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# Some topological properties of the spaces $expX$, $\lambda X$ and $NX$

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Abstract. In this paper we prove that the exponential functor $exp$ and the functor of superextension $\lambda$ preserve some topological properties with respect to the topology of any $T_1$ -space, and the functor of complete linked systems $N$ preserves some topological properties with respect to the topology of any compact space.

## 1. Introduction

In 1981 on the Prague topological symposium V.V. Fedorchuk [1] put forward the following common problems in the theory of covariant functors:

Let $P$ be some geometrical property and $F$- some covariant functor. If $X$ has a property $P$, then $F(X)$ has the same property $P$?

Or on the contrary, i.e. for what functors, if $F(X)$ possesses a property $P$, it follows that $X$ possesses the same property $P$?

In this work we prove that the exponential functor $exp$ and the functor of superextension $\lambda$ preserve the conditions (i) and (ii) with respect to the topology of any $T_1$ -space, and the functor of complete linked systems $N$ preserve the conditions (i) and (ii) with respect to the topology of any compact space, where

(i) $\tau_1 \subseteq \tau_2$;

(ii) $\tau_1$ is a $\pi$-base for $\tau_2$, i.e. for each non-empty element $O \in \tau_2$ there exists an element $V \in \tau_1$ such that $V \subset O$.

Let $X$ be a $T_1$-space. The collection of all nonempty closed subsets of $X$ we denote by $expX$. The family $B$ of all sets of the form

$$O\langle U_1, U_2, \ldots, U_n\rangle = \{F : F \in expX, F \subset \bigcup U_i, F \cap U_i \neq \emptyset, i = 1, 2, \ldots, n\},$$

where $U_1, U_2, \ldots, U_n$ is a sequence of open sets of $X$, generates the topology on the set $expX$.

This topology is called the Vietoris topology. The $expX$ with the Vietoris topology is called the exponential space or the hyperspace of $X$ [2].

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Let $X$ be a $T_1$-space. Denote by $exp_n X$ the set of all closed subsets of $X$ cardinality of which is not greater than the cardinal number $n$, i.e. $exp_n X = \{F \in expX : |F| \leq n\}$.

A system $\xi = \{F_\alpha : \alpha \in A\}$ of closed subsets of a space $X$ is called linked if every two elements of $\xi$ have non-empty intersection. By Zorn lemma any linked system can be filled up to a maximal linked system (MLS), but such completion is not unique.

**Proposition 1.1** ([2]). *A linked system $\xi$ of a space $X$ is MLS iff it has the following density property:*

*if a closed subset $A \subset X$ intersects all elements of $\xi$ then $A \in \xi$.*

The superextension $\lambda X$ of a topological space $X$ is the set $\lambda X$ of all maximal linked systems of the topological space $X$ generated by the Wallman topology, an open base of which consists of sets of the form

$$O(U_1, U_2, \ldots, U_n) = \{\xi \in \lambda X : \forall i = 1, 2, \ldots, n, \exists F_i \in \xi : F_i \subset U_i\},$$

where $U_1, U_2, \ldots, U_n$ are open subsets of $X$.

A topological space $X$ can be naturally embedded in $\lambda X$ identifying each point $x$ of $X$ with the MLS $\xi_x = \{F \in expX : x \in F\}$, where $expX$ is the exponential space of $X$.

A.V. Ivanov [3] defined the space $NX$ of complete linked systems (CLS) of a space $X$ in the following way:

**Definition 1.1** ([3]). *A linked system $\mathcal{M}$ of closed subsets of a compact $X$ is called a complete linked system (CLS) if for any closed set $F$ of $X$, the condition*

*"Every neighborhood $OF$ of the set $F$ contains of a set $\Phi \in \mathcal{M}$" implies $F \in \mathcal{M}$.*

