

A study of a generalization of bi-ideal, quasi ideal and interior ideal of semigroup

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ABSTRACT. In this paper, as a further generalization of ideals, we introduce the notion of bi-quasi-interior ideals as a generalization of bi-ideal, quasi ideal, interior ideal, bi-interior ideal and bi-quasi ideal of semigroup and study the properties of bi-quasi-interior ideals of semigroup.

1. INTRODUCTION

Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians proved important results and characterizations of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. Henriksen [2] and Shabir et al. [15] studied ideals in semirings.

We know that the notion of an one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals.

In 1952, the concept of bi-ideals was introduced by Good and Hughes [1] for semigroups. The notion of bi-ideals in rings and semigroups were introduced by Lajos and Szasz [8, 9]. Bi-ideal is a special case of $(m - n)$ ideal. In 1956, Steinfeld [16] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [5, 6, 7, 8] introduced the concept of quasi ideal for a semiring. Quasi ideals bi-ideals in Γ -semirings studied by Jagtap and Pawar [7]. Murali Krishna Rao [10, 11, 12] introduced the notion of left (right) bi-quasi ideal of semigroup, Γ -semigroup, Γ -semiring and study the properties of left bi-quasi ideals.

Murali Krishna Rao [13, 14] studied ideals in ordered Γ -semirings and introduced the notion of bi-interior ideal as a generalization of quasi ideal,

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bi-ideal and interior ideal of semigroup and study the properties of bi-interior ideals of semigroup, simple semigroup and regular semigroup.

In this paper, as a further generalization of ideals, we introduce the notion of bi- quasi-interior ideal as a generalization of bi-ideal,quasi ideal, interior ideal, bi-interior ideal and bi-quasi ideal of semigroup and study the properties of bi-quasi-interior ideals of semigroup and some characterizations of bi-quasi-interior ideals of semigroup, regular semigroup and simple semigroup.

A necessary and sufficient condition for a semigroup to be regular and simple is proved using bi- quasi-interior ideals semigroup.

2. PRELIMINARIES

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. Let M be a semigroup. If there exists $1 \in M$ such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in M$, is called an unity element of M then M is said to be semigroup with unity.

Definition 2.2. An element a of a semigroup M is called a regular element if there exists an element b of M such that $a = aba$.

Definition 2.3. A semigroup M is called a regular semigroup if every element of M is a regular element.

Definition 2.4. An element a of a semigroup M is called an idempotent element if $aa = a$.

Definition 2.5. An element b of a semigroup M is called an inverse element of a of M if $ab = ba = 1$.

Definition 2.6. A non-empty subset A of semigroup M is called

- (i) a subsemigroup of M if $AA \subseteq A$.
- (ii) a left(right) ideal of M if $MA \subseteq A(AM \subseteq A)$.
- (iii) an ideal if $MA \subseteq A$ and $AM \subseteq A$.

Definition 2.7. A non-empty subset A of semigroup M is called

- (i) an interior ideal of M if A is a subsemigroup of M and $MAM \subseteq A$.
- (ii) a quasi ideal of M if A is a subsemigroup of M and $AM \cap MA \subseteq A$.
- (iii) a bi-ideal of M if A is a subsemigroup of M and $AMA \subseteq A$.

Definition 2.8. A non-empty subset B of semigroup M is said to be bi-interior ideal of M if B is a subsemigroup of M and $MBM \cap BMB \subseteq B$.

Definition 2.9. Let M be a semigroup. A non-empty subset L of M is said to be left bi-quasi ideal (right bi-quasi ideal) of M if L is a subsemigroup of M and $ML \cap LML \subseteq L$ ($LM \cap LML \subseteq L$).

Definition 2.10. Let M be a semigroup. L is said to be bi-quasi ideal of M if it is both a left bi-quasi and a right bi-quasi ideal of M .

Example 2.1.

(i) Let Q be the set of all rational numbers. Then

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Q \right\}$$

is a semigroup with respect to usual matrix multiplication.

(a) If

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\},$$

then R is a quasi ideal of semigroup M and R is neither a left ideal nor a right ideal.

(b) If

$$S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid 0 \neq a \in Q \right\},$$

then S is a bi-ideal of semigroup M .

