Mathematica Moravica
Vol. **21**, No. **1** (2017), 37–50

A fixed point theorem for (μ, ψ)-generalized f-weakly contractive mappings in partially ordered 2-metric spaces*

Nguyen Trung Hieu and Huynh Ngoc Cam

Abstract. The purpose of this paper is to introduce the notion of a (μ, ψ)-generalized f-weakly contractive mapping in partially ordered 2-metric spaces and state a fixed point theorem for this mapping in complete, partially ordered 2-metric spaces. The main results of this paper are generalizations of the main results of [4, 10]. Also, some examples are given to illustrate the obtained results.

1. Introduction and preliminaries

In 1972, Chatterjea [5] introduced the notion of a C-contraction in metric spaces as follows.

Definition 1.1 ([5])**.** Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping. Then, T is called a *C-contraction* if there exists $\alpha \in [0, \frac{1}{2})$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha\big[d(x, Ty) + d(y, Tx)\big].$$

This notion was generalized to a weak C-contraction in metric spaces by Choudhury [6] and a (μ, ψ)-generalized f-weakly contractive mapping in metric spaces by Chandok [3]. After that, there were some fixed point results for (μ, ψ)-generalized f-weakly contractive mappings in complete metric spaces [3, Theorem 2.1] and in complete, partially ordered metric spaces [4, Theorem 2.1].

Denote by Ψ the family of lower semi-continuous functions $\psi : [0, \infty)^2 \longrightarrow [0, \infty)$ such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

2000 *Mathematics Subject Classification.* Primary: 47H10, 54H25; Secondary: 54D99, 54E99.

Key words and phrases. Fixed point; 2-metric space; (μ, ψ)-generalized f-weakly contractive mapping.

*This work is partly discussed at The Dong Thap Seminar on Mathematical Analysis.

Definition 1.2 ([6], Definition 1.3)**.** Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping. Then, T is called a *weak C-contraction* if there $\psi \in \Psi$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}\big[d(x, Ty) + d(y, Tx)\big] - \psi\big(d(x, Ty), d(y, Tx)\big).$$

Definition 1.3 ([15])**.** A function $\mu : [0, \infty) \longrightarrow [0, \infty)$ is called an *altering distance function* if the following properties are satisfied.

(1) μ is monotone increasing and continuous.
(2) $\mu(t) = 0$ if and only if $t = 0$.

Definition 1.4 ([3])**.** Let (X, d) be a metric space and $T, f : X \longrightarrow X$ be two mappings. Then, T is called a *(μ, ψ)-generalized f-weakly contractive mapping* if there exist $\psi \in \Psi$ and μ which is an altering distance function such that for all $x, y \in X$,

$$\mu\big(d(Tx, Ty)\big) \leq \mu\big(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\big) - \psi\big(d(fx, Ty), d(fy, Tx)\big).$$

Remark 1.1. If f and μ are two identify mappings, then a (μ, ψ)-generalized f-weakly contractive mapping becomes a weak C-contraction.

There were some generalizations of a metric such as a 2-metric, a D-metric, a G-metric, a cone metric and a complex-valued metric [2]. Note that in the above generalizations, only a 2-metric space has not been known to be topologically equivalent to an ordinary metric. In addition, the fixed point theorems on 2-metric spaces and metric spaces may be unrelated easily [10]. There are many fixed point results on 2-metric spaces were stated and generalized, the readers may refer to [1, 8, 9, 11, 13, 17, 18, 19] and references therein.

In 2013, Dung and Hang [10] introduced the notion of a weak C-contraction mapping in partially ordered 2-metric spaces and state some fixed point results for these mappings in complete, partially ordered 2-metric spaces [10, Theorem 2.3, Theorem 2.4, Theorem 2.5]. The notion of a weak C-contraction mapping in partially ordered 2-metric spaces was introduced in [10] as follows.

Definition 1.5 ([10], Definition 2.1)**.** Let $(X, d, \preceq)$ be a partially ordered 2-metric space and $T : X \longrightarrow X$ be a mapping. Then, T is called a *weak C-contraction* if there exists $\psi \in \Psi$ such that for all $x, y, a \in X$ with $x \succeq y$ or $x \preceq y$,

$$d(Tx, Ty, a) \leq \frac{1}{2}\big[d(x, Ty, a) + d(y, Tx, a)\big] - \psi\big(d(x, Ty, a), d(y, Tx, a)\big).$$

The purpose of this paper is to introduce the notion of a (μ, ψ)-generalized f-weakly contractive mapping in partially ordered 2-metric spaces and state a fixed point theorem for this mapping in complete, partially ordered 2-metric spaces. The main results of this paper are generalizations of the main results of [4, 10]. Also, some examples are given to illustrate the obtained results.

First, we recall some notions and lemmas which will be useful in what follows.

Definition 1.6 ([12]). Let X be a non-empty set and let $d : X \times X \times X \longrightarrow \mathbb{R}$ be a mapping satisfying the following conditions.

(1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$;
(2) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$;
(3) The symmetry: $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$;
(4) The rectangle inequality: $d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)$ for all $x, y, z, t \in X$.

