eta(b,a)}\right)^2 \right), & \text{if } |f'(a)| < 1; \\ \frac{|f'(a)||f'(b)|^{1-s}}{2} \left(\left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1 - \frac{x-a}{\eta(b,a)}\right)^2 \right), & \text{if } |f'(a)| > 1. \end{cases} \tag{9}$$

The desired result follows from (8) and (9). □

Corollary 3.1. *In Theorem 3.1, if we choose* $x = \frac{2a+\eta(b,a)}{2}$*, then we obtain the following midpoint inequality*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int\limits_{a}^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right|$$

$$\leq \eta(b,a) \begin{cases} N_{(s,\lambda)} \left(\frac{1-\lambda^{\frac{s}{2}}}{s\ln\lambda}\right)^2, & \text{if } \lambda \neq 1; \\ \frac{|f'(a)|^s}{4}, & \text{if } |f'(a)| < 1 = \lambda; \\ \frac{|f'(a)||f'(b)|^{1-s}}{4}, & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Remark 3.1. Theorem 3.1 will be reduces to Theorem 7 from [8] and Corollary 1 will be reduces to Corollary 8 from [8] and Theorem 4.1 from [14] if we put $s = 1$.

Corollary 3.2. *In Theorem 3.1, if we choose* $\eta(b,a) = b - a$*, we have the following inequality*

$$\left| f(x) - \frac{1}{b-a} \int\limits_{a}^{b} f(u)\,\mathrm{d}\,u \right| \leq (b-a)$$

$$\times\begin{cases} N_{(s,\lambda)}\left(\frac{\left(\frac{2x-(b+a)}{b-a}\right)\lambda^{s\frac{x-a}{b-a}}}{s\ln\lambda}+\frac{1+\lambda^s-2\lambda^{s\frac{x-a}{b-a}}}{(s\ln\lambda)^2}\right), & \text{if } \lambda\neq 1;\\ \frac{|f'(a)|^s}{2}\left(\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right), & \text{if } |f'(a)|<1=\lambda;\\ \frac{|f'(a)||f'(b)|^{1-s}}{2}\left(\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right), & \text{if } |f'(a)|>1=\lambda. \end{cases}$$

Moreover if we choose $s=1$ we get the following inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u\right|\leq(b-a)$$

$$\times\begin{cases} |f'(a)|\left(\frac{\left(\frac{2x-(b+a)}{b-a}\right)\lambda^{\frac{x-a}{b-a}}}{\ln\lambda}+\frac{1+\lambda-2\lambda^{\frac{x-a}{b-a}}}{(\ln\lambda)^2}\right), & \text{if } \lambda\neq 1;\\ \frac{|f'(a)|}{2}\left(\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right), & \text{if } 1=\lambda. \end{cases}$$

Corollary 3.3. *In Theorem 3.1, if we choose $\eta(b,a)=b-a$, and $x=\frac{a+b}{2}$ we have the following inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u\right|\leq(b-a)$$

$$\times\begin{cases} N_{(s,\lambda)}\left(\frac{1+\lambda^s-2\lambda^{\frac{s}{2}}}{(s\ln\lambda)^2}\right), & \text{if } \lambda\neq 1;\\ \frac{|f'(a)|^s}{4}, & \text{if } |f'(a)|<1=\lambda;\\ \frac{|f'(a)||f'(b)|^{1-s}}{4}, & \text{if } |f'(a)|>1=\lambda. \end{cases}$$

Moreover if we choose $s=1$ we get the following inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u\right|\leq(b-a)\left|f'(a)\right|\times\begin{cases}\left(\frac{1-\lambda^{\frac{1}{2}}}{\ln\lambda}\right)^2, & \text{if } \lambda\neq 1,\\ \frac{|f'(a)|}{4}, & \text{if } 1=\lambda.\end{cases}$$

Remark 3.2. The case $\lambda\neq 1$ in the last inequality of Corollary 3.3 represent Corollary 4.3 from [14].

