

A coincidence and common fixed point theorem for subsequentially continuous hybrid pairs of maps satisfying an implicit relation

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ABSTRACT. In this paper, we introduce the notion of subsequential continuity for a hybrid pair of maps and combine this concept with compatibility, to establish a coincidence and common fixed point theorem for a hybrid quadruple of maps. Our main result also demonstrates that several fixed point theorems can be unified using implicit relations. We also give two examples in support our results.

1. INTRODUCTION

Hybrid fixed point theory, which is the realm of common fixed point theorems for single valued and multivalued maps in metric spaces, has prospective applications in functional inclusions, optimization theory, fractal graphics and discrete dynamics for set valued operators. Since the work of Nadler [18], many authors gave generalizations of contraction principle of Banach for multivalued and hybrid pair of maps (for instance [1, 2, 4, 5, 14, 15, 25, 27, 28] and reference there in). Jungck [10] introduced the concept of compatibility for a pair of self maps, which has been extended to hybrid pair of maps by Kaneko and Sessa [15] and Beg and Azam [5] when $T : X \rightarrow CB(X)$. Recently, Bouhadjera and Godet Tobie [7] introduced subsequential continuity which is weaker than the reciprocal continuity introduced by pant [19]. In fact every non-vacuously pair of reciprocally continuous maps is naturally subsequentially continuous. However subsequentially continuous maps are neither sequentially continuous nor reciprocally continuous. For a brief development of variants of continuity and the relation between them one may refer to Tomar and Karapinar [30].

In this paper, we introduce the notion of subsequential continuity for a hybrid pair of maps and combine this concept with compatibility, in order to establish a common fixed point theorem for two hybrid pairs of maps satisfying implicit relation in metric spaces. Our main result also demonstrates

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that several fixed point theorems can be unified using implicit relations introduced by Popa [25].

2. PRELIMINARIES

Let (X, d) be a metric space, and let $CB(X)$ be a set of non empty, closed and bounded subsets of X . The Hausdorff metric is defined as:

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

for all $A, B \in CB(X)$ and $d(a, B) = \inf\{d(a, b), b \in B\}$.

If $A = \{a\}$ and $B = \{b\}$ (singleton), we write $H(A, B) = d(a, b)$.

Definition 2.1. Maps $f : X \rightarrow X$ and $S : X \rightarrow CB(X)$ on metric space (X, d) are compatible if and only if for all $x \in X$, $fSx \in CB(X)$ and

$$\lim_{n \rightarrow \infty} H(fSx_n, Sf x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = M \in CB(X)$ and $\lim_{n \rightarrow \infty} f x_n = t \in M$.

Example 2.1. Let $X = [0, 2]$ endowed with the euclidian metric. Define f and S as follows:

$$f x = \begin{cases} 0, & 0 \leq x < 1, \\ \frac{x+1}{2}, & 1 \leq x \leq 2, \end{cases} \quad S x = \begin{cases} \{\frac{1}{2}\}, & 0 \leq x < 1, \\ [1, x], & 1 \leq x \leq 2. \end{cases}$$

Consider a sequence $\{x_n\}$ defined by

$$x_n = 1 + \frac{1}{n}, \text{ for all } n \geq 1,$$

we have

$$f x_n = 1 + \frac{1}{2n} \rightarrow 1 \text{ as } n \rightarrow \infty,$$

$$S x_n = \left[1, 1 + \frac{1}{n}\right] \rightarrow \{1\} \text{ as } n \rightarrow \infty,$$

and $\lim_{n \rightarrow \infty} H(fSx_n, Sf x_n) = 0$, so pair (f, S) is compatible.

Singh and Mishra [29] introduced the notion of reciprocal continuity for a hybrid pair of single valued and multivalued maps as follows:

Definition 2.2. Maps $f : X \rightarrow X$ and $S : X \rightarrow CB(X)$ on metric space (X, d) are reciprocally continuous on X (resp. at $t \in X$) if and only if $fSx \in B(X)$ for each $x \in X$ (resp. $fSt \in B(X)$) and

$$\lim_{n \rightarrow \infty} fSx_n = fM, \quad \lim_{n \rightarrow \infty} Sf x_n = St,$$

where $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = M \in B(X)$, $\lim_{n \rightarrow \infty} f x_n = t \in M$.