(ii) If

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$$

and $\Gamma = M$ then M is a semigroup with respect to usual matrix multiplication and

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}.$$

Then A is not a bi-ideal of semigroup M .

3. BI-QUASI-INTERIOR IDEALS OF SEMIGROUPS

In this section we introduce the notion of bi-quasi-interior ideal as a generalization of bi-ideal, quasi-ideal and interior ideal of semigroup and study the properties of bi-quasi-interior ideal of semigroup.

Definition 3.1. A non-empty subset B of semigroup M is said to be bi-quasi-interior ideal of M if B is a subsemigroup of M and $BMBMB \subseteq B$

Every bi-quasi-interior ideal of semigroup M need not be bi-ideal, quasi-ideal, interior ideal bi-interior ideal. and bi-quasi ideals of semigroup M .

Example 3.1. Let N be a set of all natural numbers. N is a semigroup with respect to usual addition of integers. A subset $I = 2N$ of N is a bi-quasi-interior ideal of N but not bi-ideal, quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of semigroup N .

Theorem 3.1. *Every bi-ideal of semigroup M is a bi-quasi-interior ideal of semigroup M .*

Proof. Let B be a bi-ideal of semigroup M . Then $BMB \subseteq B$.

Therefore $BMBMB \subseteq BMB \subseteq B$.

Hence every bi-ideal of semigroup M is a bi-quasi-interior ideal of M . \square

Theorem 3.2. *Every interior ideal of semigroup M is a bi-quasi-interior ideal of M .*

Proof. Let I be an interior ideal of semigroup M .

Then $IMIMI \subseteq MIM \subseteq I$.

Hence I is a bi-quasi-interior ideal of semigroup M . \square

Theorem 3.3. *Let M be a semigroup. Every left ideal is a bi-quasi-interior ideal of M .*

Proof. Let B be a left ideal of semigroup M . Then $MB \subseteq B$.

Therefore $BMBMB \subseteq BBMB \subseteq BBB \subseteq B$.

Hence every left ideal of semigroup M is a bi-quasi-interior ideal of M . \square

Corollary 3.1. *Let M be a semigroup. Every right ideal is a bi-quasi-interior ideal of M .*

Corollary 3.2. *Let M be a semigroup. Every ideal is a bi-quasi-interior ideal of M .*

In the following theorems, we mention some important properties and we omit the proofs since proofs are straight forward.

Theorem 3.4. *Let M be a semigroup. Every quasi ideal is bi-quasi-interior ideal of M .*

Theorem 3.5. *Let M be a semigroup. Intersection of a right ideal and a left ideal of M is a bi-quasi-interior ideal of M .*

Theorem 3.6. *Let M be a semigroup. If L is a left ideal and R is a right ideal of semigroup M and $B = RL$ then B is a bi-quasi-interior ideal of M .*

Theorem 3.7. *Let M be a semigroup. If B is a bi-quasi-interior ideal and T is a non-empty subset of M such that BT is a subsemigroup of M then BT is a bi-quasi-interior ideal of semigroup M .*

Theorem 3.8. *Let M be a semigroup and B be a subsemigroup of M . If $BMMMB \subseteq B$ then B is a bi-quasi-interior ideal of M .*

Theorem 3.9. *Let M be a semigroup and B be a subsemigroup of M . If $MMMMB \cap BMMMM \subseteq B$ then B is a bi-quasi-interior ideal of M .*

Theorem 3.10. *Let M be a semigroup and B be a subsemigroup of M . B is a bi-quasi-interior ideal of M if and only if there exist a left ideal L and a right ideal R of M such that $RL \subseteq B \subseteq R \cap L$.*

Proof. Suppose B is a bi-quasi-interior ideal of semigroup M .

Then $BMBMB \subseteq B$. Let $R = BMBM$ and $L = MB$.

Then L and R are left and right ideals of M respectively. Therefore $RL \subseteq B \subseteq R \cap L$.

