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Common fixed points for faintly compatible mappings

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Abstract. In this paper, we obtain a generalized common fixed point theorem for four mappings using the conditions of non-compatibility and faint compatibility.

1. Introduction and Preliminaries

Generalizing Banach contraction principle, Jungck [9] initiated the study of common fixed points for a pair of commuting mappings satisfying contractive type conditions. In 1982, Sessa [14] introduced the weaker notion of commutativity which is generally known as *Weak Commutativity* and established some interesting results on the existence of common fixed points for the pair of mappings. Further, Jungck [10] generalized the concept of weak commutativity by introducing the notion of compatible mappings. Throughout this section (f, g) denotes a pair of mapping on a metric space X.

Definition 1.1 ([10]). The pair of mappings (f, g) is said to be compatible iff $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 1.2 ([10]). The pair (f, g) is said to be non-compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either non-zero or non-existent.

Again in 1996, Jungck [8] generalized the the concept of compatibility by introducing weakly compatible mappings.

Definition 1.3 ([8]). The pair (f, g) is said to be weakly compatible if the pair commutes on the set of coincidence points, i.e., $fgx = gfx$ whenever $fx = gx$ for some $x \in X$.

Al-Thagafi and Shahzad [2] introduced the concept of occasionally weakly compatible mappings by weakened the notion of weakly compatible mappings.

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Definition 1.4 ([2]). The pair (f, g) is said to be occasionally weakly compatible if there exists a coincidence point $x \in X$ such that $fx = gx$ implies $fgx = gfx$.

In 2010, Pant et al. [12] redefined the concept of occasionally weakly compatible mappings by introducing conditional commutativity.

Definition 1.5 ([12]). The pair (f, g) is said to be conditionally commuting if the pair commutes on a nonempty subset of the set of coincidence points whenever the set of coincidences is nonempty.

Again, Pant et al. [13] gave the concept of conditional compatibility which is indepedent of compatibility condition and proved that in case of existence of unique common fixed/coincidenence point, conditional compatibility can not be reduced to the compatibility condition. Further, they also proved that conditional compatibility need not imply commutativity at the coincidence points.

Definition 1.6 ([13]). The pair (f, g) is said to be conditionally compatible iff whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is nonempty, there exists a sequence $\{y_n\}$ such that

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = t \text{ and } \lim_{n\to\infty} d(fgy_n, gfy_n) = 0.$$

Over the last two decades, there are a number of common fixed/coincidence point theorems for the pair of mappings under different contractive conditions with compatibility and its weaker versions imposed on the mappings (for more details, see [4, 5, 6, 7, 1, 3] and refenences therein).

In a recent work, Bisht and Shahzad [3] gave a new notion of conditionally compatible mappings in a slighty different settings and named it as faintly compatible mappings.

Definition 1.7 ([3]). The pair (f, g) is said to be faintly compatible iff (f, g) is conditionally compatible and (f, g) commutes on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.

Bisht et al. [3] proved some interesting common fixed point theorems using the concept of faintly compatible mappings on non-complete metric spaces under defferent contractive conditions. Complementing the work of Bisht et al. [3], we give following examples for the comparative discussions on the above concepts.

(i) *Compatibilty implies faint compatibility but converse may not be true.*

Example 1.1. Let $X = [2, 4]$ and d be the usual metric on X. Define self mappings f and g on X as follows:

$$f(x) = \begin{cases} 2 & \text{if } x = 2 \, or \, x > 3 \\ x+1 & \text{if } 2 < x \leq 3 \end{cases} \quad \text{and } g(x) = \begin{cases} 2 & \text{if } x = 2 \\ \frac{x+4}{2} & \text{if } 2 < x \leq 3 \\ \frac{x+1}{2} & \text{if } x > 3. \end{cases}$$

In this example, f and g are faintly compatible but not compatible. For if, we consider the constant sequence $\{x_n = 2\}$, then f and g are faintly compatible. On the other hand, if we choose a sequence $\{y_n = 3 + \frac{1}{n}\}$, then $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = 2$ and $\lim_{n\to\infty} d(fgy_n, gfy_n) = 1\,(\neq 0)$. Hence f and g are not compatible.

(ii) *Faint compatibility and non-compatibility are independent concepts.*

Example 1.2. Let $X = [2, 8]$ and d be the usual metric on X. Define self mappings f and g on X as follows:

$$f(x) = \begin{cases} 6 & \text{if } 2 \leq x \leq 4 \\ 2 & \text{if } x > 4 \end{cases} \quad \text{and } g(x) = \begin{cases} 2 & \text{if } 2 \leq x < 4 \\ x - 2 & \text{if } x \geq 4. \end{cases}$$

In this example, f and g are non-compatible but not faintly compatible. To see this, we consider a sequence $\{x_n = 4 + \frac{1}{n}\}$, then $\lim_{n\to\infty} fx_n = 2 = \lim_{n\to\infty} gx_n$ but $\lim_{n\to\infty} d(fgx_n, gfx_n) = 4$. So, f and g are non-compatible.