Definition 2.3 ([7]). Maps f and g on a metric space (X, d) are said to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that:

$$\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = z,$$

for some $z \in X$ and

$$\lim_{n \rightarrow \infty} f g x_n = f z$$

$$\lim_{n \rightarrow \infty} g f x_n = g z.$$

Let \mathcal{F} is the set of of all continuous functions $F : \mathbb{R}_+^6 \rightarrow \mathbb{R}_+$ satisfying the following conditions:

(F_1) : F non increasing in t_5, t_6 .

(F_2) : For any $v > 0$, we have $F(v, u, 0, 0, u, u) > 0$, for every $u > 0$.

Example 2.2.

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4, t_5, t_6\},$$

where $0 \leq \alpha < 1$.

Example 2.3.

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha t_2 - \beta \max\{t_3, t_4\} - \gamma \max\{t_2 + t_3, t_4\} - \delta \max\{t_5, t_6\},$$

where $\alpha, \beta, \gamma > 0$, $\delta \geq 0$ and $\alpha + \beta + \gamma + \delta < 1$.

Example 2.4.

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)\{at_5 + bt_6\},$$

where $0 \leq \alpha < 1$, $a, b \geq 0$ and $a + b < 1$.

Example 2.5.

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi(t_2, t_3, t_4, t_5, t_6),$$

where $\phi : \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$ is an upper semi continuous function such that $\phi(0) = 0$ and $\phi(t, 0, 0, t, t) < t$ for each $t > 0$.

Example 2.6.

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(\max\{t_2, t_3, t_4, t_5, t_6\}),$$

where $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing function such $\psi(t) < t$ for each $t > 0$.

Example 2.7.

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - [at_2^p + bt_3^p + ct_4^p]^{\frac{1}{p}} - dt_5 t_6$$

, where $a > (1 + d)^p$, $d \geq 0, 0 \leq c, b < 1, p \in \mathbb{N}$.

3. MAIN RESULTS

First, we introduce subsequentially continuity for a hybrid pair of single valued and multivalued maps.

Definition 3.1. Maps $f : X \rightarrow X$ and $S : X \rightarrow CB(X)$ on metric space (X, d) are subsequentially continuous if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Sx_n = M \in CB(X) \quad \text{and} \quad \lim_{n \rightarrow \infty} fx_n = z \in M,$$

for some $z \in X$ and $\lim_{n \rightarrow \infty} fSx_n = fM$, $\lim_{n \rightarrow \infty} Sfx_n = Sz$.

Notice that continuity or reciprocal continuity implies subsequential continuity, but the converse may be not true.

Example 3.1. Let $X = [0, 2]$ and d the Euclidian metric, we define f, S by

$$fx = \begin{cases} 2 - x, & 0 \leq x \leq 1, \\ \frac{1}{2}, & 1 < x \leq 2, \end{cases} \quad Sx = \begin{cases} [0, x], & 0 \leq x \leq 1, \\ [x - 1, x], & 1 < x \leq 2. \end{cases}$$

We consider a sequence $\{x_n\}$ defined for each $n \geq 1$ by: $x_n = 1 - \frac{1}{n}$. Clearly $\lim_{n \rightarrow \infty} fx_n = 1 \in [0, 1]$ and $\lim_{n \rightarrow \infty} Sx_n = [0, 1] \in CB(X)$. Also we have:

$$\lim_{n \rightarrow \infty} fSx_n = \lim_{n \rightarrow \infty} \left[1 + \frac{1}{n}, 2\right] = [1, 2] = f[0, 1],$$

and

$$\lim_{n \rightarrow \infty} Sfx_n = \lim_{n \rightarrow \infty} \left[\frac{1}{n}, 1 + \frac{1}{n}\right] = [0, 1] = S1,$$

then (f, S) is subsequentially continuous.