Conversely suppose that there exist L and R are left and right ideals of M respectively such that $RL \subseteq B \subseteq R \cap L$. Then

$$BMBMB \subseteq (R \cap L)M(R \cap L)M(R \cap L) \subseteq RMRML \subseteq RL \subseteq B.$$

Hence B is a bi-quasi-interior ideal of semigroup M . \square

Theorem 3.11. *The intersection of a bi-quasi-interior ideal B of semigroup M and a right ideal A of M is always bi-quasi-interior ideal of M .*

Proof. Suppose $C = B \cap A$.

$$CMCMC \subseteq BMBMB \subseteq BCMCMC \subseteq AMAMA \subseteq A,$$

since A is a left ideal of M , therefore, $CMCMC \subseteq B \cap A = C$.

Hence the intersection of a bi-quasi-interior ideal B of semigroup M and a subsemigroup A of M is always a bi-quasi-interior ideal of M . \square

Theorem 3.12. *Let A and C be bi-quasi-interior ideals of semigroup M and $B = AC$. If $CC = C$ then B is a bi-quasi-interior ideal of M .*

Proof. Let A and C be bi-quasi-interior ideals of semigroup M and $B = AC$
 $BB = ACAC = ACCAC \subseteq ACMCMC \subseteq AC = B$.

Therefore $B = AC$ is a subsemigroup of M

$$BMBMB = ACMACMAC \subseteq AMAMAC \subseteq AC = B.$$

Hence B is a bi-quasi-interior ideal of M . \square

Corollary 3.3. *Let A and C be bi-quasi-interior ideals of semigroup M and $B = CA$. If $CC = C$ then B is a bi-quasi-interior ideal of M .*

Theorem 3.13. *Let A and C be subsemigroups of M and $B = AC$. If A is the left ideal then B is a bi-quasi-interior ideal of M .*

Proof. Let A and C be subsemigroups of M and $B = AC$ Suppose A is the left ideal of M . $BB = ACAC \subseteq AC = B$.

$$BMBMB = ACMACMAC \subseteq AC = B.$$

Hence B is a bi-quasi-interior ideal of M . \square

Corollary 3.4. *Let A and C be subsemigroups of semigroup M and $B = AC$. If C is a right ideal then B is a bi-quasi-interior ideal of M .*

Theorem 3.14. *Let M be a semigroup and T be a non-empty subset of M . Then every subsemigroup of T containing $TMTMT$ is a bi-quasi-interior ideal of semigroup M .*

Proof. Let B be a subsemigroup of T containing $TMTMT$. Then

$$BMBMB \subseteq TMTMT \subseteq B.$$

Therefore $BMBMB \subseteq B$.

Hence B is a bi-quasi-interior ideal of M . □

Theorem 3.15. *B is a bi-quasi-interior ideal of semigroup M if and only if B is a left ideal of some right ideal of semigroup M .*

Proof. Suppose B is a left ideal of some right ideal R of semigroup M . Then $RB \subseteq B, RM \subseteq R$. Hence $BMBMB \subseteq BMB \subseteq RMB \subseteq RB \subseteq B$. Therefore B is a bi-quasi-interior ideal of semigroup M . Conversely suppose that B is a bi-quasi-interior ideal of semigroup M . Then $BMBMB \subseteq B$.

Therefore B is a left ideal of right ideal $BMBM$ of semigroup M . □

Corollary 3.5. *B is a bi-quasi-interior ideal of semigroup M if and only if B is a right ideal of some left ideal of semigroup M .*

Theorem 3.16. *If B is a bi-quasi-interior ideal of semigroup M , T is a subsemigroup of M and $T \subseteq B$ then BT is a bi-quasi-interior ideal of M .*

Proof. Suppose B is a bi-quasi-interior ideal of semigroup M , T is a subsemigroup of M and $T \subseteq B$. $BTBT \subseteq BT$.

Hence BT is a subsemigroup of M .

We have $MBTM \subseteq MBM$ and $BTMBT \subseteq BMB$, then $BTMBTMBT \subseteq BMBMBT \subseteq BT$.