Then, d is called a 2-*metric* on X and (X, d) is called a 2-*metric space* which will be sometimes denoted by X if there is no confusion. Every member $x \in X$ is called a *point* in X.

Definition 1.7 ([13]). Let $\{x_n\}$ be a sequence in a 2-metric space (X, d). Then

(1) $\{x_n\}$ is called *convergent* to x in (X, d), written as $\lim\limits_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim\limits_{n\to\infty} d(x_n, x, a) = 0$.
(2) $\{x_n\}$ is called *Cauchy* in X if for all $a \in X$, $\lim\limits_{n,m\to\infty} d(x_n, x_m, a) = 0$, that is, for each $\varepsilon > 0$, there exists n_0 such that $d(x_n, x_m, a) < \varepsilon$ for all $n, m \geq n_0$.
(3) (X, d) is called *complete* if every Cauchy sequence in (X, d) is a convergent sequence.

Definition 1.8 ([16], Definition 8). A 2-metric space (X, d) is called *compact* if every sequence in X has a convergent subsequence.

Lemma 1.1 ([16], Lemma 3). *Every 2-metric space is a T_1-space.*

Lemma 1.2 ([16], Lemma 4). $\lim\limits_{n\to\infty} x_n = x$ *in a 2-metric space* (X, d) *if and only if* $\lim\limits_{n\to\infty} x_n = x$ *in the 2-metric topological space* X.

Lemma 1.3 ([16], Lemma 5). *If* $T : X \longrightarrow Y$ *is a continuous map from a 2-metric space* X *to a 2-metric space* Y, *then* $\lim\limits_{n\to\infty} x_n = x$ *in* X *implies* $\lim\limits_{n\to\infty} Tx_n = Tx$ *in* Y.

Remark 1.2. (1) It is straightforward from Definition 1.6 that every 2-metric is non-negative and every 2-metric space contains at least three distinct points.

(2) A 2-metric $d(x,y,z)$ is sequentially continuous in one argument. Moreover, if a 2-metric $d(x,y,z)$ is sequentially continuous in two arguments, then it is sequentially continuous in all three arguments, see [19, p.975].
(3) A convergent sequence in a 2-metric space need not be a Cauchy sequence, see [19, Remark 01 and Example 01]
(4) In a 2-metric space (X,d), every convergent sequence is a Cauchy sequence if d is continuous, see [19, Remark 02].
(5) There exists a 2-metric space (X,d) such that every convergent sequence is a Cauchy sequence but d is not continuous, see [19, Remark 02 and Example 02].

Definition 1.9 ([7], Definition 2.1)**.** Let $(X,\preceq)$ is a partially ordered set and $T,f:X\longrightarrow X$ be two mappings. Then, T is called *monotone f-nondecreasing* if for all $x,y\in X$, $fx\preceq fy$ implies $Tx\preceq Ty$. If f is an identity mapping, then T is called *monotone nondecreasing.*

Definition 1.10 ([14])**.** Let (X,d) be a metric space and $T,f:X\longrightarrow X$ be two mappings. Them, the pair (T,f) is called *weakly compatible* if they commute at their coincidence points, that is, $Tfx=fTx$ for all $x\in X$ with $Tx=fx$.

2. Main results

First, we introduce the notion of a (μ,ψ)-generalized f-weakly contractive mapping in partially ordered 2-metric spaces.

Definition 2.1. Let $(X,d,\preceq)$ be a partially ordered 2-metric space and $T,f:\ X\longrightarrow X$ be two mappings. Then, T is called a (μ,ψ)- *generalized f-weakly contractive mapping* if there exist $\psi\in\Psi$ and μ which is an altering distance function such that for all $x,y,a\in X$ with $fx\succeq fy$ or $fx\preceq fy$,

$$\begin{aligned}(1)\qquad & \mu\big(d(Tx,Ty,a)\big)\\ \leq\ & \mu\Big(\frac{1}{2}[d(fx,Ty,a)+d(fy,Tx,a)]\Big)-\psi\big(d(fx,Ty,a),d(fy,Tx,a)\big).\end{aligned}$$

Remark 2.1. If f and μ are two identify mappings, then a (μ,ψ)-generalized f-weakly contractive mapping in partially ordered 2-metric spaces becomes a weak C-contraction mapping in partially ordered 2-metric spaces in Definition 1.5.

The following result is a sufficient condition for the existence and the uniqueness of the common fixed point for (μ,ψ)- generalized f-weakly contractive mappings in partially ordered 2-metric spaces.

Theorem 2.1. *Let $(X,\preceq,d)$ be a complete, partially ordered 2-metric space and $T,f:X\longrightarrow X$ be two mappings such that*

(1) *$TX\subset fX$ and fX is closed.*

(2) *T is a monotone f-nondecreasing mapping.*
(3) *T is a (μ, ψ)-generalized f-weakly contractive mapping.*
(4) *If $\{fx_n\} \subset X$ is a nondecreasing sequence such that $\lim_{n\to\infty} fx_n = fz \in fX$, then $fx_n \preceq fz$ and $fz \preceq f(fz)$ for every $n \in \mathbb{N} \cup \{0\}$.*
(5) *There exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$.*

Then, T and f have coincidence point. Further, if T and f are weakly compatible, then T and f have a common fixed point. Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point.