A set $NX$ of all complete linked systems of a compact $X$ is called the space $NX$ of CLS of $X$. This space is equipped with the topology, the open basis of which is formed by sets of the form of

$$\begin{gathered} E = O(U_1, U_2, \ldots, U_n)\langle V_1, V_2, \ldots, V_s\rangle = \big\{\mathcal{M} \in NX : \text{ for any} \\ i = 1, 2, \ldots, n \text{ there exists } F_i \in \mathcal{M} \text{ such that } F_i \subset U_i, \\ \text{and for any } j = 1, 2, \ldots, s, F \cap V_j \neq \emptyset \text{ for any } F \in \mathcal{M}\big\}, \end{gathered}$$

where $U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_s$ are nonempty open in $X$ sets.

A complete linked system was defined by Ivanov [3] for compacta. Functor $N$ is well defined in the category *Comp*. In current paper we define CLS for an arbitrary $T_1$ - space. For $T_1$ - spaces the functor $N$ is not defined. But the space $NX$ is well defined for $T_1$ - space.

**Definition 1.2.** *A linked system $\mathcal{M}$ of closed subsets of a $T_1$ - space $X$ is called a complete linked system (CLS) if for any closed set $F$ of $X$, the condition*

*"Every neighborhood $OF$ of the set $F$ contains of a set $\Phi \in \mathcal{M}$" implies $F \in \mathcal{M}$.*

## 2. Main Results

**Theorem 2.1.** *Suppose $\tau_1$ and $\tau_2$ are two topologies on $X$. If the topologies $\tau_1$ and $\tau_2$ satisfy the following conditions:*

*(i) $\tau_1 \subseteq \tau_2$;*

*(ii) $\tau_1$ is a $\pi$-base for $\tau_2$, i.e. for each non-empty element $O \in \tau_2$ there exists an element $V \in \tau_1$ such that $V \subset O$.*

*Then the topologies $\exp \tau_1$ and $\exp \tau_2$ also satisfies conditions (i) and (ii) on $\exp X$.*

*Proof.* (i) Let $O\langle U_1, U_2, \ldots, U_n\rangle$ be an arbitrary element of $\exp \tau_1$, where $U_1, U_2, \ldots, U_n \in \tau_1$. By the condition $\tau_1 \subseteq \tau_2$. This implies that $U_1, U_2, \ldots, U_n \in \tau_2$. In this case, by the definition of the Vietoris topology on $\exp X$, we have $O\langle U_1, U_2, \ldots, U_n\rangle \in \exp \tau_2$.

(ii) Let $O\langle V_1, V_2, \ldots, V_k\rangle$ be an arbitrary element of $\exp \tau_2$, where $V_1, V_2, \ldots, V_k \in \tau_2$. Since the system $\tau_1$ is $\pi$-base, by condition (ii), we see that there are nonempty elements $U_1, U_2, \ldots, U_k \in \tau_1$ such that $U_1 \subset V_1, U_2 \subset V_2, \ldots, U_k \subset V_k$. Then $O\langle U_1, U_2, \ldots, U_k\rangle \subset O\langle V_1, V_2, \ldots, V_k\rangle$. Indeed, suppose $F \in O\langle U_1, U_2, \ldots, U_k\rangle$ is an arbitrary element. Then $F \subset \bigcup_{i=1}^{k} U_i$ and $F \cap U_i \neq \emptyset$, $i = 1, 2, \ldots, k$. Therefore, $F \subset \bigcup_{i=1}^{k} U_i \subset \bigcup_{i=1}^{k} V_i$ and $F \cap V_i \neq \emptyset$, $i = 1, 2, \ldots, k$. Hence, we have $F \in O\langle V_1, V_2, \ldots, V_k\rangle$. Thus $\exp \tau_1$ is a $\pi$-base for $\exp \tau_2$. We have proved that the topologies $\exp \tau_1$ and $\exp \tau_2$ satisfies conditions (i) and (ii) on $\exp X$. Theorem 2.1 is proved. □

Let $O = O\langle U_1, U_2, ..., U_n\rangle$ be a nonempty open basic element of hyperspace $expX$. For $O = O\langle U_1, U_2, ..., U_n\rangle$ the class $K(O) = \{U_1, U_2, ..., U_n\}$ is called a frame of $O$.