On the other hand, consider a sequence $\{y_n\}$ defined for all $n \geq 1$ by: $y_n = 1 + \frac{1}{n}$, we have

$$\lim_{n \rightarrow \infty} fy_n = \frac{1}{2} \in [0, 1], \quad \text{and} \quad \lim_{n \rightarrow \infty} Sy_n = [0, 1] \in CB(X),$$

however

$$\lim_{n \rightarrow \infty} fSy_n = \lim_{n \rightarrow \infty} f \left(\left[\frac{1}{n}, 1 + \frac{1}{n} \right] \right) \neq f([0, 1]),$$

then f and S are never reciprocally continuous. Also $\lim_{n \rightarrow \infty} H(fSx_n, Sfx_n) \neq 0$, i.e, pair (f, S) is not compatible.

Example 3.2. Let $X = \mathbb{R}_+$ endowed with the euclidian metric, we define f, S by

$$fx = \begin{cases} \frac{x+2}{2}, & 0 \leq x \leq 2, \\ x - 1, & x > 2, \end{cases} \quad Sx = \begin{cases} \{2\}, & 0 \leq x \leq 2, \\ \{1\}, & x > 2. \end{cases}$$

Let $\{x_n\}$ be sequence such that $x_n = 2 + e^{-n}$, for each $n \geq 1$. Clearly $\lim_{n \rightarrow \infty} f x_n = 1 \in \{1\}$ and $\lim_{n \rightarrow \infty} S x_n = \{1\} \in CB(X)$. Also we have:

$$\lim_{n \rightarrow \infty} f S x_n = f\{1\},$$

and

$$\lim_{n \rightarrow \infty} S f x_n = \{2\} = S1,$$

then (f, S) is subsequentially continuous.

Now, let $\{y_n\}$ be a sequence such $y_n = \frac{1}{n}$, for each $n \geq 1$. We have

$$\lim_{n \rightarrow \infty} f y_n = 2 \in \{2\}, \quad \text{and} \quad \lim_{n \rightarrow \infty} S y_n = \{2\} \in CB(X),$$

however

$$\lim_{n \rightarrow \infty} S f y_n = \{1\} \neq S2,$$

then f and S are never reciprocally continuous.

Also $\lim_{n \rightarrow \infty} H(f S x_n, S f x_n) = 0$, i.e, pair (f, S) is compatible.

Remark 3.1. In Example 3.1 hybrid pair of maps is subsequentially continuous but is neither compatible nor reciprocal continuous. However in Example 3.2, hybrid pair is both subsequentially continuous and compatible but not reciprocally continuous. Hence subsequential continuity for a hybrid pair of single valued and multivalued maps is independent of compatibility. Further one may notice that continuity and reciprocal continuity for hybrid pair of maps implies subsequential continuity, but the converse may be not be true.

Theorem 3.1. *Let $f, g : X \rightarrow X$, be single valued maps and $S, T : X \rightarrow CB(X)$ multivalued maps on a metric space (X, d) such that the pairs $(f, S), (g, T)$ are subsequentially continuous and compatible for all $x, y \in X$. Then the hybrid pair of maps (f, S) and (g, T) have a coincidence point. Moreover f, g, S and T have a common fixed point in X provided that maps satisfy:*

$$(1) \quad F \left(H(Sx, Ty), d(fx, gy), d(fx, Sx), \right. \\ \left. d(gy, Ty), d(fx, Ty), d(gy, Sx) \right) \leq 0,$$

for all $x, y \in X$, where $F \in \mathcal{F}$.

Proof. Since f and S are subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} f S x_n = fM \quad \text{and} \quad \lim_{n \rightarrow \infty} S f x_n = Sz$$

and $\lim_{n \rightarrow \infty} S x_n = M \in CB(X)$, $\lim_{n \rightarrow \infty} f x_n = z \in M$.

Also, the compatibility of the pair (f, S) implies that

$$\lim_{n \rightarrow \infty} H(f S x_n, S f x_n) = H(fM, Sz) = 0,$$

i.e, $fM = Sz$, and z is a coincidence point for f and S .