Theorem 3.2. *Let $f:K\to(0,\infty)$ be a differentiable function such that $f'\in L([a,a+\eta(b,a)])$, and let $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|f'|^q$ is s-logarithmically preinvex function in the second sense for some fixed $s\in(0,1]$ with $|f'(a)|\neq 0$, we have the following inequality*

$$\left|f(x)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u\right|\leq\frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}}$$

$$\times\begin{cases} N^{\frac{1}{q}}_{(s,q,\lambda)}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}}\left(\frac{\lambda^{qs\frac{x-a}{\eta(b,a)}}-1}{qs\ln\lambda}\right)^{\frac{1}{q}}\right. \\ \quad\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}}\left(\frac{\lambda^{qs}-\lambda^{qs\frac{x-a}{\eta(b,a)}}}{qs\ln\lambda}\right)^{\frac{1}{q}}\right), & \text{if } \lambda\neq 1; \\ |f'(a)|^{s}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{2}+\left(1-\frac{x-a}{\eta(b,a)}\right)^{2}\right), & \text{if } |f'(a)|\leq 1=\lambda; \\ |f'(a)|^{2-s}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{2}+\left(1-\frac{x-a}{\eta(b,a)}\right)^{2}\right), & \text{if } |f'(a)|> 1=\lambda, \end{cases}$$

where λ is defined as in (1)*, and*

$$(10)\qquad N_{(s,q,\lambda)}=\begin{cases} |f'(a)|^{qs}, & \text{if } |f'(a)|,\ |f'(b)|<1; \\ |f'(a)|^{qs}|f'(b)|^{q-qs}, & \text{if } |f'(a)|\leq 1\leq |f'(b)|; \\ |f'(a)|^{q}, & \text{if } |f'(b)|\leq 1\leq |f'(a)|; \\ |f'(a)|^{q}|f'(b)|^{q-qs}, & \text{if } |f'(a)|,\ |f'(b)|>1, \end{cases}$$

Proof. From Lemma 2.2, property of modulus, and Hölder's inequality, we have

$$\begin{aligned} &\left|f(x)-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(u)\,\mathrm{d}u\right| \\ \leq\ & \eta(b,a)\left(\left(\int_{0}^{\frac{x-a}{\eta(b,a)}}t^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{x-a}{\eta(b,a)}}\left|f'(a+t\eta(b,a))\right|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{p}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}\left|f'(a+t\eta(b,a))\right|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\right) \\ =\ & \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}}\left(\int_{0}^{\frac{x-a}{\eta(b,a)}}\left|f'(a+t\eta(b,a))\right|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\right. \\ &\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}}\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}\left|f'(a+t\eta(b,a))\right|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\right). \end{aligned}$$

Using the fact that $|f'|^q$ is s-logarithmically preinvex and Lemma 2.1, we obtain

$$\begin{aligned}
&\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right| \\
&\leq \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} \left|f'(a)\right|^{q(1-t)^s} \left|f'(b)\right|^{qt^s} \mathrm{d}\,t \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(1 - \frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 \left|f'(a)\right|^{q(1-t)^s} \left|f'(b)\right|^{qt^s} \mathrm{d}\,t \right)^{\frac{1}{q}} \right) \\
&\leq \frac{\eta(b,a) N_{(s,q,\lambda)}^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} \lambda^{qst}\,\mathrm{d}\,t \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(1 - \frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 \lambda^{qst}\,\mathrm{d}\,t \right)^{\frac{1}{q}} \right). \qquad (11)
\end{aligned}$$

For $\lambda \neq 1$, (11) becomes

$$\begin{aligned}
&\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right| \\
&\leq \frac{\eta(b,a)\, N_{(s,q,\lambda)}^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}} \left(\frac{\lambda^{qs\frac{x-a}{\eta(b,a)}} - 1}{qs\ln\lambda} \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(1 - \frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}} \left(\frac{\lambda^{qs} - \lambda^{qs\frac{x-a}{\eta(b,a)}}}{qs\ln\lambda} \right)^{\frac{1}{q}} \right), \qquad (12)
\end{aligned}$$

where λ and $N_{(s,q,\lambda)}$ are defined as in (1) and (10) respectively, and the fact that

$$\int_0^{\frac{x-a}{\eta(b,a)}} \lambda^{qst}\,\mathrm{d}\,t = \frac{\lambda^{qs\frac{x-a}{\eta(b,a)}}-1}{qs\ln\lambda},$$

$$\int_{\frac{x-a}{\eta(b,a)}}^{1} \lambda^{qst}\,\mathrm{d}\,t = \frac{\lambda^{qs}-\lambda^{qs\frac{x-a}{\eta(b,a)}}}{qs\ln\lambda}.$$