**Theorem 2.2.** *Suppose $\tau_1$ and $\tau_2$ are two topologies on a $T_1$ space $X$. If the topologies $\exp \tau_1$ and $\exp \tau_2$ satisfy the conditions (i) and (ii) in Theorem 2.1, then the topologies $\tau_1$ and $\tau_2$ also satisfy conditions (i) and (ii) on $X$.*

*Proof.* Let $\exp \tau_1 = \{O\langle U_1, U_2, \ldots, U_n\rangle : n \in A\}$ and $\exp \tau_2 = \{O\langle V_1, V_2, \ldots, V_k\rangle : k \in B\}$ be two topologies and satisfy the conditions (i) and (ii), where $A$, $B$ are index sets. Consider the frame $\tau_1 = K(\exp \tau_1) = \{\{U_1, U_2, \ldots, U_n\} : n \in A\}$ for each $O\langle U_1, U_2, \ldots, U_n\rangle \in \exp \tau_1$ and the frame $\tau_2 = K(\exp \tau_2) = \{\{V_1, V_2, \ldots, V_k\} : k \in B\}$ for each $O\langle V_1, V_2, \ldots, V_k\rangle \in$

$\exp\tau_2$. Since the system $\exp\tau_1$ is a $\pi$-base for $\exp\tau_2$, we see that for each element $O\langle V_1, V_2, \ldots, V_k\rangle \in \exp\tau_2$ there exists an element $O\langle U_1, U_2, \ldots, U_n\rangle \in \exp\tau_1$ such that $O\langle U_1, U_2, \ldots, U_n\rangle \subset O\langle V_1, V_2, \ldots, V_k\rangle$. Now, we shall show that for each $V_i$, $i = 1, 2, \ldots, k$ there is $U_s$, $s = 1, 2, \ldots, n$ such that $U_s \subset V_i$. Suppose that for $V_i$ and for each $U_1, U_2, \ldots, U_n$ we have $U_s \not\subset V_1$, $s = 1, 2, \ldots, n$. Choose a point $x_s \in U_s \backslash V_i$, $s = 1, 2, \ldots, n$ for each $s = 1, 2, \ldots, n$. Then $F = \{x_1, x_2, \ldots, x_n\} \in O\langle U_1, U_2, \ldots, U_n\rangle$. But $F \notin O\langle V_1, V_2, \ldots, V_k\rangle$, since $F \cap V_i = \emptyset$. This is in contradiction to $O\langle U_1, U_2, \ldots, U_n\rangle \subset O\langle V_1, V_2, \ldots, V_k\rangle$. So, for each element $V$ from $\tau_2$ there is $U$ from $\tau_1$ such that $U \subset V$. It means that the system $\tau_1$ is a $\pi$-base for the system $\tau_2$. (ii) is proved.

Now we prove condition (i). Let $U_s$ be an arbitrary nonempty element from $\tau_1$. Then there exists an element $O\langle U_1, \ldots, U_s \ldots, U_n\rangle$ from $\exp\tau_1$ such that contains an element $U_s$. From the condition of the theorem, we have $O\langle U_1, \ldots, U_s \ldots, U_n\rangle \in \exp\tau_2$. Then $K(O\langle U_1, \ldots, U_s \ldots, U_n\rangle) = \{U_1, \ldots, U_s \ldots, U_n\} \in \tau_2$. Hence we have $U_s \in \tau_2$. Since the element $U_s \in \tau_1$ is arbitrary, we have $\tau_1 \subset \tau_2$. Condition (i) is satisfied. Theorem 2.2 is proved. □

Joining Theorems 2.1 and 2.2 we obtain following

**Theorem 2.3.** *Suppose $\tau_1$ and $\tau_2$ are two topologies on a set $X$. Topologies $\tau_1$ and $\tau_2$ satisfy the conditions (i) and (ii) in Theorem 2.1, iff the topologies $\exp\tau_1$ and $\exp\tau_2$ also satisfy conditions (i) and (ii) on $\exp X$.*

Let $O = O(U_1, U_2, \ldots, U_n)$ be an element of the base of the superextension $\lambda X$. The frame of $O$ in $X$ is the system $K(O) = \{U_1, U_2, \ldots, U_n\}$.