Similarly, since pair (g, T) is subsequentially continuous and compatible $gN = Tt$. Hence t is a coincidence point for g and T .

We claim $fz = gt$, if not by using (1) we get:

$$F(H(Sz, Tt), d(fz, gt), d(fz, Sz), d(gy, Tt), d(fz, Tt), d(gt, Sz)) \leq 0,$$

or

$$F(H(Sz, Tt), d(fz, gt), 0, 0, d(fz, Tt), d(gt, Sz)) \leq 0.$$

Since F is non increasing in t_5, t_6 , we get

$$F(H(gz, gt), d(fz, gt), 0, 0, d(fz, gt), d(gt, fz)) \leq 0$$

which is a contradiction with (F_2) , hence $fz = gt$.

Nextly, we show that $z = fz$, if not by using (1) we get:

$$F(H(Sx_n, Tt), d(fx_n, gt), d(fx_n, Sx_n), d(gt, Tt), d(fx_n, Tt), d(gt, Sx_n)) \leq 0.$$

Letting $n \rightarrow \infty$, we get:

$$F(H(M, Tt), d(z, fz), 0, 0, d(z, Tt), d(fz, M)) \leq 0.$$

Since $z \in M$ and $fz \in Sz$,

$$F(d(z, fz), d(z, fz), 0, 0, d(z, fz), d(z, fz)) \leq 0,$$

which is a contradiction, hence $z = fz \in Sz$ and z is common fixed point of f and S .

Now we prove $z = t$, if not by using (1) we get:

$$F\left(H(Sx_n, Ty_n), d(fx_n, gy_n), d(fx_n, Sx_n),\right. \\ \left. d(gy_n, Ty_n), d(fx_n, Ty_n), d(gy_n, Sx_n)\right) \leq 0.$$

Letting $n \rightarrow \infty$, we get:

$$F(H(M, N), d(z, t), 0, 0, d(z, N), d(t, M)) \leq 0.$$

Since $z \in M$ and $t \in N$, we get:

$$F(H(z, t), d(z, t), 0, 0, d(z, t), d(z, t)) \leq 0,$$

which is a contradiction. Hence $z = t$, i.e., z is common fixed point for f, g, S and T . \square

If $f = g$ and $S = T$, we obtain the following corollary:

Corollary 3.1. *Let $f : X \rightarrow X$ be a single valued map and $S : X \rightarrow CB(X)$ be multivalued map on a metric space (X, d) such that the hybrid pair (f, S) is subsequentially continuous and compatible for all $x, y \in X$. Then the hybrid pair of maps (f, S) has a coincidence point. Moreover f and S have a common fixed point in X provided that maps satisfy:*

$$F(H(Sx, Sy), d(fx, fy), d(fx, Sx), d(fy, Sy), f(fx, Sy), d(fy, Sx)) \leq 0,$$

for $x, y \in X$, where $F \in \mathcal{F}$.

Example 3.3. Let $X = \mathbb{R}_+$ endowed with the euclidian metric, define maps f and S as follows:

$$fx = \begin{cases} 2, & 0 \leq x < 1, \\ \frac{x+1}{2}, & x \geq 1, \end{cases} \quad Sx = \begin{cases} [0, \frac{3}{4}], & 0 \leq x < 1, \\ \{1\}, & x \geq 1. \end{cases}$$

We consider a sequence $\{x_n\}$ defined for each $n \geq 1$ by:

$x_n = 1 + \frac{1}{n}$, clearly $\lim_{n \rightarrow \infty} Sx_n = \{1\}$ and $\lim_{n \rightarrow \infty} fx_n = 1 \in \{1\}$. So:

$$\lim_{n \rightarrow \infty} fSx_n = \{1\} = f\{1\}, \quad \lim_{n \rightarrow \infty} Sfx_n = S(1 + \frac{1}{2n}) = S1,$$

then $\{f, S\}$ is subsequentially continuous. Also, we have:

$$\lim_{n \rightarrow \infty} H(fSx_n, Sfx_n) = H(\{1\}, \{1\}) = 0,$$

so f and S are compatible.