In the case where $\lambda = 1$, (11) becomes

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right|$$

$$\leq \begin{cases} \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}} \left|f'(a)\right|^s \left(\left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1-\frac{x-a}{\eta(b,a)}\right)^2 \right), & \text{if } |f'(a)| \leq 1; \\ \frac{\eta(b,a) N_{(s,q,\lambda)}^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} \left|f'(a)\right|^{2-s} \left(\left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1-\frac{x-a}{\eta(b,a)}\right)^2 \right), & \text{if } |f'(a)| > 1. \end{cases}$$

From (12) and (13) we get the desired result. □

Corollary 3.4. *In Theorem 3.2, if we choose $x = \frac{2a+\eta(b,a)}{2}$, then we obtain the following midpoint inequality*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right|$$

$$\leq \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}} \times \begin{cases} \frac{N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1+\frac{1}{p}}} \left(\left(\frac{\lambda^{\frac{qs}{2}}-1}{qs\ln\lambda}\right)^{\frac{1}{q}} + \left(\frac{\lambda^{qs}-\lambda^{\frac{qs}{2}}}{qs\ln\lambda}\right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1, \\ \frac{1}{2}\left|f'(a)\right|^s, & \text{if } |f'(a)| \leq 1 = \lambda, \\ \frac{1}{2}\left|f'(a)\right|^{2-s}, & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Remark 3.3. Theorem 3.2 will be reduces to Theorem 11 from [8], and Corollary 3.4 will be reduces to Corollary 12 from [8] and Theorem 4.2 from [14] if we put $s = 1$.

Corollary 3.5. *In Theorem 3.2, if we choose $\eta(b,a) = b-a$, then we obtain the following inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,\mathrm{d}\,u \right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}$$

$$\times\begin{cases} N_{(s,q,\lambda)}^{\frac{1}{q}}\left(\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{p}}\left(\frac{\lambda^{qs\frac{x-a}{b-a}}-1}{qs\ln\lambda}\right)^{\frac{1}{q}}\right. \\ \quad\left.+\left(\frac{b-x}{b-a}\right)^{1+\frac{1}{p}}\left(\frac{\lambda^{qs}-\lambda^{qs\frac{x-a}{b-a}}}{qs\ln\lambda}\right)^{\frac{1}{q}}\right), & \text{if } \lambda\neq 1; \\ |f'(a)|^{s}\left(\left(\frac{x-a}{b-a}\right)^{2}+\left(\frac{b-x}{b-a}\right)^{2}\right), & \text{if } |f'(a)|\leq 1=\lambda; \\ |f'(a)|^{2-s}\left(\left(\frac{x-a}{b-a}\right)^{2}+\left(\frac{b-x}{b-a}\right)^{2}\right), & \text{if } |f'(a)|>1=\lambda. \end{cases}$$

Moreover if we choose $s=1$ we get the following inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u\right|\leq\frac{b-a}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|$$

$$\times\begin{cases}\left(\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{p}}\left(\frac{\lambda^{q\frac{x-a}{b-a}}-1}{q\ln\lambda}\right)^{\frac{1}{q}}+\left(\frac{b-x}{b-a}\right)^{1+\frac{1}{p}}\left(\frac{\lambda^{q}-\lambda^{q\frac{x-a}{b-a}}}{q\ln\lambda}\right)^{\frac{1}{q}}\right), & \text{if } \lambda\neq 1; \\ \left(\left(\frac{x-a}{b-a}\right)^{2}+\left(\frac{b-x}{b-a}\right)^{2}\right), & \text{if } 1=\lambda. \end{cases}$$

Corollary 3.6. *In Theorem 3.2, if we choose $\eta(b,a)=b-a$, and $x=\frac{a+b}{2}$ we have the following inequality*

$$\left|f\left(\tfrac{b+a}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u\right|\leq\frac{b-a}{(p+1)^{\frac{1}{p}}}$$

$$\times\begin{cases}\frac{N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1+\frac{1}{p}}}\left(\left(\frac{\lambda^{\frac{qs}{2}}-1}{qs\ln\lambda}\right)^{\frac{1}{q}}+\left(\frac{\lambda^{qs}-\lambda^{\frac{qs}{2}}}{qs\ln\lambda}\right)^{\frac{1}{q}}\right), & \text{if } \lambda\neq 1; \\ \frac{1}{2}\left|f'(a)\right|^{s}, & \text{if } |f'(a)|\leq 1=\lambda; \\ \frac{1}{2}\left|f'(a)\right|^{2-s}, & \text{if } |f'(a)|>1=\lambda. \end{cases}$$

