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On $(\sigma, \delta) - (S, 1)$ rings and their extensions

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Abstract. Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. We recall that R is called an $(S,1)$-ring if for a, $b \in R$, $ab = 0$ implies $aRb = 0$. We involve σ and δ to generalize this notion and say that R is a $(\sigma, \delta) - (S, 1)$ ring if for a, $b \in R$, $ab = 0$ implies $aRb = 0$, $\sigma(a)Rb = 0$, $aR\sigma(b) = 0$ and $\delta(a)Rb = 0$. In case σ is identity, R is called a $\delta - (S, 1)$ ring.

In this paper we study the associated prime ideals of Ore extension $R[x; \sigma, \delta]$ and we prove the following in this direction:

Let R be a semiprime right Noetherian ring, which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers), σ an automorphism of R and δ a σ-derivation of R such that R is a $(\sigma, \delta) - (S, 1)$ ring. Then P is an associated prime ideal of $R[x; \sigma, \delta]$ (viewed as a right module over itself) if and only if there exists an associated prime ideal U of R (viewed as a right module over itself) such that $(P \cap R)[x; \sigma, \delta] = P$ and $P \cap R = U$.

1. Introduction

All rings are associative with identity $1 \neq 0$ unless otherwise stated. The ring of integers, the field of rational numbers and the field of real numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively, unless otherwise stated. $Spec(R)$ denotes the set of prime ideals of R. $MinSpec(R)$ denotes the set of minimal prime ideals of R. The Prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. For any subset J of a right R-module M, annihilator of J is denoted by $Ann(J)$. The set of associated prime ideals of R (viewed as a right module over itself) is denoted by $Ass(R_R)$. Let R be a right Noetherian ring. For any uniform right R-module J, the assassinator of J is denoted by $Assas(J)$. Let M be a right R-module. Consider the set

$$\{Assas(J) \mid J \text{ is a uniform right } R\text{-submodule of } M\}.$$

We denote this set by $\mathbb{A}(M_R)$.

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Remark 1.1. If R is viewed as a right module over itself, we note that $Ass(R_R) = \mathbb{A}(R_R)$ ($5Y$ of Goodearl and Warfield [3]).

Endomorphisms and derivations.

Let R be a ring, σ be an endomorphism of R and δ be a σ-derivation of R. Then $\delta : R \to R$ is an additive map such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$.

Example 1.1. (1) Let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a σ-derivation of R if and only if ϕ is a homomorphism.

(2) For any endomorphism τ of a ring R and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$ is a τ-derivation of R.

In case σ is the identity map, δ is called just a derivation of R. For example let F be a field and $R = F[X]$. Then the usual differential operator $\frac{d}{dx}$ is a derivation of R.

Ore extensions.

Ore extension (skew polynomial ring) over R in an indeterminate x is: $R[x; \sigma, \delta] = \{f(x) = \sum_{i=0}^{n} x^i a_i \mid a_i \in R\}$ with addition as usual and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x; \sigma, \delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$ as followed in McConnell and Robson [6]. This definition of non-commutative polynomial rings was first introduced by Ore in 1933, who combined earlier ideas of Hilbert (in the case $\delta = 0$) and Schlessinger (in the case $\sigma = 1$). We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. An ideal I of a ring R is called σ-stable if $\sigma(I) \subseteq I$ and is called δ-invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ-stable and δ-invariant, then $I[x; \sigma, \delta]$ is an ideal of $O(R)$ and as usual we denote it by $O(I)$.

Theorem 1.1. *(Hilbert Basis Theorem, namely Theorem 2.6 of Goodearl and Warfield* [3]*). Let R be a right/left Noetherian ring. Let σ and δ be as above. Then the Ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian.*

2. PRELIMINARIES

Completely prime ideals.

Definition 2.1. (McCoy [7]) An ideal P of a ring R is said to be completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$.

Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $P = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ is completely prime ideal of R.

Note: In commutative case completely prime ideal and prime have the same meaning. In general (non-commutative) situation every completely prime ideal of a ring R is a prime ideal, but converse is not be true.

Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbf{Z})$. If p is a prime number, then the ideal $P = M_2(p\mathbf{Z})$ is a prime ideal of R, but is not completely prime, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

$\sigma(*)$**-rings.**

Definition 2.2. (Kwak [4]). Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 2.1. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}; a, b, c \in F, \text{ a field } \right\}$.

Now $P(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}; a, c \in F \right\}$.

Define $\sigma : R \to R$ a map by $\sigma\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then σ is an endomorphism of R, and it can be easily seen that R is a $\sigma(*)$-ring.

Remark 2.1. It can be seen that a $\sigma(*)$-ring R is 2-primal, for let $a \in R$ be such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)^2(a) \in \sigma(P(R)) = P(R).$$

Therefore, $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. So $P(R)$ is completely semiprime and hence R is 2-primal.

Weak σ-rigid rings.

Ouyang in [8] introduced weak σ-rigid rings, where σ is an endomorphism of ring R. These rings are related to 2-primal rings.