Hence BT is a bi-quasi-interior ideal of semigroup M . □

Theorem 3.17. *Let B be bi-ideal of semigroup M and I be an interior ideal of M . Then $B \cap I$ is a bi-quasi-interior ideal of M .*

Proof. Obviously $B \cap I$ is subsemigroup of M . Suppose B is a bi-ideal of M and I is an interior ideal of M . Then

$$\begin{aligned} (B \cap I)M(B \cap I)M(B \cap I) &\subseteq BMBMB \subseteq B, \\ (B \cap I)M(B \cap I)M(B \cap I) &\subseteq IMIMI \subseteq I \end{aligned}$$

Therefore $(B \cap I)M(B \cap I)M(BI)M \subseteq B \cap I$.

Hence $B \cap I$ is a bi-quasi-interior ideal of M . □

Theorem 3.18. *Let M be a semigroup and T be a subsemigroup of M . Then every subsemigroup of T containing $TMTMT$ is a bi-quasi-interior ideal of M .*

Proof. Let C be a subsemigroup of T containing $TMTMT$. Then

$$CMCMC \subseteq TMTMT \subseteq C.$$

Hence C is a bi-quasi-interior ideal of semigroup M . □

Theorem 3.19. *Let M be a semigroup. If $M = Ma$, for all $a \in M$, then every bi-quasi-interior ideal of M is a quasi ideal of M .*

Proof. Let B be a bi-quasi-interior ideal of a semigroup M and $a \in B$. Then

$$\begin{aligned} BMBMB &\subseteq B, \\ Ma &\subseteq MB, \\ M &\subseteq MB \subseteq M, \\ MB &= M, \\ BM = BMB &\subseteq BMBMB \subseteq B, \\ MB \cap BM &\subseteq M \cap BM \subseteq B. \end{aligned}$$

Therefore B is a quasi ideal of M . Hence the theorem. \square

Theorem 3.20. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ bi-quasi-interior ideals of a semigroup M is a bi-quasi-interior ideal of M .*

Proof. Let $B = \bigcap_{\lambda \in A} B_\lambda$. Then B is a subsemigroup of M .

Since B_λ is a bi-quasi-interior ideal of M , we have

$$\begin{aligned} B_\lambda MB_\lambda MB_\lambda &\subseteq B_\lambda, \quad \text{for all } \lambda \in A, \\ (\cap B_\lambda)M(\cap B_\lambda)M(\cap B_\lambda) &\subseteq \cap B_\lambda, \\ BMBMB &\subseteq B. \end{aligned}$$

Hence B is a bi-quasi-interior ideal of M . \square

Theorem 3.21. *Let B be a bi-quasi-interior ideal of semigroup M , $e \in B$ and e be an idempotent. Then eB is a bi-quasi-interior ideal of M .*

Proof. Let B be a bi-quasi-interior ideal of semigroup M . Suppose $x \in B \cap eM$. Then $x \in B$ and $x = ey, y \in M$.

$$x = ey = eey = e(ey) = ex \in eB.$$

Therefore,

$$\begin{aligned} B \cap eM &\subseteq eB, \\ eB &\subseteq B, \quad eB \subseteq eM, \\ eB &\subseteq B \cap eM, \\ eB &= B \cap eM. \end{aligned}$$

Hence eB is a bi-quasi-interior ideal of M . \square

Corollary 3.6. *Let M be a semigroup M and e be an idempotent. Then eM and Me are bi-quasi-interior ideals of M .*

Theorem 3.22. *If B be a left bi-quasi ideal of semigroup M , then B is a bi-quasi-interior ideal of M .*

Proof. Suppose B is a left bi-quasi ideal of semigroup M . Then $BMBMB \subseteq MB$ and $BMBMB \subseteq BMB$.

Therefore $BMBMB \subseteq MB \cap MBM \subseteq B$.

Hence B is a bi-quasi-interior ideal of M . \square

Corollary 3.7. *If B be a right bi-quasi ideal of semigroup M , then B is a bi-quasi-interior ideal of M .*

Corollary 3.8. *If B be a bi-quasi ideal of semigroup M , then B is a bi-quasi-interior ideal of M .*

Theorem 3.23. *If B be a bi-interior ideal of semigroup M , then B is a bi-quasi-interior ideal of M .*

Proof. Suppose B is a bi-interior ideal of semigroup M . Then $BMBMB \subseteq MBM$ and $BMBMB \subseteq MBM$.

Therefore $BMBMB \subseteq MBM \cap MBM \subseteq B$.