Moreover if we choose $s=1$ we get the following inequality

$$\left|f\left(\tfrac{b+a}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u\right|\leq\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left|f'(a)\right|$$

$$\times\begin{cases}\frac{1}{2^{\frac{1}{p}}}\left(\left(\frac{\lambda^{\frac{q}{2}}-1}{q\ln\lambda}\right)^{\frac{1}{q}}+\left(\frac{\lambda^{q}-\lambda^{\frac{q}{2}}}{q\ln\lambda}\right)^{\frac{1}{q}}\right), & \text{if } \lambda\neq 1; \\ 1, & \text{if } 1=\lambda. \end{cases}$$

Remark 3.4. The case $\lambda\neq 1$ in the last inequality of Corollary 3.6 represent Corollary 4.4 from [14].

Theorem 3.3. *Let $f:K\to(0,\infty)$ be a differentiable function such that $f'\in L([a,a+\eta(b,a)])$ and let $q>1$. If $|f'|^q$ is s-logarithmically preinvex* *function in the second sense for some fixed $s \in (0,1]$ with $|f'(a)| \neq 0$, then we have the following inequality*

$$\left| f(x) - \frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right|$$

$$\leq \begin{cases} \frac{\eta(b,a) N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}} \left(\frac{\frac{x-a}{\eta(b,a)} \lambda^{qs\frac{x-a}{\eta(b,a)}}}{qs\ln\lambda} + \frac{1-\lambda^{qs\frac{x-a}{\eta(b,a)}}}{(qs\ln\lambda)^2} \right)^{\frac{1}{q}} \right. \\ \left. + \left(1-\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}} \left(\frac{\lambda^{qs}-\lambda^{qs\frac{x-a}{\eta(b,a)}}}{(qs\ln\lambda)^2} - \frac{\left(1-\frac{x-a}{\eta(b,a)}\right)\lambda^{qs\frac{x-a}{\eta(b,a)}}}{qs\ln\lambda} \right)^{\frac{1}{q}} \right), \\ \qquad \text{if } \lambda \neq 1, \\ \frac{\eta(b,a)|f'(a)|^s}{2} \left(\left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1-\frac{x-a}{\eta(b,a)}\right)^2 \right), \quad \text{if } |f'(a)| \leq 1 = \lambda, \\ \frac{\eta(b,a)|f'(a)|^{2-s}}{2} \left(\left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1-\frac{x-a}{\eta(b,a)}\right)^2 \right), \quad \text{if } |f'(a)| > 1 = \lambda, \end{cases}$$

where λ and $N_{(s,q,\lambda)}$ are defined as in (1) *and* (10) *respectively.*

Proof. From Lemma 2.2, property of modulus, power mean inequality, s-logarithmically preinvexity of $|f'|^q$, and Lemma 1, we get

$$\left| f(x) - \frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u \right|$$

$$\leq \eta(b,a) \left(\left(\int\limits_0^{\frac{x-a}{\eta(b,a)}} t\,\mathrm{d}\,t \right)^{1-\frac{1}{q}} \left(\int\limits_0^{\frac{x-a}{\eta(b,a)}} t \left|f'(a+t\eta(b,a))\right|^q \mathrm{d}\,t \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int\limits_{\frac{x-a}{\eta(b,a)}}^1 (1-t)\,\mathrm{d}\,t \right)^{1-\frac{1}{q}} \left(\int\limits_{\frac{x-a}{\eta(b,a)}}^1 (1-t) \left|f'(a+t\eta(b,a))\right|^q \mathrm{d}\,t \right)^{\frac{1}{q}} \right)$$

$$= \frac{\eta(b,a)}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)} \left(\int\limits_0^{\frac{x-a}{\eta(b,a)}} t \left|f'(a+t\eta(b,a))\right|^q \mathrm{d}\,t \right)^{\frac{1}{q}} \right.$$

$$+\left(1-\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)\left|f'(a+t\eta(b,a))\right|^{q}\,\mathrm{d}\,t\right)^{\frac{1}{q}}\Bigg)$$

$$\leq \frac{\eta(b,a)}{2^{1-\frac{1}{q}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}}\left(\int_{0}^{\frac{x-a}{\eta(b,a)}}t\left|f'(a)\right|^{q(1-t)^{s}}\left|f'(b)\right|^{qt^{s}}\,\mathrm{d}\,t\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}}\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)\left|f'(a)\right|^{q(1-t)^{s}}\left|f'(b)\right|^{qt^{s}}\,\mathrm{d}\,t\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{\eta(b,a)\,N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1-\frac{1}{q}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}}\left(\int_{0}^{\frac{x-a}{\eta(b,a)}}t\lambda^{qst}\,\mathrm{d}\,t\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}}\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)\lambda^{qst}\,\mathrm{d}\,t\right)^{\frac{1}{q}}\right), \tag{13}$$

where λ and $N_{(s,q,\lambda)}$ are defined as in (1) and (10) respectively.