Definition 2.3. (Ouyang [8]). Let R be a ring and σ an endomorphism of R such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$. Then R is called a weak σ-rigid ring.

Example 2.2. Assume that $W_1[F]$ is the first Weyl algebra over a field F of characteristic zero. Then $W_1[F] = F[\mu, \lambda]$, the polynomial ring with

indeterminates μ and λ with $\lambda\mu=\mu\lambda+1$.
Now let R be the ring

$$\begin{pmatrix} W_1[F] & W_1[F] \\ 0 & 0 \end{pmatrix}.$$

Now the prime radical $P(R)$ of R is

$$\begin{pmatrix} 0 & W_1[F] \\ 0 & 0 \end{pmatrix}.$$

Define an endomorphism $\sigma : R \to R$ by

$$\sigma\Big(\begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}\Big)=\begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}.$$

Then R is a weak σ-rigid ring.

3. $(\sigma,\delta)-(S,1)$ RINGS

Definition 3.1. (Kim and Lee [5]). A ring R is called an $(S,1)$-ring if for a, $b\in R$, $ab=0$ implies $aRb=0$.

This notion was actually introduced by Shin (i.e. a ring satisfying SI property, Lemma 1.2 of [9]). In this article we generalize the notion of $(S,1)$-rings by involving a derivation δ of R.

Definition 3.2. Let R be a ring and δ a derivation of R. Then R is called a $\delta-(S,1)$ ring if for $a,b\in R)$, $ab=0$ implies that $aRb=0$ and $\delta(a)Rb=0$.

Example 3.1. Let $R=S\times S$, S a ring and $A=(u,v)\in R$. Define $\delta : R\longrightarrow R$ by $\delta_A(a,b)=(au-ua,bv-vb)$. Then δ_A is a derivation of R.

Now the only elements $\alpha,\beta\in R$ such that $\alpha\beta=0$ are of the form $\alpha=(a,0)$, and $\beta=(0,b)$ and for all $\gamma=(r,s)\in R$,

$\alpha\gamma\beta=0$ and

$$\begin{aligned}\delta_A(\alpha)\gamma\beta &= \delta((a,0))(r,s)(0,b)\\ &=(au-ua,0)(r,s)(0,b)\\ &=((au-ua)r,0)(0,b)\\ &=(0,0)\end{aligned}$$

So, R is a $\delta_A-(S,1)$ ring.

We note that a $\delta-(S,1)$ ring is an $(S,1)$-ring.

Definition 3.3. Let R be a ring (not necessarily with 1), σ an endomorphism of R and δ a σ derivation of R. Then R is called a $(\sigma,\delta)-(S,1)$ ring if for $a,b\in R$, $ab=0$ implies that $aRb=0$, $\sigma(a)Rb=0$, $aR\sigma(b)=0$ and $\delta(a)Rb=0$. In case δ is the zero map, $(\sigma,\delta)-(S,1)$ is called a $\sigma-(S,1)$.

Example 3.2. Let S be a ring ((not necessarily with 1) and $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a, b \in S \right\}$.

Define $\sigma : R \to R$ a map by $\sigma\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then σ is an endomorphism of R.

Define $\delta : R \to R$ a map by $\delta\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$. Then δ is a σ-derivation of R.

Now the only matrices $A \in R$ and $B \in R$ satisfying $AB = 0$ are of the type
$A = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}$; $u, v \in \mathbb{H}$.

Now for all $J = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in R$,
$AJB = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

and

(1) $\sigma(A)JB = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

(2) $AJ\sigma(B) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

(3) $\delta(A)JB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Therefore, R is a $(\sigma, \delta) - (S, 1)$ ring.

By definition, we see that a $(\sigma, \delta) - (S, 1)$ ring is an $(S, 1)$ ring, but the converse is not true.

Example 3.3. Let S be a domain and $R = S \times S$. Let $\sigma : R \longrightarrow R$ be defined by $\sigma(a, b) = (b, a)$. Then σ is an endomorphism of R. Let $A = (u, v) \in R$, define $\delta_A : R \longrightarrow R$ a map by

$$\delta_A\left((a, b)\right) = (au - ub, bv - va)$$

Then δ_A is a σ-derivation of R.

Now the only elements $p, q \in R$ such that $pq = 0$ are of the form $p = (a, 0)$ and $q = (0, b)$, for all $a, b \in S$. Now for all $t = (r, s) \in R$, $ptq = (0, 0)$. So R is an $(S, 1)$-ring, but for nonzero $a, b, u, v, r, s \in S$,

$\sigma(p)tq = \sigma((a, 0))(r, s)(0, b) = (0, a)(r, s)(0, b) = (0, asb) \neq 0$,
$pt\sigma(q) = (a, 0)(r, s)\sigma((0, b)) = (a, 0)(r, s)(b, 0) = (arb, 0) \neq 0$, and
$\delta_A(p)tq = \delta_A((a, 0))(r, s)(0, b) = (au, -va)(r, s)(0, b) = (0, -vasb) \neq 0$.

Thus, R is not a $(\sigma, \delta) - (S, 1)$ ring.