Proof. Let $x_0 \in X$ such that $fx_0 \preceq Tx_0$. Since $TX \subset fX$, we can choose $x_1 \in X$ such that $fx_1 = Tx_0$. Since $Tx_1 \in fX$, there exists $x_2 \in X$ such that $fx_2 = Tx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Since $fx_0 \preceq Tx_0 = fx_1$ and T is a monotone f-nondecreasing mapping, we have $Tx_0 \preceq Tx_1$. Continuing, we obtain

$$Tx_0 \preceq Tx_1 \preceq \ldots Tx_n \preceq Tx_{n+1} \preceq \ldots$$

Then, $fx_{n+1} \succeq fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Due to T is a (μ, ψ)-generalized f-weakly contractive mapping, we get

$$\begin{aligned}
& \mu\big(d(Tx_{n+1}, Tx_n, a)\big) \\
\leq\ & \mu\big(\frac{1}{2}[d(fx_{n+1}, Tx_n, a) + d(fx_n, Tx_{n+1}, a)]\big) \\
& \quad -\psi\big(d(fx_{n+1}, Tx_n, a), d(fx_n, Tx_{n+1}, a)\big) \\
=\ & \mu\big(\frac{1}{2}[d(Tx_n, Tx_n, a) + d(Tx_{n-1}, Tx_{n+1}, a)]\big) \\
& \quad \psi\big(d(Tx_n, Tx_n, a), d(Tx_{n-1}, Tx_{n+1}, a)\big) \\
=\ & \mu\big(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1}, a)\big) - \psi\big(0, d(Tx_{n-1}, Tx_{n+1}, a)\big) \\
\leq\ & \mu\big(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1}, a)\big) \qquad (2)
\end{aligned}$$

for all $a \in X$. Since μ is a monotone increasing, from (2), we get

$$d(Tx_{n+1}, Tx_n, a) \leq \frac{1}{2}d(Tx_{n-1}, Tx_{n+1}, a) \qquad (3)$$

for all $a \in X$. By choosing $a = Tx_{n-1}$ in (3), we get $d(Tx_{n+1}, Tx_n, Tx_{n-1}) \leq 0$ and hence

$$d(Tx_{n+1}, Tx_n, Tx_{n-1}) = 0. \qquad (4)$$

It follows from (3) and (4) that

$$\begin{aligned} & d(Tx_{n+1}, Tx_n, a) \\ \leq\ & \frac{1}{2} d(Tx_{n-1}, Tx_{n+1}, a) \\ \leq\ & \frac{1}{2}\left(d(Tx_{n-1}, Tx_n, a) + d(Tx_n, Tx_{n+1}, a) + d(Tx_{n-1}, Tx_{n+1}, Tx_n)\right) \\ (5)\ \leq\ & \frac{1}{2}\left(d(Tx_{n-1}, Tx_n, a) + d(Tx_n, Tx_{n+1}, a)\right). \end{aligned}$$

It implies that

$$d(Tx_{n+1}, Tx_n, a) \leq d(Tx_{n-1}, Tx_n, a). \tag{6}$$

Thus, $\{d(Tx_n, Tx_{n+1}, a)\}$ is a decreasing sequence of non-negative real numbers and hence it is convergent. Let

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}, a) = r. \tag{7}$$

Taking the limit as $n \to \infty$ in (5) and using (7), we get

$$r \leq \frac{1}{2} \lim_{n\to\infty} d(Tx_{n-1}, Tx_{n+1}, a) \leq \frac{1}{2}(r + r) = r.$$

It implies that

$$\lim_{n\to\infty} d(Tx_{n-1}, Tx_{n+1}, a) = 2r. \tag{8}$$

Taking the limit as $n \to \infty$ in (2) and using (7), (8), we get

$$\mu(r) \leq \mu(r) - \psi(0, 2r) \leq \mu(r).$$

It implies that $\psi(0, 2r) = 0$, that is, $r = 0$. Then, (7) becomes

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}, a) = 0. \tag{9}$$

From (6), if we have $d(Tx_{n-1}, Tx_n, a) = 0$, then $d(Tx_n, Tx_{n+1}, a) = 0$. Since $d(Tx_0, Tx_1, Tx_0) = 0$, we have $d(Tx_n, Tx_{n+1}, Tx_0) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $d(Tx_{m-1}, Tx_m, Tx_m) = 0$, we get

$$d(Tx_n, Tx_{n+1}, Tx_m) = 0 \tag{10}$$

for all $n \geq m - 1$. For all $0 \leq n < m - 1$, noting that $m - 1 \geq n - 1$, from (10), we obtain

$$d(Tx_{m-1}, Tx_m, Tx_{n+1}) = d(Tx_{m-1}, Tx_m, Tx_n) = 0.$$

It implies that

$$\begin{aligned} & d(Tx_n, Tx_{n+1}, Tx_m) \\ \leq\ & d(Tx_n, Tx_{n+1}, Tx_{m-1}) + d(Tx_{n+1}, Tx_m, Tx_{m-1}) + d(Tx_m, Tx_n, Tx_{m-1}) \\ =\ & d(Tx_n, Tx_{n+1}, Tx_{m-1}). \end{aligned}$$