Example 1.3. Let $X = [1, \infty)$ and let d be the usual metric on X. Define self mappings f and g on X as follows:

$$f(x) = x \quad \forall x \in X \quad \text{and} \quad g(x) = 3x - 2 \quad \forall x \in X.$$

In this one, f and g are faintly compatible but not non-compatible.

(iii) *Weakly compatible implies faint compatibility, but converse is not true in general.*

Example 1.4. Let $X = [0, \frac{2}{3}]$ with the usual metric d. Define self mappings f and g on X as follows:

$$f(x) = \frac{1}{3} - \left|\frac{1}{3} - x\right| \quad \text{and} \quad g(x) = \frac{2 - 3x}{9}.$$

In this example the mappings f and g are faintly compatible but not weakly compatible. To see this, we take a constant sequence $\{x_n = \frac{1}{12}\}$ and they are commuting at the coincidence point $x = \frac{1}{12}$. On the other hand, f and g do not commute at the coincidence point $x = \frac{2}{3}$, hence they are not weakly compatible.

(iv) *Occasionally weakly compatible implies faintly compatible but the converse may not be true.*

Example 1.5. Let $X = [0, \infty)$ with usual metric d on X. Define self mappings f and g on X as follows:

$$f(x) = \frac{x}{2} \quad \forall x \in X \quad \text{and} \quad g(x) = \frac{x + 3}{2} \quad \forall x \in X.$$

In this example, mappings f and g are trivially faintly compatible but not occasionally compatible.

In one of the interesting paper, Jungck [11] established a common fixed point theorem for four mappings in a complete metric space. Now, we prove our main result for the existence of common fixed point for four mappings in a non-complete metric space using the concept of faintly compatible mappings which is analogous to the result of Jungck [11].

2. Main Results

Theorem 2.1. *Let A, B, S and T be continuous self mappings of a metric space (X, d). Suppose*

(i) *pairs (A, S) and (B, T) are non-compatible and faintly compatible,*
(ii) *$AX \subset TX$ and $BX \subset SX$.*

If there exists $k \in (0, 1)$ such that

$$d(Ax, By) \leq k \max\Big\{d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\Big\} \tag{1}$$

for $x, y \in X$. Then there is a unique point $z \in X$ such that $Az = Bz = Sz = Tz = z$.

Proof. As the pair (A, S) is non-compatible, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either non-zero or non-existent. Since A and S are faintly compatible and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, there exists a sequence $\{z_n\}$ in X satisfying $\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sz_n = u(say)$ such that

$$\lim_{n\to\infty} d(ASz_n, SAz_n) = 0. \tag{2}$$

Further, since A is continuous, $\lim_{n\to\infty} AAz_n = Au$ and $\lim_{n\to\infty} ASz_n = Au$. These last three limits together imply $\lim_{n\to\infty} SAz_n = Au$. The inclusion $AX \subset TX$ implies that $Au = Tv$ for some $v \in X$ and $\lim_{n\to\infty} AAz_n = Tv$, $\lim_{n\to\infty} SAz_n = Tv$.

Similarly, non-compatibility of the pair B, T implies that there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t'$ for some $t' \in X$ but $\lim_{n\to\infty} d(BTy_n, TBy_n)$ is either non-zero or non-existent. Now faint compatibility of B and T will imply that there exists a sequence $\{w_n\}$ in X satisfying $\lim_{n\to\infty} Bw_n = \lim_{n\to\infty} Tw_n = u'(say)$ such that

$$\lim_{n\to\infty} d(BTw_n, TBw_n) = 0. \tag{3}$$

Again, B is continuous so $\lim_{n\to\infty} BBw_n = Bu'$ and $\lim_{n\to\infty} BTw_n = Bu'$. These last three limits together imply $\lim_{n\to\infty} TBw_n = Bu'$. The inclusion $BX \subset SX$ implies that $Bu' = Sv'$ for some $v' \in X$ and $\lim_{n\to\infty} BBw_n = Sv'$, $\lim_{n\to\infty} TBw_n = Sv'$.