**Theorem 2.4.** *Let $\tau_1$ and $\tau_2$ be two topologies on $T_1$- spaces $X$ and satisfy the conditions (i), (ii) in Theorem 2.1. Then the topologies $\lambda(\tau_1)$ and $\lambda(\tau_2)$ also satisfies conditions (i) and (ii) on $\lambda X$.*

*Proof.* Suppose $\tau_1 = \{U_\alpha : \alpha \in A\}$ and $\tau_2 = \{V_\beta : \beta \in B\}$ are topologies on $X$ satisfying conditions (i) and (ii). Consider the family $R_1 = \{W_\alpha : \alpha \in A\}$ of all finite unions of elements of $\tau_1$. Let $P_\infty(R_1) = \{M \subset R_1 : |M| < \aleph_0\}$ be the system of all finite subfamilies of the family $R_1$. Put $O(M) = O(W_1, W_2, \ldots, W_n)$, where , $W_i \in R_1$, $i = 1, 2, \ldots, n$. It is clear that the system $\lambda(\tau_1) = \{O(W_1, W_2, \ldots, W_n) : W_i \in \tau_1, i = 1, 2, \ldots, n\}$ is a topology on $\lambda X$. Suppose $\lambda(\tau_2) = \{O(V_1, V_2, \ldots, V_k) : V_j \in \tau_2, j = 1, 2, \ldots, k\}$ is a topology on $\lambda X$, where $\tau_2$ is the topology on $X$.

We shall prove that topologies $\lambda(\tau_1)$ and $\lambda(\tau_2)$ satisfy conditions (i) and (ii).

(i) Suppose $O(W_1, W_2, \ldots, W_n)$ is an arbitrary element of $\lambda(\tau_1)$, where $W_1, W_2, \ldots, W_n \in R_1$ and $W_1, W_2, \ldots, W_n$ are finite unions of elements $\tau_1$. By the condition we have $\tau_1 \subseteq \tau_2$. This implies that $\{W_1, W_2, \ldots, W_n\} \in \tau_2$, hence $O(W_1, W_2, \ldots, W_n) \in \lambda(\tau_2)$.

(ii) We will show that the topology $\lambda(\tau_1)$ is a $\pi$-base for the topology $\lambda(\tau_2)$. Let $O = O(V_1, V_2, \dots, V_k)$ be an arbitrary element of $\lambda(\tau_2)$, where $V_1, V_2, \dots, V_k \in \tau_2$. Consider the pairwise trace $S(O)$ of $O$ in $X$, i.e. the system $\{V_1', V_2', \dots, V_l'\} = S(O)$ of all pairwise intersections of elements of the class $K(O) = \{V_1, V_2, \dots, V_k\}$, where $K(O)$ is the frame of $O$ in $X$. Since sets $V_1', V_2', \dots, V_l'$ are open and $\tau_1$ is a $\pi$-base for $X$, we see that there exists a system $L = \{U_1, U_2, \dots, U_l\}$ of elements of the $\pi$-base such that $U_1 \subset V_1', U_2 \subset V_2', \dots, U_l \subset V_l'$.