Hence B is a bi-quasi-interior ideal of M . □

We introduce the notion of bi-quasi-interior simple semigroup and characterize the bi-quasi-interior simple semigroup using bi-quasi-interior ideals of semigroup and study the properties of minimal bi-quasi-interior ideals of semigroup..

Definition 3.2. A semigroup M is said to be bi-quasi-interior simple semigroup if M has no bi-quasi-interior ideals other than M itself.

Theorem 3.24. *Let M be a simple semigroup. Every bi-quasi-interior ideal is bi-ideal of M .*

Proof. Let B be a bi-quasi-interior ideal of a simple semigroup M .

Then $BMBMB \subseteq B$ and MBM is an ideal of M .

Since M is a simple semigroup, we have $MBM = M$. Hence

$$\begin{aligned} BMBMB &\subseteq B, \\ BMB &\subseteq B. \end{aligned}$$

Hence the theorem is proved. □

Theorem 3.25. *Let M be a semigroup. Then M is a bi-quasi-interior simple semigroup if and only if $(a)_{bqi} = M$, for all $a \in M$, where $(a)_{bqi}$ is the bi-quasi-interior ideal generated by a .*

Proof. Let M be a semigroup. Suppose that $(a)_{bqi}$ is a bi-quasi-interior ideal generated by a and M is a bi-quasi-interior simple semigroup. Then $(a)_{bqi} = M$, for all $a \in M$.

Conversely suppose that B is a bi-quasi-interior ideal of semigroup M and $(a)_{bqi} = M$, for all $a \in M$. Let $b \in B$.

Then $(b)_{bqi} \subseteq B$ implies $M = (b)_{bqi} \subseteq B \subseteq M$.

Therefore M is a bi-quasi-interior simple semigroup. □

Theorem 3.26. *Let M be a semigroup. M is a bi-quasi-interior simple semigroup if and only if $\langle a \rangle = M$, for all $a \in M$ and where $\langle a \rangle$ is the smallest bi-quasi-interior ideal generated by a .*

Proof. Let M be a semigroup. Suppose M is a bi-quasi-interior simple semigroup, $a \in M$ and $B = Ma$.

Then B is a left ideal of M .

Therefore, by Theorem [3.5], B is a bi-quasi-interior ideal of M .

Therefore $B = M$. Hence $Ma = M$, for all $a \in M$.

$$Ma \subseteq \langle a \rangle \subseteq M,$$

$$M \subseteq \langle a \rangle \subseteq M,$$

$$M = \langle a \rangle.$$

Suppose $\langle a \rangle$ is the smallest bi-quasi-interior ideal of M generated by a , $\langle a \rangle = M$, A is the bi-quasi-interior ideal and $a \in A$. Then

$$\langle a \rangle \subseteq A \subseteq M,$$

$$M \subseteq A \subseteq M.$$

Therefore $A = M$. Hence M is a bi-quasi-interior simple semigroup. \square

Theorem 3.27. *Let M be a semigroup. Then M is a bi-quasi-interior simple Γ -semigroup if and only if $aMaMa = M$, for all $a \in M$.*

Proof. Suppose M is left bi-quasi simple semigroup and $a \in M$.

Therefore $aMaMa$ is a bi-quasi-interior ideal of M .

Hence $aMaMa = M$, for all $a \in M$.

Conversely suppose that $aMaMa = M$, for all $a \in M$.

Let B be a bi-quasi-interior ideal of semigroup M and $a \in B$.

$$M = aMaMa \subseteq BMBMB \subseteq B.$$

Therefore, $M = B$. Hence M is a bi-quasi-interior simple semigroup. \square

Definition 3.3. A semigroup M is a left (right) simple semigroup if M has no proper left (right) ideal of M .

A semigroup M is said to be simple semigroup if M has no proper ideals.

Theorem 3.28. *If semigroup M is left simple semigroup then every bi-quasi-interior ideal of M is a right ideal of M .*

Proof. Let B be a bi-quasi-interior of left simple semigroup. Then MB is a left ideal of M and $MB \subseteq M$. Therefore, $MB = M$. Then $BM = BMB$, and $BM \subseteq BMBMB \subseteq B$, so $BM \subseteq B$.