Using the condition (1), we get

$$\begin{aligned} d(u,u') &= \lim_{n\to\infty} d(Az_n, Bw_n) \\ &\le k \lim_{n\to\infty} \max\big\{d(Az_n,Sz_n), d(Bw_n,Tw_n), d(Sz_n,Tw_n), \\ &\qquad \frac{1}{2}[d(Az_n,Tw_n)+d(Bw_n,Sz_n)\big\} \\ &= k \max\big\{d(u,u'), d(u',u'), d(u,u'), \frac{1}{2}[d(u,u')+d(u',u)]\big\}. \end{aligned}$$

Thus

$$d(u,u') \le k\, d(u,u') \quad\Rightarrow\quad d(u,u')=0 \quad\Rightarrow\quad u=u'$$

So, $Au = Tv$ and $Bu = Sv'$.

Now, $\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Bw_n = \lim_{n\to\infty} Tw_n = u$.
Continuity of S and T together with conditions (2) and (3) imply

$$\lim_{n\to\infty} SSz_n = \lim_{n\to\infty} SAz_n = Su \Rightarrow \lim_{n\to\infty} SSz_n = \lim_{n\to\infty} ASz_n = Su,$$

and $\lim_{n\to\infty} TBw_n = \lim_{n\to\infty} TTw_n = Tu \Rightarrow \lim_{n\to\infty} TTw_n = \lim_{n\to\infty} BTw_n = Tu.$

Now,

$$\begin{aligned} d(ASz_n, BTw_n) \le k \max\big\{&d(ASz_n,SSz_n), d(BTw_n,TTw_n), d(SSz_n,TTw_n), \\ &\frac{1}{2}[d(ASz_n,TTw_n)+d(BTw_n,SSz_n)\big\}. \end{aligned}$$

Taking $n\to\infty$, we get

$$\begin{aligned} d(Su,Tu) \le k \max\big\{d(Su,Su), d(Tu,Tu), d(Su,Tu), \frac{1}{2}[d(Su,Tu)+d(Tu,Su)]\big\} \\ \Rightarrow\quad d(Su,Tu) \le k\, d(Su,Tu) \quad\Rightarrow\quad d(Su,Tu)=0 \end{aligned}$$

$$\Rightarrow Su = Tu. \tag{4}$$

Also,

$$\begin{aligned} d(Au,Tu) &= \lim_{n\to\infty} d(Au, BTw_n) \\ &\le \max \lim_{n\to\infty} \big\{d(Au,Su), d(BTw_n,TTw_n), d(Su,TTw_n), \\ &\qquad \frac{1}{2}[d(Au,TTw_n)+d(BTw_n,Su)\big\} \\ \Rightarrow\quad d(Au,Tu) &\le k \max\big\{d(Au,Su), d(Tu,Tu), d(Su,Tu), \\ &\qquad \frac{1}{2}[d(Au,Tu)+d(Tu,Su)]\big\} \\ \Rightarrow\quad d(Au,Tu) &\le k \max\big\{d(Au,Tu), \frac{1}{2}d(Au,Tu)\big\} \\ \Rightarrow\quad d(Au,Tu) &= 0 \end{aligned}$$

$$(5) \qquad \Rightarrow \quad Au = Tu.$$

Using (2.1) with $x = y = u$, we get

$$\begin{aligned} d(Au, Bu) &\leq k \max\Big\{d(Au, Su), d(Bu, Tu), d(Su, Tu), \\ &\quad \frac{1}{2}[d(Au, Tu) + d(Bu, Su)]\Big\} \\ \Rightarrow \quad d(Au, Bu) &\leq k \max\Big\{d(Bu, Au), \\ &\quad \frac{1}{2}d(Bu, Au)\Big\} \\ \Rightarrow \quad d(Au, Bu) &= 0 \end{aligned}$$

$$(6) \qquad \Rightarrow \quad Au = Bu.$$

From (4), (5) and (6), we have $Au = Bu = Su = Tu$. In fact, u is a common fixed point A, B, S and T. To see this,

$$\begin{aligned} d(u, Bu) &= \lim_{n\to\infty} d(Az_n, Bu) \\ &\leq k \max \lim_{n\to\infty} \Big\{d(Az_n, Sz_n), d(Bu, Tu), d(Sz_n, Tu), \\ &\quad \frac{1}{2}[d(Az_n, Tu) + d(Bu, Sz_n)]\Big\} \\ \Rightarrow \quad d(u, Bu) &\leq k \max\Big\{d(u, Tu), \frac{1}{2}[d(u, Tu) + d(Bu, u)]\Big\} \\ \Rightarrow \quad d(u, Bu) &\leq k \max\Big\{d(u, Bu), \frac{1}{2}d(u, Bu)\Big\} \\ \Rightarrow \quad d(u, Bu) &= 0 \quad \Rightarrow \quad Bu = u. \end{aligned}$$

Hence, $Au = Bu = Su = Tu = u$.