It implies that

$$d(Tx_n, Tx_{n+1}, Tx_m) \le d(Tx_n, Tx_{n+1}, Tx_{n+1}) \tag{11}$$

for all $0 \le n < m-1$. Since $d(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0$, from (11), we have

$$d(Tx_n, Tx_{n+1}, Tx_m) = 0 \tag{12}$$

for all $0 \le n < m-1$. From (10) and (12), we have $d(Tx_n, Tx_{n+1}, Tx_m) = 0$ for all $n, m \in \mathbb{N} \cup \{0\}$. Now for all $i, j, k \in \mathbb{N}$ with $i < j$, we have

$$d(Tx_{j-1}, Tx_j, Tx_i) = d(Tx_{j-1}, Tx_j, Tx_k) = 0.$$

Therefore,

$$\begin{aligned} d(Tx_i, Tx_j, Tx_k) &\le d(Tx_i, Tx_j, Tx_{j-1}) + d(Tx_j, Tx_k, Tx_{j-1}) \\ &\qquad + d(Tx_k, Tx_i, Tx_{j-1}) \\ &= d(Tx_i, Tx_{j-1}, Tx_k) \\ &\le ... \\ &= d(Tx_i, Tx_i, Tx_k) \\ &= 0. \end{aligned}$$

This proves that for all $i, j, k \in \mathbb{N} \cup \{0\}$,

$$d(Tx_i, Tx_j, Tx_k) = 0. \tag{13}$$

In what follows, we will prove that $\{Tx_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{Tx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ where $n(k)$ is the smallest integer such that $n(k) > m(k) > k$ and

$$d(Tx_{n(k)}, Tx_{m(k)}, a) \ge \varepsilon \tag{14}$$

for all $k \in \mathbb{N}$. Therefore,

$$d(Tx_{n(k)-1}, Tx_{m(k)}, a) < \varepsilon. \tag{15}$$

By using (13), (14) and (15), we have

$$\begin{aligned} \varepsilon &\le d(Tx_{n(k)}, Tx_{m(k)}, a) \\ &\le d(Tx_{n(k)}, Tx_{n(k)-1}, a) + d(Tx_{n(k)-1}, Tx_{m(k)}, a) \\ &\qquad + d(Tx_{n(k)}, Tx_{m(k)}, Tx_{n(k)-1}) \\ &= d(Tx_{n(k)}, Tx_{n(k)-1}, a) + d(Tx_{n(k)-1}, Tx_{m(k)}, a) \\ &< d(Tx_{n(k)}, Tx_{n(k)-1}, a) + \varepsilon. \end{aligned} \tag{16}$$

Taking the limit as $k \to \infty$ in (16) and using (9), we have

$$\lim_{k\to\infty} d(Tx_{n(k)}, Tx_{m(k)}, a) = \lim_{k\to\infty} d(Tx_{n(k)-1}, Tx_{m(k)}, a) = \varepsilon. \tag{17}$$

Also, from (13), we have

$$\begin{aligned} & d(Tx_{m(k)}, Tx_{n(k)-1}, a) \\ \le\ & d(Tx_{m(k)}, Tx_{m(k)-1}, a) + d(Tx_{m(k)-1}, Tx_{n(k)-1}, a) \end{aligned}$$

$$
\begin{aligned}
&\quad +d(Tx_{m(k)},Tx_{n(k)-1},Tx_{m(k)-1})\\
&= d(Tx_{m(k)},Tx_{m(k)-1},a)+d(Tx_{m(k)-1},Tx_{n(k)-1},a)\\
&\le d(Tx_{m(k)},Tx_{m(k)-1},a)+d(Tx_{m(k)-1},Tx_{n(k)},a)\\
&\quad +d(Tx_{n(k)-1},Tx_{n(k)},a)+d(Tx_{m(k)-1},Tx_{n(k)-1},Tx_{n(k)})\\
&= d(Tx_{m(k)},Tx_{m(k)-1},a)+d(Tx_{m(k)-1},Tx_{n(k)},a)\\
&\qquad +d(Tx_{n(k)-1},Tx_{n(k)},a) \qquad (18)
\end{aligned}
$$

and

$$
\begin{aligned}
&d(Tx_{m(k)-1},Tx_{n(k)},a)\\
&\le d(Tx_{m(k)-1},Tx_{m(k)},a)+d(Tx_{n(k)},Tx_{m(k)},a)\\
&\quad +d(Tx_{m(k)-1},Tx_{n(k)},Tx_{m(k)})\\
&= d(Tx_{m(k)-1},Tx_{m(k)},a)+d(Tx_{n(k)},Tx_{m(k)},a). \qquad (19)
\end{aligned}
$$