Now, by taking

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{4}{5} \max\{t_2, t_3, t_4, t_5, t_6\}$$

in corollary 3.1, and we show that the following inequality hold. We have the following cases:

$$H(Sx, Sy) \leq \frac{4}{5} \max\{d(fx, fy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(fy, Sx)\}.$$

- (1) For $x, y \in [0, 1)$, we have $H(Sx, Sy) = 0$, so obviously the inequality holds.
- (2) For $x \in [0, 1)$ and $y \geq 1$, we have

$$H(Sx, Sy) = \frac{1}{4} \leq \frac{4}{5} = \frac{4}{5} d(fx, Sx).$$

- (3) For $x \in [1, \infty)$ and $y \in [0, 1)$, we have

$$H(Sx, Sy) = \frac{1}{4} \leq \frac{4}{5} = \frac{4}{5} d(fy, Sy).$$

- (4) For $x, y \in [1, \infty)$, we have $H(Sx, Sy) = 0$, so the inequality holds.

Hence all hypotheses of corollary 3.1 are satisfied and the point 1 is a common fixed for f and S .

Corollary 3.2. *Let $f, g : X \rightarrow X$ be single valued maps and let $S, T : X \rightarrow CB(X)$ be multivalued maps on a metric space (X, d) such that the hybrid pairs (f, S) and (g, T) are subsequentially continuous and compatible for all $x, y \in X$. Then the hybrid pair of maps (f, S) and (g, T) has a coincidence point. Moreover f, g, S and T have a common fixed point in X provided that maps satisfy*

$$H(Sx, Ty) \leq \alpha \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)\},$$

where $0 \leq \alpha < 1$.

Proof. Proof follows immediately on taking

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4, t_5, t_6\}$$

in theorem 3.1, where $0 \leq \alpha < 1$. \square

Corollary 3.3. *Let $f, g : X \rightarrow X$ be single valued maps and let $S, T : X \rightarrow CB(X)$ be multivalued maps on a metric space (X, d) such that the hybrid pairs (f, S) and (g, T) are subsequentially continuous and compatible for all $x, y \in X$. Then the hybrid pair of maps (f, S) and (g, T) has a coincidence point. Moreover f, g, S and T have a common fixed point in X provided that maps satisfy*

$$H(Sx, Ty) \leq \phi(d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)),$$

where $\phi : \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$ is an upper semicontinuous function such $\phi(0) = 0$ and $\phi(t, 0, 0, t, t) < t$ for each $t > 0$.

Proof. Proof follows immediately on taking

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi(t_2, t_3, t_4, t_5, t_6)$$

in theorem 3.1, where $\phi : \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$ is an upper semi continuous function such that $\phi(0) = 0$ and $\phi(t, 0, 0, t, t) < t$ for each $t > 0$. \square

Corollary 3.4. *Let $f, g : X \rightarrow X$ be single valued maps and let $S, T : X \rightarrow CB(X)$ be multivalued maps on a metric space (X, d) such that the hybrid pairs (f, S) and (g, T) are subsequentially continuous and compatible for all $x, y \in X$. Then the hybrid pair of maps (f, S) and (g, T) has a coincidence point. Moreover f, g, S and T have a common fixed point in X provided that maps satisfy*

$H(Sx, Ty) \leq \alpha \max\{d(fx, gy), d(fx, Sx), d(gy, Ty)\} - (1 - \alpha)(ad(fx, Ty) + bd(gy, Sx))$, where $0 \leq \alpha < 1$, $a + b < 1$.