For $\lambda \neq 1$, (13), becomes

$$\left|f(x)-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(u)\,\mathrm{d}\,u\right|$$

$$\leq \frac{\eta(b,a)N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1-\frac{1}{q}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}}\left(\frac{\frac{x-a}{\eta(b,a)}\lambda^{qs\frac{x-a}{\eta(b,a)}}}{qs\ln\lambda}+\frac{1-\lambda^{qs\frac{x-a}{\eta(b,a)}}}{(qs\ln\lambda)^{2}}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}}\left(\frac{\lambda^{qs}-\lambda^{qs\frac{x-a}{\eta(b,a)}}}{(qs\ln\lambda)^{2}}-\frac{\left(1-\frac{x-a}{\eta(b,a)}\right)\lambda^{qs\frac{x-a}{\eta(b,a)}}}{qs\ln\lambda}\right)^{\frac{1}{q}}\right), \tag{14}$$

where we have used the fact that

$$\int_{0}^{\frac{x-a}{\eta(b,a)}}t\lambda^{qst}\,\mathrm{d}\,t=\frac{\frac{x-a}{\eta(b,a)}\lambda^{qs\frac{x-a}{\eta(b,a)}}}{qs\ln\lambda}+\frac{1-\lambda^{qs\frac{x-a}{\eta(b,a)}}}{(qs\ln\lambda)^{2}}$$

and

$$\int_{\frac{x-a}{\eta(b,a)}}^{1} (1-t)\,\lambda^{qst}\,\mathrm{d}\,t = \frac{\lambda^{qs}-\lambda^{qs\frac{x-a}{\eta(b,a)}}}{(qs\ln\lambda)^2} - \frac{\left(1-\frac{x-a}{\eta(b,a)}\right)\lambda^{qs\frac{x-a}{\eta(b,a)}}}{qs\ln\lambda}.$$

In the case where $\lambda = 1$, (13) gives

$$\begin{aligned}&\left|f(x) - \frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u\right| \\ &\le \begin{cases} \frac{\eta(b,a)|f'(a)|^s}{2}\left(\left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1-\frac{x-a}{\eta(b,a)}\right)^2\right), & \text{if } |f'(a)| \le 1 = \lambda; \\ \frac{\eta(b,a)|f'(a)|^{2-s}}{2}\left(\left(\frac{x-a}{\eta(b,a)}\right)^2 + \left(1-\frac{x-a}{\eta(b,a)}\right)^2\right), & \text{if } |f'(a)| > 1 = \lambda. \end{cases}\end{aligned} \tag{15}$$

Thus, from (14) and (15) we get the desired result. □

Corollary 3.7. *In Theorem 3.3, if we choose* $x = \frac{2a+\eta(b,a)}{2}$*, then we obtain the following midpoint inequality where*

$$\begin{aligned}&\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(u)\,\mathrm{d}\,u\right| \\ &\le \begin{cases} \frac{\eta(b,a)N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{3-\frac{3}{q}}}\left(\left(\frac{\lambda^{\frac{qs}{2}}}{2qs\ln\lambda} + \frac{1-\lambda^{\frac{qs}{2}}}{(qs\ln\lambda)^2}\right)^{\frac{1}{q}} + \left(\frac{\lambda^{qs}-\lambda^{\frac{qs}{2}}}{(qs\ln\lambda)^2} - \frac{\lambda^{\frac{qs}{2}}}{2qs\ln\lambda}\right)^{\frac{1}{q}}\right), \\ \qquad\qquad \textit{if } \lambda \neq 1; \\ \frac{\eta(b,a)|f'(a)|^s}{4}, \quad \textit{if } |f'(a)| \le 1 = \lambda; \\ \frac{\eta(b,a)|f'(a)|^{2-s}}{4}, \quad \textit{if } |f'(a)| > 1 = \lambda. \end{cases}\end{aligned}$$

Remark 3.5. Theorem 3.3 and Corollary 3.7 will be reduces to Theorem 15 and Corollary 16 from [8] if we put $s = 1$.