Put $W_i = \bigcup\{U_j \in L : U_j \subset V_i\}, i = 1, 2, \dots, k$. Then, obviously, the system $\mu = \{W_1, W_2, \dots, W_k\}$ is linked and is contained to $P_\infty(R_1) \in \tau_1$. Hence $O(\mu) = O(W_1, W_2, \dots, W_k) \neq \emptyset$. We shall prove $O(W_1, W_2, \dots, W_k) \subset O(V_1, V_2, \dots, V_k)$.

Take an arbitrary point $\xi \in O(W_1, W_2, \dots, W_k)$. Then there exist linked closed sets $F_i \in \xi, i = 1, 2, \dots, k$ such that $F_i \subset W_i, i = 1, 2, \dots, k$, therefore $W_i \subset V_i, i = 1, 2, \dots, k$. This implies that $\xi \in O(V_1, V_2, \dots, V_k)$. So, the system $\lambda(\tau_1)$ is a $\pi$-base for $\lambda(\tau_2)$. Theorem 2.4 is proved. □

**Theorem 2.5.** *Let $\tau_1$ and $\tau_2$ are two topologies on $X$. If the topologies $\lambda(\tau_1)$ and $\lambda(\tau_2)$ satisfy the conditions (i) and (ii) in Theorem 2.1, then the topologies $\tau_1$ and $\tau_2$ also satisfy conditions (i) and (ii) on $X$.*

*Proof.* Assume that $\lambda(\tau_1) = \{O(U_1, U_2, \dots, U_n) : n \in A\}$ and $\lambda(\tau_2) = \{O(V_1, V_2, \dots, V_k) : k \in B\}$ are two topology on $\lambda X$ and satisfy the conditions (i) and (ii), where $A, B$ are sets of indexes.

Consider the frame $\tau_1 = K(\lambda(\tau_1)) = \{\{U_1, U_2, \dots, U_n\} : n \in A\}$ for each $O(U_1, U_2, \dots, U_n) \in \lambda(\tau_1)$ and $\tau_2 = K(\lambda(\tau_2)) = \{\{V_1, V_2, \dots, V_k\} : k \in B\}$ for each $O(V_1, V_2, \dots, V_k) \in \lambda(\tau_2)$. Since the system $\lambda(\tau_1)$ is a $\pi$-base for $\lambda(\tau_2)$, we see that for each element $O(V_1, V_2, \dots, V_k) \in \lambda(\tau_2)$ there exists an element $O(U_1, U_2, \dots, U_n) \in \lambda(\tau_1)$ such that $O(U_1, U_2, \dots, U_n) \subset O(V_1, V_2, \dots, V_k)$ .

We now prove that if $O(U_1, U_2, \dots, U_n) \subset O(V_1, V_2, \dots, V_k)$ then for each $V_i, i = 1, 2, \dots, k$ there exists $U_s, s = 1, 2, \dots, n$ such that $U_s \subset V_i$.

Suppose opposite, i.e. there exists $V_s, s = 1, 2, \dots, k$ such that $U_k \not\subset V_s, k = 1, 2, \dots, n$. Then for any $k = 1, 2, \dots, n$ we have $U_k \setminus V_s \neq \emptyset$. Take points $x_i \in U_k \setminus V_s$ for each $i = 1, 2, \dots, n$. Since sets $U_i, i = 1, 2, \dots, n$ are linked, we can take points $x_{ij} \in U_i \cap U_j, i = 1, 2, \dots, n, j = 1, 2, \dots, n, i \neq j$, from each set $U_i \cap U_j$. Consider sets $F_1 = \{x_1, x_{12}, x_{13}, \dots, x_{1n}\}$, $F_2 = \{x_2, x_{21}, x_{23}, \dots, x_{2n}\}$, $\dots$, $F_n = \{x_n, x_{n1}, x_{n2}, \dots, x_{nn-1}\}$ and $F_{n+1} = \{x_1, x_2, x_3, \dots, x_n\}$. It is clear that $\mu = \{F_1, F_2, \dots, F_{n+1}\}$ is linked system of closed sets. Extend $\mu$ to a MLS $\xi$. For each $i = 1, 2, \dots, n$ we have $F_i \subset U_i$ and $F_i \in \xi$. Therefore $\xi \in O(U_1, U_2, \dots, U_n)$. Let's show $\xi \notin O(V_1, V_2, \dots, V_k)$.