Taking the limit as $k\to\infty$ in (18), (19) and using (9), (17), we obtain

$$\lim_{k\to\infty} d(Tx_{m(k)-1},Tx_{n(k)},a)=\varepsilon. \tag{20}$$

Since $n(k)>m(k)$, we have $fx_{n(k)-1}\succeq fx_{m(k)-1}$. Since T is a (μ,ψ)-generalized f-weakly contractive mapping, we have

$$
\begin{aligned}
\mu(\varepsilon) &\le \mu(Tx_{m(k)},Tx_{n(k)},a)\\
&\le \mu\Big(\frac{1}{2}[d(fx_{m(k)},Tx_{n(k)},a)+d(fx_{n(k)},Tx_{m(k)},a)]\Big)\\
&\quad -\psi\big(d(fx_{m(k)},Tx_{n(k)},a),d(fx_{n(k)},Tx_{m(k)},a)\big)\\
&= \mu\Big(\frac{1}{2}[d(Tx_{m(k)-1},Tx_{n(k)},a)+d(Tx_{n(k)-1},Tx_{m(k)},a)]\Big)\\
&\quad -\psi\big(d(Tx_{m(k)-1},Tx_{n(k)},a),d(Tx_{n(k)-1},Tx_{m(k)},a)\big) \qquad (21)
\end{aligned}
$$

Taking the limit as $k\to\infty$ in (21) and using (17), (20) and the property of μ,ψ, we have $\mu(\varepsilon)\le\mu(\varepsilon)-\psi(\varepsilon,\varepsilon)$ and consequently $\psi(\varepsilon,\varepsilon)\le 0$, which is contradiction. Thus, $\{Tx_n\}$ is a Cauchy sequence. Since $fx_n=Tx_{n-1}$, $\{fx_n\}$ is also a Cauchy sequence in fX. Since fX is closed, there exists $z\in X$ such that

$$\lim_{n\to\infty} fx_{n+1}=\lim_{n\to\infty} Tx_n=fz. \tag{22}$$

Since $\{fx_n\}$ is a nondecreasing sequence and $\lim_{n\to\infty} fx_{n+1}=fz$, by the assumption 4, we have $fx_n\preceq fz$ and $fz\preceq f(fz)$ for all $n\ge 0$. On the other hand, we have

$$
\begin{aligned}
\mu\big(d(Tz,fx_{n+1},a)\big) &= \mu\big(d(Tz,Tx_n,a)\big)\\
&\le \mu\Big(\frac{1}{2}[d(fz,Tx_n,a)+d(fx_n,Tz,a)]\Big)\\
&\qquad -\psi\big(d(fz,Tx_n,a),d(fx_n,Tz,a)\big). \qquad (23)
\end{aligned}
$$

Taking the limit as $k \to \infty$ in (23) and using (22) and the property of μ, ψ, we have

$$\begin{aligned}
& \mu\big(d(Tz, fz, a)\big) \\
\le\ & \mu\big(\frac{1}{2}[d(fz, fz, a) + d(fz, Tz, a)]\big) - \psi\big(d(fz, fz, a), d(fz, Tz, a)\big) \\
=\ & \mu\big(\frac{1}{2}d(fz, Tz, a)\big) - \psi\big(0, d(fz, Tz, a)\big) \\
\le\ & \mu\big(\frac{1}{2}d(fz, Tz, a)\big).
\end{aligned}$$

This implies that $d(Tz, fz, a) = 0$ for all $a \in X$. Therefore $Tz = fz$, that is, z is a coincidence point of T and f.

Now, suppose that T and f are weakly compatible. Let $w = fz = Tz$. Then $Tw = T(fz) = f(Tz) = f(w)$. Since $fz \preceq f(fz) = f(w)$ and T is a (μ, ψ)-generalized f-weakly contractive mapping, we have

$$\begin{aligned}
& \mu\big(d(Tz, Tw, a)\big) \\
\le\ & \mu\big(\frac{1}{2}[d(fz, Tw, a) + d(fw, Tz, a)]\big) - \psi\big(d(fz, Tw, a), d(fw, Tz, a)\big) \\
=\ & \mu\big(\frac{1}{2}[d(Tz, Tw, a) + d(Tw, Tz, a)]\big) - \psi\big(d(Tz, Tw, a), d(Tw, Tz, a)\big) \\
=\ & \mu\big(d(Tw, Tz, a)]\big) - \psi\big(d(Tz, Tw, a), d(Tw, Tz, a)\big).
\end{aligned}$$

It implies that $d(Tz, Tw, a) = 0$ for all $a \in X$. Therefore $Tz = Tw = w$, that is, $Tw = fw = w$. It means w is a common fixed point of T and f.