Corollary 3.8. *In Theorem 3.3, if we choose* $\eta(b,a) = b-a$*, then we obtain the following inequality*

$$\left|f(x) - \frac{1}{b-a}\int_{a}^{b} f(u)\,\mathrm{d}\,u\right|$$

$$\leq \begin{cases} \frac{(b-a)N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{b-a}\right)^{2-\frac{2}{q}} \left(\frac{\frac{x-a}{b-a}\lambda^{qs\frac{x-a}{b-a}}}{qs\ln\lambda} + \frac{1-\lambda^{qs\frac{x-a}{b-a}}}{(qs\ln\lambda)^2} \right)^{\frac{1}{q}} \right. \\ \left. + \left(\frac{b-x}{b-a}\right)^{2-\frac{2}{q}} \left(\frac{\lambda^{qs}-\lambda^{qs\frac{x-a}{b-a}}}{(qs\ln\lambda)^2} - \frac{\left(\frac{b-x}{b-a}\right)\lambda^{qs\frac{x-a}{b-a}}}{qs\ln\lambda} \right)^{\frac{1}{q}} \right), & \textit{if } \lambda \neq 1; \\ \frac{(b-a)|f'(a)|^s}{2} \left(\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2 \right), & \textit{if } |f'(a)| \leq 1 = \lambda; \\ \frac{(b-a)|f'(a)|^{2-s}}{2} \left(\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2 \right), & \textit{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Moreover if we choose $s = 1$ we get the following inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u \right|$$

$$\leq \begin{cases} \frac{(b-a)|f'(a)|}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{b-a}\right)^{2-\frac{2}{q}} \left(\frac{\frac{x-a}{b-a}\lambda^{q\frac{x-a}{b-a}}}{q\ln\lambda} + \frac{1-\lambda^{q\frac{x-a}{b-a}}}{(q\ln\lambda)^2} \right)^{\frac{1}{q}} \right. \\ \left. + \left(\frac{b-x}{b-a}\right)^{2-\frac{2}{q}} \left(\frac{\lambda^{q}-\lambda^{q\frac{x-a}{b-a}}}{(q\ln\lambda)^2} - \frac{\left(\frac{b-x}{b-a}\right)\lambda^{q\frac{x-a}{b-a}}}{q\ln\lambda} \right)^{\frac{1}{q}} \right), & \textit{if } \lambda \neq 1, \\ \frac{(b-a)|f'(a)|}{2} \left(\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2 \right), & \textit{if } 1 = \lambda. \end{cases}$$

Corollary 3.9. *In Theorem 3.3, if we choose $\eta(b,a) = b - a$, and $x = \frac{a+b}{2}$ we have the following inequality*

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u \right|$$

$$\leq \begin{cases} \frac{(b-a)N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{3-\frac{3}{q}}} \left(\left(\frac{\lambda^{\frac{qs}{2}}}{2qs\ln\lambda} + \frac{1-\lambda^{\frac{qs}{2}}}{(qs\ln\lambda)^2} \right)^{\frac{1}{q}} + \left(\frac{\lambda^{qs}-\lambda^{\frac{qs}{2}}}{(qs\ln\lambda)^2} - \frac{\lambda^{\frac{qs}{2}}}{2qs\ln\lambda} \right)^{\frac{1}{q}} \right), \\ \qquad \textit{if } \lambda \neq 1; \\ \frac{(b-a)|f'(a)|^s}{4}, \quad \textit{if } |f'(a)| \leq 1 = \lambda; \\ \frac{(b-a)|f'(a)|^{2-s}}{4}, \quad \textit{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Moreover if we choose $s = 1$ we get the following inequality

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u \right|$$

$$\leq \begin{cases} \frac{(b-a)|f'(a)|}{2^{3-\frac{3}{q}}} \left(\left(\frac{\lambda^{\frac{q}{2}}}{2q\ln\lambda} + \frac{1-\lambda^{\frac{q}{2}}}{(q\ln\lambda)^2} \right)^{\frac{1}{q}} + \left(\frac{\lambda^q-\lambda^{\frac{q}{2}}}{(q\ln\lambda)^2} - \frac{\lambda^{\frac{q}{2}}}{2q\ln\lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1, \\ \frac{(b-a)|f'(a)|}{4}, & \text{if } 1 = \lambda. \end{cases}$$

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