Hence every bi-quasi-interior ideal is a right ideal of M . \square

Corollary 3.9. *If semigroup M is right simple semigroup, then every bi-quasi-interior ideal of M is a left ideal of M .*

Corollary 3.10. *Every bi-quasi-interior ideal of left and right simple semigroup M is an ideal of M .*

Theorem 3.29. *Let M be a semigroup and B be bi-quasi-interior ideal of M . Then B is minimal bi-quasi-interior ideal of M if and only if B is a bi-quasi-interior simple subsemigroup.*

Proof. Let B be a minimal bi-quasi-interior ideal of semigroup M and C be a bi-quasi-interior ideal of B . Then $CBCBC \subseteq C$.

Therefore $CBCBC$ is a bi-quasi-interior ideal of M .

Since C is a bi-quasi-interior ideal of B , then $B = CBCBC \subseteq C$, so $B = C$.

Conversely, suppose that B is a bi-quasi-interior simple subsemigroup of M . Let C be a bi-quasi-interior ideal of M and $C \subseteq B$. Then

$$C = CBCBC \subseteq CMCMC \subseteq BMBMB \subseteq B,$$

so $B = C$, since B is a bi-quasi-interior simple semigroup.

Hence B is a minimal bi-quasi-interior ideal of M . □

Definition 3.4. A group M is said to be bi-quasi-interior simple group if M has no bi-quasi-interior ideals other than M itself.

Theorem 3.30. *If M is a group then M is bi-quasi-interior simple group.*

Proof. Let B be a proper bi-quasi-interior ideal of group M and $1 \neq a \in B$. Since M is a group, there exists $b \in M$ such that $ab = 1$. Then $abx = x = xab$, for all $x \in M$. Then $x \in BM$. Therefore $M \subseteq BM$. We have $BM \subseteq M$. Hence $M = BM$.

Similarly we can prove $MB = M$. Further, we have

$$M = MB = BMBMB \subseteq BM \subseteq B.$$

Therefore, $M = B$.

Hence group M has no proper bi-quasi-interior ideals. □

Theorem 3.31. *Let M be a semigroup and $B = RL$, where L and R are minimal left ideal and right ideal of M respectively. Then B is a minimal bi-quasi-interior ideal of M .*

Proof. Obviously $B = RL$ is bi-quasi-interior ideal of M . Let A be bi-quasi-interior ideal of M such that $A \subseteq B$.

We have MA is a right ideal. Then

$$MA \subseteq MB = MRL \subseteq L,$$

since L is a left ideal of M .

Similarly, we can prove $AM \subseteq R$. Therefore $MA = L$, $AM = R$. Hence $B = AMMA \subseteq AMAMA \subseteq A$. Therefore $A = B$. Hence B is a minimal bi-quasi-interior ideal of M . □

We characterize regular semigroup using bi-quasi-interior ideals of semigroup.

Theorem 3.32. *Let M be a regular semigroup. Then every bi-quasi-interior ideal of M is an ideal of M .*

Proof. Let B be a bi-quasi-interior ideal of M . Then $BMBMB \subseteq B$, and since M is regular, holds $BM \subseteq BMB$, so $BM \subseteq BMBMB \subseteq B$.

Similarly, we can show that $MB \subseteq BMBMB \subseteq B$.

Hence the theorem is proved. \square

Theorem 3.33. *M is regular semigroup if and only if $AB = A \cap B$ for any right ideal A and left ideal B of semigroup M .*

Theorem 3.34. *Let M be a regular semigroup. Then B is a bi-quasi-interior ideal of M if and only if $BMBMB = B$, for all bi-quasi-interior ideals B of M .*

Proof. Suppose M is a regular semigroup, B is a bi-interior ideal of M and $x \in B$. Then $BMBMB \subseteq B$ and there exists $y \in M$ such that $x = xyxyx \in BMBMB$. Therefore $x \in BMBMB$. Hence $BMBMB = B$.

Conversely suppose that $BMBMB = B$, for all bi-quasi-interior ideals B of M . Let $B = R \cap L$, where R is a right ideal and L is a left ideal of M . Then B is a bi-interior ideal of M .