Now, suppose that the set of common fixed points of T and f is well ordered. We claim that common fixed points of T and f is unique. If otherwise, then there exists $u \neq v$ such that $Tu = fu = u$ and $Tv = fv = v$. Then

$$\begin{aligned}
& \mu\big(d(u, v, a)\big) \\
=\ & \mu\big(d(Tu, Tv, a)\big) \\
\le\ & \mu\big(\frac{1}{2}[d(fu, Tv, a) + d(fv, Tu, a)]\big) - \psi\big(d(fu, Tv, a), d(fv, Tu, a)\big) \\
=\ & \mu\big(\frac{1}{2}[d(u, v, a) + d(v, u, a)]\big) - \psi\big(d(u, v, a), d(v, u, a)\big).
\end{aligned}$$

This implies that $d(u, v, a) = 0$ for all $a \in X$. Therefore $u = v$, that is, that common fixed points of T and f is unique. Conversely, if T and f have only one common fixed point then the set of common fixed points of T and f being singleton is well ordered. □

From Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let $(X, d, \preceq)$ be a complete, partially ordered 2-metric space and $T : X \longrightarrow X$ be a mapping such that*

(1) *T is a monotone nondecreasing mapping.*
(2) *There exist $\psi \in \Psi$ and μ which is an altering distance function such that for all $x, y, a \in X$ with $x \succeq y$ or $x \preceq y$,*

$$\begin{aligned} & \mu\big(d(Tx, Ty, a)\big) \\ (24) \quad \leq \; & \mu\Big(\frac{1}{2}[d(x, Ty, a) + d(y, Tx, a)]\Big) - \psi\big(d(x, Ty, a), d(y, Tx, a)\big). \end{aligned}$$

(3) *If $\{x_n\} \subset X$ is a nondecreasing sequence such that $\lim_{n\to\infty} x_n = z \in X$, then $x_n \preceq z$ for every $n \in \mathbb{N} \cup \{0\}$ or T is continuous.*
(4) *There exists an $x_0 \in X$ with $x_0 \preceq Tx_0$.*

Then, T has a fixed point. Moreover, if for arbitrary two points $x, y \in X$, there exists $w \in X$ such that w is comparable with both x and y, then T has a unique fixed point.

Proof. We assume that if $\{x_n\} \subset X$ is a nondecreasing sequence such that $\lim_{n\to\infty} x_n = z \in X$, then $x_n \preceq z$ for every $n \in \mathbb{N} \cup \{0\}$. By using Theorem 2.1 with f is an identity mapping, we conclude that T has a fixed point. Now, we assume that T is continuous. Then, the proceeding as in Theorem 2.1 with f is an identity mapping we see that $\{Tx_n\}$ is a Cauchy sequence. Then, there exists $z \in X$ such that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z$. Since T is continuous, we have $z = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tz$, that is, z is a fixed point of T.

Now, let u and v be two fixed points of T such that $u \neq v$. We consider the following two cases.

Case 1. u and v are comparable. Then, from (24), we have

$$\begin{aligned} \mu\big(d(u, v, a)\big) &= \mu\big(d(Tu, Tv, a)\big) \\ &\leq \mu\Big(\frac{1}{2}[d(u, Tv, a) + d(v, Tu, a)]\Big) - \psi\big(d(u, Tv, a), d(v, Tu, a))\big) \\ &= \mu\Big(\frac{1}{2}[d(u, v, a) + d(v, u, a)]\Big) - \psi\big(d(u, v, a), d(v, u, a))\big) \\ &= \mu\big(d(v, u, a)\big) - \psi\big(d(u, v, a), d(v, u, a))\big). \end{aligned}$$

It implies that $d(u, v, a) = 0$ for all $a \in X$. Therefore $u = v$.

Case 2. u and v are not comparable. Then, there exists $w \in X$ such that w is comparable with both u and v. If u is comparable with w, then $u = T^n u$ is comparable with $T^n w$ for each $n \in \mathbb{N} \cup \{0\}$. From (24), we have

$$\begin{aligned} & \mu\big(d(u, T^n w, a)\big) \\ = \; & \mu\big(d(T^n u, T^n w, a)\big) \\ = \; & \mu\big(d(TT^{n-1}u, TT^{n-1}w, a)\big) \end{aligned}$$

$$
\begin{aligned}
&\le \mu\Big(\frac{1}{2}[d(T^{n-1}u, T^n w, a) + d(T^{n-1}w, T^n u, a)]\Big) \\
&\quad -\psi\big(d(T^{n-1}u, T^n w, a), d(T^{n-1}w, T^n u, a)\big)\Big) \\
&= \mu\Big(\frac{1}{2}[d(u, T^n w, a) + d(T^{n-1}w, u, a)]\Big) \\
&\quad -\psi\big(d(u, T^n w, a), d(T^{n-1}w, u, a)\big)\Big) \\
&\le \mu\Big(\frac{1}{2}[d(u, T^n w, a) + d(T^{n-1}w, u, a)]\Big). \qquad (25)
\end{aligned}
$$

It implies that $d(u, T^n w, a) \le d(u, T^{n-1}w, a)$. This prove that $\{d(u, T^n w, a)\}$ is a decreasing sequence of nonnegative real numbers. Thus, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(u, T^n w, a) = r. \qquad (26)$$

Then, taking the limit as $n \to \infty$ in (25), using (26) and property of μ, ψ, we have $\mu(r) \le \mu(r) - \psi(r, r) \le \mu(r)$. It implies that $\psi(r, r) = 0$, that is, $r = 0$. Consequently, $\lim_{n\to\infty} d(u, T^n w, a) = 0$. It means $\lim_{n\to+\infty} T^n w = u$.

Similarly, if v is comparable with w, then we can prove that $\lim_{n\to\infty} T^n w = v$. Since the limit is unique, we get $u = v$.