For the uniqueness of the common fixed point, let w be another common fixed point of A, B, S and T, i.e., $Aw = Bw = Sw = Tw = w$. We have

$$\begin{aligned} d(u, w) &= A(Au, Bw) \leq k \max\Big\{d(Au, Bu), d(Bw, Tw), d(Su, Tw), \\ &\quad \frac{1}{2}[d(Au, Tw) + d(Bw, Su)]\Big\} \\ \Rightarrow \quad d(u, w) &\leq k\, d(u, w) \quad \Rightarrow \quad u = w. \end{aligned}$$

□

Remark 2.1. In Theorem 2.1, taking (A, S) and (B, T) as compatible pairs of mappings and metric space as complete, we get the result of Jungck [11].

Now, we give an example in the support of our main result.

Example 2.1. Let $X = [0, 10]$ with usual metric d on X. Define self mappings A, B, S and T on X as follows:

$$Ax = \begin{cases} 5 & \text{if } x \leq 5 \\ 10 - x & \text{if } x > 5 \end{cases} \qquad Bx = \begin{cases} \frac{3x+10}{5} & \text{if } x \leq 5 \\ 10 - x & \text{if } x > 5 \end{cases}$$

$$Sx = \begin{cases} \frac{-3x+40}{5} & \text{if } x \leq 5 \\ 10 - x & \text{if } x > 5 \end{cases} \qquad\qquad Tx = 10 - x \quad \forall x \in X.$$

In this example, pairs (A, S) and (B, T) on X are faintly compatible mappings. For this, we consider the constant sequence $\{x_n = 5\}$ and $AS(5) = SA(5)$. Also, pairs (A, S) and (B, T) are non-compatible mappings. To see this, consider the sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} x_n = 10$, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 0$$

and

$$\lim_{n\to\infty} d(ASx_n, SAx_n) \neq 0, \quad \lim_{n\to\infty} d(BTx_n, TBx_n) \neq 0.$$

It can be verified that the mappings A, B, S and T on X are satisfying the condition (1) with $k = \frac{3}{8}$ and 5 is the only common fixed point of A, B, S and T.

Taking $A = B$ and $S = T$ in Theorem 2.1, we obtain following corollary.

Corollary 2.1. *Let A, S be self mappings of a metric space (X, d). Suppose*

(i) *A and S are continuous,*
(ii) *pairs (A, S) is non-compatible faintly compatible,*
(iii) *$AX \subset SX$.*

If there exists $k \in (0, 1)$ such that

$$d(Ax, Ay) \leq k \max\Big\{d(Ax, Sx), d(Ay, Sy), d(Sx, Sy), \frac{1}{2}[d(Ax, Sy) + d(Ay, Sx)]\Big\}$$

for $x, y \in X$. Then A and S have unique common fixed point in X.

Remark 2.2. Observe that Corollary 2.1 is a generalization of the result due to Bisht et al. ([3], Theorem 2.1) .

Corollary 2.2. *Let A, B, S and T be continuous self mappings of a metric space (X, d). Suppose the pairs (A, S) and (B, T) are non-compatible faintly compatible and $AX \subset TX$ and $BX \subset SX$. If there exists $k \in (0, 1)$ such that any of the following inequalities holds*

(i) $d(Ax, By) \leq k \max\big\{d(Ax, Sx), d(By, Ty)\big\}$
(ii) $d(Ax, By) \leq k \max\big\{d(Ax, Sx), d(By, Ty), d(Sx, Ty)\big\}$
(iii) $d(Ax, By) \leq k \max\big\{d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(Ax, Ty)+d(By, Sx)]\big\}$

for all $x, y \in X$. Then there is a unique point $z \in X$ such that $Az = Bz = Sz = Tz = z$.

If we consider $A = T$ and $B = S$ in Theorem 2.1, we have following corollary.

Corollary 2.3. *Let A and B be continuous self mappings of a metric space (X, d), the pair (A, B) be non-compatible faintly compatible and $AX \subset BX$. If there exists $k \in (0, 1)$ such that*

$$d(Ax, By) \leq k \max\left\{d(Ax, Bx), d(Ay, By), d(Bx, Ay), \frac{1}{2}[d(Ax, Ay) + d(Bx, By)]\right\}$$

for all $x, y \in X$. Then A and B have a unique common fixed point in X.

Taking $T = S = I$ (*identity map*) in Theorem 2.1, we obtain following result as corollary.

Corollary 2.4. *Let A and B be continuous self mappings of a metric space (X, d), the pair (A, B) be non-compatible faintly compatible and $AX \subset BX$. If there exists $k \in (0, 1)$ such that*

$$d(Ax, By) \leq k \max\left\{d(x, y), d(x, Ax), d(y, By), \frac{1}{2}[d(y, Ax) + d(x, By)]\right\}$$

for all $x, y \in X$. Then A and B have a unique common fixed point in X.

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