Assume to the contrary that $\xi \in O(V_1, V_2, \dots, V_k)$. Then for each $j = 1, 2, \dots, k$ there exist closed sets $M_j \in \xi$ such that $M_j \subset V_j$. The set $F_{n+1} = \{x_1, x_2, x_3, \ \dots, x_n\}$ consists of finite points $x_i \in U_k \setminus V_s, i = 1, 2, \dots, n.$

For any set $M_j \in \xi, j = 1,2,\dots,k$ we have $M_j \cap F_{n+1} = \emptyset$. So, $\xi \notin O(V_1, V_2, \dots, V_k)$.

This contradiction proves that for each $V_i, i = 1,2,\dots,k$ there exists at least one element $U_s, s = 1,2,\dots,n$ such that $U_s \subset V_i$. Therefore, the topology $\tau_1$ is a $\pi$-base of the topology $\tau_2$. (ii) is proved.

Now we prove condition (i). Let $U_s$ be an arbitrary element of the topology $\tau_1$. Then there exists an element $O(U_1, U_2, \dots, U_s, \dots, U_n) \in \lambda(\tau_1)$ from the system $\lambda(\tau_1)$, which contains $U_s$, since the topologies $\lambda(\tau_1)$ and $\lambda(\tau_2)$ satisfy the conditions (i) and (ii) on $\lambda X$. From condition (i) we have $O(U_1, U_2, \dots, U_s, \dots, U_n) \in \lambda(\tau_2)$. Consider the frame $K(O(U_1, U_2, \dots, U_s, \dots, U_n)) = \{U_1, U_2, \dots, U_s, \dots, U_n\} \in \tau_2$. Then we have $U_s \in \tau_2$. The element $U_s \in \tau_1$ being arbitrary, we have $\tau_1 \subseteq \tau_2$. Condition (i) holds. Theorem 2.5 is proved. □

Uniting Theorems 2.4 and 2.5 we obtain the following theorem.

**Theorem 2.6.** *Let $\tau_1$ and $\tau_2$ are two topologies on $T_1$-spaces $X$. Topologies $\tau_1$ and $\tau_2$ satisfy the conditions (i) and (ii) in Theorem 2.1, iff the topologies $\lambda(\tau_1)$ and $\lambda(\tau_2)$ also satisfies conditions (i) and (ii) on $\lambda X$.*

Let $E = O(U_1, U_2, \dots, U_n)\langle V_1, V_2, \dots, V_s\rangle$ be an element of the base of the complete linked system $NX$ of a space $X$. The frame of $E$ in $X$ is the system $K(O) = \{U_1, U_2, \dots, U_n, V_1, V_2, \dots, V_s\}$.

We will call a paired trace of a basic element $E$ the $X$ following system opened in $X$ subsets:

$$S(E) = \{U_i \cap V_j : i = 1,2,\dots,n, \quad j = 1,2,\dots,s\} \bigcup S(O),$$

where $S(O)$ is a paired trace of an element $O(U_1, U_2, \dots, U_n)$ of $X$.

**Proposition 2.1.** [4]. *Let $\mu = \{\Phi_1, \Phi_2, \dots, \Phi_n\}$ be a finite linked system of closed subsets of a space $X$. Then the system $M = \{F \in expX : \exists \Phi_i \in \mu, \Phi_i \subset F\}$ is a complete linked system of $X$.*

**Theorem 2.7.** *Let $\tau_1$ and $\tau_2$ be two topologies on $T_1$ - spaces $X$.If the topologies $\tau_1$ and $\tau_2$ satisfy the conditions (i), (ii) in Theorem 2.1. Then the topologies $N(\tau_1)$ and $N(\tau_2)$ also satisfies conditions (i) and (ii) on $NX$.*