From above cases, we conclude that T has a unique fixed point. □

Remark 2.2. By taking $\mu(t) = t$ for all $t \ge 0$ in Corollary 2.1, we get [10, Theorem 2.3], [10, Theorem 2.4] and [10, Theorem 2.5].

From Lemma 2.1 with $\mu(t) = t$ for all $t \ge 0$ and $\psi(x, y) = \Big(\frac{1}{2} - k\Big)(x + y)$ for all $x, y \in [0, +\infty)$ and for some $k \in [0, \frac{1}{2})$, we get the following corollary which is a version of the main result of [5] in the context of partially ordered 2-metric spaces.

Corollary 2.2. *Let $(X, d, \preceq)$ be a complete, partially ordered 2-metric space and $T : X \longrightarrow X$ be a mapping such that*

1. *T is a monotone nondecreasing mapping.*
2. *There exists $k \in [0, \frac{1}{2})$ such that for all $x, y, a \in X$ with $x \succeq y$ or $x \preceq y$,*
$$d(Tx, Ty, a) \le k[d(x, Ty, a) + d(y, Tx, a)].$$
3. *If $\{x_n\} \subset X$ is a nondecreasing sequence such that $\lim_{n\to\infty} x_n = z \in X$, then $x_n \preceq z$ for every $n \in \mathbb{N} \cup \{0\}$ or T is continuous.*
4. *There exists an $x_0 \in X$ with $x_0 \preceq Tx_0$.*

Then, T has a fixed point. Moreover, if for arbitrary two points $x, y \in X$, there exists $w \in X$ such that w is comparable with both x and y, then T has a unique fixed point.

Finally, in order to support the useability of our results, let us introduce some the following examples.

Example 2.1. Let $X = \{0,1,2\}$ with the usual order $\preceq$ on $\mathbb{R}$. Define a 2-metric d on X as follows.

$$d(x,y,z) = \min\{|x-y|, |y-z|, |z-x|\}$$

for all $x,y,z \in X$. Then $(X,d,\preceq)$ is a partially ordered, complete 2-metric space. Let $T, f : X \longrightarrow X$ be defined by

$$T0 = T1 = T2 = 0$$

and

$$f0 = 0, f1 = f2 = 2.$$

Define the function $\mu(t) = t$ for all $t \geq 0$ and $\psi(a,b) = \dfrac{a+b}{3}$ for all $a,b \geq 0$. Then, for all $x,y,a \in X$ with $fx \succeq fy$, we have

$$d(Tx,Ty,a) = d(0,0,a) = 0$$

and

$$\begin{aligned} & \mu\Big(\frac{1}{2}\big[d(fx,Ty,a)+d(fy,Tx,a)\big]\Big) - \psi\big(d(fx,Ty,a), d(fy,Tx,a)\big) \\ = \;& \mu\Big(\frac{1}{2}\big[d(fx,0,a)+d(fy,0,a)\big]\Big) - \psi\big(d(fx,0,a), d(fy,0,a)\big) \\ = \;& \frac{1}{6}\big[d(fx,0,a)+d(fy,0,a)\big] \geq 0. \end{aligned}$$

It implies that the condition (1) is satisfied. This proves that T is a (μ,ψ)-generalized f-weakly contractive mapping. Moreover, other assumptions of Theorem 2.1 also are satisfied. Therefore, Theorem 2.1 is applicable to T, f, (X,d) and μ,ψ.

The following example shows that Theorem 2.1 can be used to prove the existence of a common fixed point when standard arguments in metric spaces in [4] fail, even for trivial maps. The idea of this example appears in [10].

Example 2.2. Let $X = \{0,1,2,\ldots,n,\ldots\}$ with the usual order,

$$d(x,y,z) = \begin{cases} 1 & \text{if} \quad x \neq y \neq z \\ & \quad \text{and there exists } \; n \geq 1 \text{with} \; \{n, n+1\} \subset \{x,y,z\} \\ 0 & \text{if} \quad \text{otherwise,} \end{cases}$$

and $Tx = fx = 0$ for all $x \in X$. Then

(1) (X,d) is a complete, totally ordered 2-metric space.
(2) (X,d) is not completely metrizable.
(3) T is a (μ,ψ)-generalized f-weakly contractive mapping on the 2-metric space X. Moreover, other assumptions of Theorem 2.1 are satisfied.

Proof. (1) and (2). See [10, Example 2.13].
(3). By choosing $\psi(a,b) = \dfrac{a+b}{2}$ for all $a, b \geq 0$ and $\mu(t) = t$ for all $t \geq 0$, we conclude that condition (1) holds. This prove that T is a (μ, ψ)-generalized f-weakly contractive mapping on the 2-metric space (X, d). □

Remark 2.3. In 2010, Tasković [20] formulated some monotone principles of fixed point. Notice that Theorem 2.1 states the existence of common fixed point for two mappings while [20, Theorem 15, Theorem 16, Corollary 36] only state the existence of the fixed point of a mapping. For example, Theorem 2.1 is applicable to T and f in Example 2.1 but [20, Theorem 15, Theorem 16, Corollary 36] can not be applicable to T and f. We also see that Corollary 2.1 and Corollary 2.2 are particular cases of Theorem 2.1. These results state the existence and the uniqueness of the fixed point while [20, Theorem 15, Theorem 16, Corollary 36] only state the existence of the fixed point.