4. Completely prime ideals of $(\sigma, \delta) - (S, 1)$ rings

Proposition 4.1. *Let R be a ring, σ an automorphism of R such that R is a $\sigma - (S, 1)$ ring. Then R is 2-primal.*

Proof. R is a $\sigma - (S, 1)$ ring. Therefore by Theorem 1.5 of [9] R is 2-primal, which implies that $P(R)$ is completely semiprime. We give a sketch of proof.

Let $a \in N(R)$, say $a^n = 0$. If $a \notin P$ for some prime ideal P, then $ax_1a \notin P$ for some element $x_1 \in R$. Continuing the process we can find elements $x_i \in R$ such that P does not contain $b = ax_1a...x_{n-1}a$. But, R is an $(S, 1)$-ring, so $a^n = 0$ implies $b = 0$, hence $b \in P$, a contradiction. □

Proposition 4.2. *Let R be a ring, σ an automorphism of R such that R is a $\sigma - (S, 1)$ ring. Then R is a $\sigma(*)$-ring.*

Proof. R is 2-primal and $P(R)$ is completely semiprime by Proposition 4.1.

We will show that R is a weak σ-rigid ring. Let $a \in R$ be such that $a\sigma(a) \in N(R)$. Now $a\sigma(a)\sigma^{-1}(a\sigma(a)) \in N(R)$ implies that $a^2 \in N(R)$, and so $a \in N(R)$. Therefore, R is a weak σ-rigid ring, and is also a $\sigma(*)$-ring. □

Remark 4.1. Converse of Proposition 4.1 and Proposition 4.2 is not true. For example, the ring in Example 2.1 is a $\sigma(*)$-ring and, therefore, is also a 2-primal ring, but it is not a $\sigma - (S, 1)$ ring (even not an $(S, 1)$ ring).

Proposition 4.3. *Let R be a right Noetherian which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a $(\sigma, \delta) - (S, 1)$ ring. Then $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in MinSpec(R)$.*

Proof. R is 2-primal by Proposition 4.1 and is a $\sigma(*)$- ring by Proposition 4.2.

Now the result follows by Proposition 2.1 of [2]. □

Proposition 4.4. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a $(\sigma, \delta) - (S, 1)$ ring. Then $U \in MinSpec(R)$ implies that U is a completely prime ideal of R.*

Proof. Suppose that U is not completely prime. Then there exist $a, b \in R \backslash U$ with $ab \in U$. Consider U_i as in 4.3. Let c be any element of $b(U_2 \cap U_3 \cap ... \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n} U_i = P(R)$. So $c \in P(R)$ and, thus $b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U$. Therefore, $bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U$ and, as U is prime, $a \in U$, $U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime. □

Proposition 4.5. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a $(\sigma, \delta) - (S, 1)$ ring. Then $U \in Min.Spec(R)$ implies that $O(U)$ is a completely prime ideal of $O(R)$.*

Proof. Let $U \in Min.Spec(R)$. Now Proposition 4.3 implies that $\sigma(U) = U$ and $\delta(U) \subseteq U$. Also Now Proposition 4.4 implies that U is a completely prime ideal of R. Now the result follows by Theorem 2.4 of [1]. □

Theorem 4.1. *Let R be a semiprime right Noetherian ring. Let σ be an automorphism of R such that R is a $(\sigma, \delta) - (S, 1)$ ring. Then $P \in Ass(O(R)_{O(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $O(P \cap R) = P$ and $P \cap R = U$.*

Proof. R being right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$ (Remark 1.1). Now R a $(\sigma, \delta) - (S, 1)$ ring implies that $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in Min.Spec(R)$ by Proposition 4.3.

$O(R)$ is right Noetherian by Hilbert Basis Theorem. Let $J \in Ass(O(R)_{O(R)})$. Now by Remark 1.1 $Ass(O(R)_{O(R)}) = \mathbb{A}(O(R)_{O(R)})$. Let $P = Ann(I) = Assas(I)$ for some ideal I of $O(R)$ such that I is uniform as a right $O(R)$-module. Choose $f \in I$ to be nonzero of minimal degree (with leading coefficient a_n). Let $U = Ann(a_n R) = Assas(a_n R)$. Now R is right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$ and since R is semiprime, $U \in Min.Spec(R)$ by Proposition (2.2.14) of McConnell and Robson [6]. Now $\sigma(U) = U$, and $\delta(U) \subseteq U$ by Proposition 4.3. So $O(U)$ is an ideal of $O(R)$. Now $fU = 0$. Therefore $fO(R)U \subseteq fUO(R) = 0$. So $U \subseteq P \cap R$. But it is clear that $P \cap R \subseteq U$. Thus $P \cap R = U$.

Conversely let $U = Ann(cR) = Assas(cR)$, $c \in R$. Now R is right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$. Now $\sigma(U) = U$, $\delta(U) \subseteq U$ and it can be easily seen that $O(U) = Ann(chO(R))$ for all $h \in O(R)$. Therefore $O(U) = Ann(cO(R)) = Assas(cO(R))$. □

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