*Proof.* Suppose $\tau_1 = \{U_\alpha : \alpha \in A\}$ and $\tau_2 = \{V_\beta : \beta \in B\}$ are two topology on $X$ such that the topologies satisfies conditions (i) and (ii). Consider the family $R_1 = \{W_\alpha : \alpha \in A\}$ of all finite unions of elements of $\tau_1$. Let $P_\infty(R_1) = \{M \subset R_1 : |M| < \aleph_0\}$ be the system of all finite subfamilies of the family $R_1$. Since $\tau_1$ is a topology on $X$, then $R_1 \subset \tau_1$.

Put $N(\tau_1) = \{O_\alpha(W_1, W_2, \dots, W_b)\langle W_1', W_2', \dots, W_f'\rangle : W_s, W_p' \in \tau_1; s = 1,2,\dots,b; p = 1,2,\dots,f; \alpha \in A\}$ is a topology on $NX$ of the topology $\tau_1$. Let $N(\tau_2) = \{O_\beta(V_1, V_2, \dots, V_k)\langle V_1', V_2', \dots, V_l'\rangle : V_p, V_q' \in \tau_2; p = 1,2,\dots,k; q = 1,2,\dots,l; \beta \in B\}$ is a topology on $NX$ of the topology $\tau_2$.

We shall prove that topologies $N(\tau_1)$ and $N(\tau_2)$ satisfy conditions (i) and (ii).

We will show condition (i). Suppose $O(W_1, W_2, \ldots, W_b)\langle W'_1, W'_2, \ldots, W'_f\rangle$ is an arbitrary element of $N(\tau_1)$, where $W_1, W_2, \ldots, W_b, W'_1, W'_2, \ldots, W'_f$ are nonempty open in $X$ sets, and $W_1, W_2, \ldots, W_b, W'_1, W'_2, \ldots, W'_f \in \tau_1$. By the condition we have $\tau_1 \subseteq \tau_2$. This implies that $W_1, W_2, \ldots, W_b, W'_1, W'_2, \ldots, W'_f \in \tau_2$, hence $O(W_1, W_2, \ldots, W_b)\langle W'_1, W'_2, \ldots, W'_f\rangle \in N(\tau_2)$.

Now we will show condition (ii). We will show that the topology $N(\tau_1) = \{O_\alpha(W_1, W_2, \ldots, W_b)\langle W'_1, W'_2, \ldots, W'_f\rangle : W_s, W'_p \in \tau_1; s = 1, 2, \ldots, b; p = 1, 2, \ldots, f; \alpha \in A\}$ is a $\pi$-base for the topology $N(\tau_2)$. Let $E = O(V_1, V_2, \ldots, V_k)\langle V'_1, V'_2, \ldots, V'_l\rangle$ be an arbitrary base element of $N(\tau_2)$, where $V_1, V_2, \ldots, V_k, V'_1, V'_2, \ldots, V'_l \in \tau_2$. Consider the pairwise trace of $E$ in $X$:

$$S(E) = \{V_i \cap V_j : i = 1, 2, \ldots, k; \quad j = 1, 2, \ldots, l\} \bigcup S(O),$$

where $S(O)$ is the pairwise trace of $O(V_1, V_2, \ldots, V_k)$ in $X$. Since sets $V_i, i = 1, 2, \ldots, k$ are linked we have $V_i \cap V_j \neq \emptyset$ for any $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, k$. Since the topology $\tau_1$ is a $\pi$- base for the topology $\tau_2$, then there exist element $U_{ii'} \in \tau_1$ such that $U_{ii'} \subset V_i \cap V_{i'}, i = 1, 2, \ldots, k, i' = 1, 2, \ldots, k$ and $U_{im} \subset V_i \cap V'_m, i = 1, 2, \ldots, k, m = 1, 2, \ldots, l$.