3. Acknowledgements

The authors sincerely thank anonymous referees for their remarkable comments, especially on adding comparing the obtained results with the main results of [20]. Also, the authors sincerely thank The Dong Thap Seminar on Mathematical Analysis and Its Applications for the discussion on this article.

References

[1] A. Aliouche, C. Simpson, *Fixed points and lines in 2-metric spaces*, Adv. Math., Vol. **229** (2012), pp. 668–690.

[2] T. V. An, N. V. Dung, Z. Kadelburg, S. Radenović, *Various generalizations of metric spaces and fixed point theorems*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, Vol. **109** (2015), pp. 175–198.

[3] S. Chandok, *Some common fixed point theorems for generalized nonlinear contractive mappings*, Comput. Math. Appl., **62** (2011), pp. 3692–3699.

[4] S. Chandok, *Some common fixed point results for generalized weak contractive mappings in partially ordered metric spaces*, J. Nonlinear Anal. Optim., Vol. **4**, No. **1** (2003), pp. 45–52.

[5] S. K. Chatterjea, *Fixed Point Theorems*, Rend. Acad. Bulgare. Sci., Vol. **25** (1972), pp. 727–730.

[6] B. S. Choudhury, *Unique fixed point theorem for weakly C-contractive mappings*, Kathmandu Univ. J. Sci. Eng. Tech. Vol., **5**, No. **1** (2009), pp. 6–13.

[7] L. Ćirić, N. Cakić, M. Rajović, J. S. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl., Vol. **2008** (2008), pp. 1–11.

[8] B. Deshpande, S. Chouhan, *Common fixed point theorems for hybrid pairs of mappings with some weaker conditions in 2-metric spaces*, Fasc. Math., Vol. **46** (2011), pp. 37–55.

[9] B. Deshpande, R. Pathak, *Hybrid pairs of mappings with some weaker conditions in consideration of common fixed point on* 2*-metric spaces*, Math. Morav., Vol. **16**, No. **2** (2012), pp. 1–12.

[10] N. V. Dung, V. T. L. Hang, *Fixed point theorems for weak C-contractions in partially ordered* 2*-metric spaces*, Fixed Point Theory Appl., Vol. **2013:161** (2013), pp. 1–14.

[11] N. V. Dung, N. T. Hieu, N. T. T. Ly, V. D. Thinh, *Remarks on the fixed point problem of* 2*-metric spaces*, Fixed Point Theory Appl., Vol. **2013:167** (2013), pp. 1–8.

[12] V. S. Gähler, 2*-metrische Räume und ihre topologische struktur*, Math. Nachr. Vol. **26** (1963/64), pp. 115–118.

[13] K. Iseki, *Fixed point theorems in 2-metric spaces*, Math. Seminar Notes, Kobe Univ. **3** (1975), pp. 133–136.

[14] G. Jungck, B. E. Rhoades, *Fixed points for set valued functions without continuity*, J. Pure Appl. Math., **29**, No. **3** (1998), pp. 227–238.

[15] M. S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc., Vol. **30**, No. **1** (1984), pp. 1–9.

[16] B. K. Lahiri, P. Das, L. K. Dey, *Cantor's theorem in* 2*-metric spaces and its applications to fixed point problems*, Taiwanese J. Math., Vol. **15**, No. **1** (2011), pp. 337–352.

[17] S. N. Lai, A. K. Singh, *An analogue of Banach's contraction principle of* 2*-metric spaces*, Bull. Austral. Math. Soc., Vol. **18** (1978), pp. 137–143.

[18] S. N. Mishra, R. Pant, R. Panicker, *Sequences of (ψ, ϕ)-weakly contractive mappings and stability of fixed points in* 2*-metric spaces*, Math. Morav., Vol. **17**, No. **2** (2003), pp. 1–14.

[19] S. V. R. Naidu, J. R. Prasad, *Fixed point theorems in* 2*-metric spaces*, Indian J. Pure Appl. Math., Vol. **17**, No. **8** (1986), pp. 974–993.

[20] M. R. Tasković, *Transversal theory of fixed point, fixed apices, and forked points*, Math. Morav., Vol. **14**, No. **2** (2000), pp. 19–97.

NGUYEN TRUNG HIEU
FACULTY OF MATHEMATICS AND INFORMATION
TECHNOLOGY TEACHER EDUCATION
DONG THAP UNIVERSITY
CAO LANH CITY
DONG THAP PROVINCE
VIETNAM
E-mail address: `ngtrunghieu@dthu.edu.vn`

HUYNH NGOC CAM
FACULTY OF MATHEMATICS AND INFORMATION
TECHNOLOGY TEACHER EDUCATION
DONG THAP UNIVERSITY
CAO LANH CITY
DONG THAP PROVINCE
VIETNAM
E-mail address: `huynhngoccam@dthu.edu.vn`