Proof. Proof follows immediately on taking

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)\{at_5 + bt_6\}$$

in theorem 3.1, where $0 \leq \alpha < 1$, $a, b \geq 0$ and $a + b < 1$. \square

Example 3.4. Let $X = [0, 4]$ endowed with the euclidian metric, define maps f, g, S and T as follows:

$$fx = \begin{cases} \frac{x+2}{2}, & 0 \leq x \leq 2, \\ 0, & 2 < x \leq 4, \end{cases} \quad gx = \begin{cases} x + 1, & 0 \leq x \leq 1, \\ 2, & x = 2, \\ 4, & 2 < x \leq 4, \end{cases}$$

$$Sx = \begin{cases} [2, 4 - x], & 0 \leq x \leq 2, \\ \{\frac{7}{4}\}, & 2 < x \leq 4, \end{cases} \quad Tx = \begin{cases} \{2\}, & 0 \leq x \leq 2, \\ [0, 1], & 2 < x \leq 4. \end{cases}$$

We consider a sequence $\{x_n\}$ defined for each $n \geq 1$ by:

$x_n = 2 - \frac{1}{n}$, clearly $\lim_{n \rightarrow \infty} Sx_n = \{2\}$ and $\lim_{n \rightarrow \infty} fx_n = 2 \in \{2\}$, also we have:

$$\lim_{n \rightarrow \infty} fSx_n = \{2\} = f\{2\}$$

$$\lim_{n \rightarrow \infty} Sfx_n = S2 = \{2\}.$$

Also $\lim_n H(fSx_n, Sfx_n) = 0$. Hence (f, S) is subsequentially continuous and compatible.

For g and T , consider a sequence $\{y_n\}$ defined by: $y_n = 2 - e^{-n}$, for all $n \geq 1$. It is clear that $\lim_{n \rightarrow \infty} gy_n = 2$, $\lim_{n \rightarrow \infty} Ty_n = \{2\}$, and $\lim_{n \rightarrow \infty} gTy_n = g\{2\} = \{2\}$, $\lim_{n \rightarrow \infty} Tgy_n = T2 = \{2\}$,

$$\lim_{n \rightarrow \infty} H(gTy_n Tgy_n) = 0,$$

i.e., (g, T) is subsequentially continuous and compatible.

By taking $\alpha = \frac{2}{3}$ and $a = b = \frac{1}{4}$ in corollary 3.4, we show that the inequality is satisfied. We have the following cases:

(1) For $x, y \in [0, 2]$, we have $H(Sx, Ty) = 0$, obviously inequality is satisfied.

(2) For $x \in [0, 2]$ and $y \in (2, 4]$, we have

$$H(Sx, Ty) = 1 \leq 2 = \frac{2}{3}d(gy, Ty)$$

(3) For $x \in (2, 4]$ and $y \in [0, 2]$, we have

$$H(Sx, Ty) = \frac{1}{4} \leq \frac{7}{6} = \frac{2}{3}d(fx, Sx)$$

(4) For $x, y \in (2, 4]$, we have

$$H(Sx, Ty) = \frac{3}{4} \leq \frac{8}{3} = \frac{2}{3}d(fx, gy),$$

then all hypotheses of corollary 3.4 are satisfied and the point 2 is a common fixed for f, g, S and T .

Remark 3.2. (i) Clearly subsequentially continuous and compatible hybrid quadruple of maps on a non- complete metric space have a coincidence point. However it is believed that a pair of continuous and commuting (and hence compatible) self maps of a complete metric space may not have a coincidence point.[for instance, $gx = 1 + x$ and $fx = x, x \in [0, 1]$].

(ii) It is clear from Example 3.4 that (S, f) and (g, T) have different coincidence point, i.e., $f2 \in S2, f2 = g1 \in T1$.

(iii) None of the map is continuous in the illustrating Examples 3.3 and 3.4, i.e., continuity requirements of all the involved maps is completely relaxed whereas most of the earlier theorems require the continuity of at least one involved map.

- (iv) It is interesting to note that Examples 3.3 and 3.4 cannot be covered by all those coincidence and common fixed point theorems which require containment of range space and continuity of involved maps along with completeness (or closedness) of underlying space/subspace. Subsequential continuity is indeed a weaker notion than the continuity of one of the map. However containment of the range space of the involved maps is indispensable in [4] and [27]. Our results generalize, extend and improve multitude of common fixed point results existing in the literature (for instance Altun and Türkoğlu [4], Popa [25], Djoudi [8] and Sedghi et al.[27])and guarantee the existence of coincidence and common fixed point for discontinuous hybrid quadruple of maps in non-complete metric space without containment of the range space of involved maps.

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