Therefore

$$\begin{aligned} R \cap L &= (R \cap L)M(R \cap L)M(R \cap L) \\ &\subseteq RMLML \subseteq RL \subseteq R \cap L, \end{aligned}$$

since $RL \subseteq L$ and $RL \subseteq R$.

Therefore $R \cap L = RL$. Hence M is a regular semigroup. \square

Theorem 3.35. *Let B be the bi-quasi-interior ideal of regular semigroup M . If B is a bi-quasi-interior ideal of M and B is regular subsemigroup of M then any bi-quasi-interior ideal of B is a bi-quasi-interior ideal of M .*

Proof. Let A be a bi-quasi-interior ideal of regular subsemigroup B of M . Then by Theorem [3.49], $ABABA = A$. We have $BMBMB = B$ and $A \subseteq AB$. Also,

$$\begin{aligned} AMA\Gamma MA &\subseteq BMBMB = B, \\ ABABA &= A \subseteq B, \\ A &= ABABA \subseteq AMA\Gamma MA, \\ AMAMA &= A. \end{aligned}$$

Hence A is a bi-quasi-interior ideal of M . \square

Theorem 3.36. *Let B be a subsemigroup of regular semigroup M . If B can be represented as $B = RL$, where R is a right ideal and L is a left ideal of M then B is a bi-quasi-interior ideal of M .*

Proof. Suppose $B = RL$, where R is a right ideal of M and L is a left ideal of M . The following holds

$$BMBMB = RLMRLMRL \subseteq RL = B.$$

Hence B is a bi-quasi-interior ideal of semigroup M .

Conversely suppose that B is a bi-quasi-interior ideal of regular semigroup M .

We have $BMBMB = B$. Let $R = BM$ and $L = MB$. Then $R = BM$ is a right ideal of M and $L = MB$ is a left ideal of M . The following holds

$$\begin{aligned} BM \cap MB &\subseteq BMBMB = B, \\ BM \cap MB &\subseteq B, \\ R \cap L &\subseteq B. \end{aligned}$$

We have $B \subseteq BM = R$ and $B \subseteq MB = L$, i.e., $B = R \cap L = RL$, since M is a regular semigroup.

Hence B can be represented as RL , where R is the right ideal and L is the left ideal of M .

Hence the theorem is proved. □

The following theorem is a necessary and sufficient condition for semigroup M to be regular using bi-quasi-interior ideal.

Theorem 3.37. *M is a regular semigroup if and only if $B \cap I \cap L \subseteq BIL$, for any bi-quasi-interior ideal B , ideal I and left ideal L of M .*

Proof. Suppose M be a regular semigroup, B, I and L are bi-quasi-interior ideal, ideal and left ideal of M respectively.

Let $a \in B \cap I \cap L$. Then $a \in aMa$, since M is regular, and

$$a \in aMa \subseteq aMaMa \subseteq BIL.$$

Hence $B \cap I \cap L \subseteq BIL$.

Conversely suppose that $B \cap I \cap L \subseteq BIL$, for any bi-quasi-interior ideal B , ideal I and left ideal L of M . Let R be a right ideal and L be a left ideal of M . Then by assumption $R \cap L = R \cap M \cap L \subseteq RML \subseteq RL$. We have $RL \subseteq R, RL \subseteq L$. Therefore $RL \subseteq R \cap L$. Hence $R \cap L = RL$.

Thus M is a regular semigroup. □

4. CONCLUSION

As a further generalization of ideals, we introduced the notion of bi-quasi-interior ideal of semigroup as a generalization of ideal, left ideal, right ideal, bi-ideal, quasi ideal and interior ideal of semigroup and studied some of their properties. We introduced the notion of bi-quasi-interior simple semigroup and characterized the bi-quasi-interior simple semigroup, regular semigroup using bi-quasi-interior ideals of semigroup. We proved every bi-quasi ideal of semigroup and bi-interior ideal of semigroup are bi-quasi-interior ideals and studied some of the properties of bi-interior ideals of semigroup. In continuity of this paper, we study prime bi-quasi-interior ideals, prime, maximal and minimal bi-quasi-interior ideals of semigroup.

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