Put $L = \{U_{ii'}, U_{im} : i, i' = 1, 2, \ldots, k; m = 1, 2, \ldots, l\}$ and

$$(1) \qquad W_i = \bigcup\{U_{ii'} : U_{ii'} \subset V_i \cap V_{i'}\}, i = 1, 2, \ldots, k; \quad i' = 1, 2, \ldots, k, 1$$

$$(2) \qquad W'_m = \bigcup\{U_{im} : U_{im} \subset V_i \cap V'_m\}, i = 1, 2, \ldots, k; \quad m = 1, 2, \ldots, l.2$$

Then, obviously, the system $\mu = \{W_i, W'_m : i = 1, 2, \ldots, k; m = 1, 2, \ldots, l\}$ is linked and is contained in $P_\infty(R_1)$.

We shall prove $O(W_1, W_2, \ldots, W_k)\langle W'_1, W'_2, \ldots, W'_l\rangle \neq \emptyset$.

Indeed, from each set $\{W_i : i = 1, 2, \ldots, k\}$ we can take points $x_{ii'} \in W_i \cap W_{i'}, i, i' = 1, 2, \ldots, k$ and from each set $\{W_i, W'_m : i = 1, 2, \ldots, k; m = 1, 2, \ldots, l\}$ we can take points $x_{im} \in W_i \cap W'_m, i = 1, 2, \ldots, k, m = 1, 2, \ldots, l$. Let $\Phi = \{x_{ii'}, x_{im} : i, i' = 1, 2, \ldots, k; m = 1, 2, \ldots, l\}$. Put $F_i = \{x_{im} \in \Phi : x_{im} \in W_i\}$ and $F_m = \{x_{im} \in \Phi : x_{im} \in W'_m\}$, where $i = 1, 2, \ldots, k, m = 1, 2, \ldots, l$. Then $\mu = \{F_1, F_2, \ldots, F_k, F_{k+1}, \ldots, F_{k+l}\}$ is a linked system of closed subsets in $X$. Consider $M = \{F \in expX : \exists \Phi_i \in \mu : \Phi_i \subset F\}$, in that case, by Proposition 2.1 in [4], $M$ is complete linked system of a space $X$ and $M \in O(W_1, W_2, \ldots, W_k)\langle W'_1, W'_2, \ldots, W'_l\rangle \neq \emptyset$.

We will show $O(W_1, W_2, \ldots, W_k)\langle W'_1, W'_2, \ldots, W'_l\rangle \subset O(V_1, V_2, \ldots, V_k)\langle V'_1, V'_2, \ldots, V'_l\rangle$.

Let $\eta \in O(W_1, W_2, \ldots, W_k)\langle W'_1, W'_2, \ldots, W'_l\rangle$. Then for any $i = 1, 2, \ldots, k$; $\exists F_i \in \eta$ such that $F_i \subset W_i$ and for any $F \in \eta$ we have $F \cap W'_m \neq \emptyset, m = 1, 2, \ldots, l$. By (1) we have $F_i \subset W_i \subset V_i$ and by (2) we have $F \cap V'_m \neq \emptyset, m = 1, 2, \ldots, l$. Hence $\eta \in O(V_1, V_2, \ldots, V_k)\langle V'_1, V'_2, \ldots, V'_l\rangle$. Theorem 2.7 is proved. □

**Theorem 2.8.** *Let $\tau_1$ and $\tau_2$ be two topologies on $T_1$ - spaces $X$. If the topologies $N(\tau_1)$ and $N(\tau_2)$ satisfy the conditions (i) and (ii) in Theorem 2.1, then the topologies $\tau_1$ and $\tau_2$ also satisfies conditions (i) and (ii) in $X$.*

*Proof.* Assume that $N(\tau_1) = \{O_\alpha(U_1, U_2, \ldots, U_n)\langle U'_1, U'_2, \ldots, U'_{n'}\rangle : n, n' \in N; \alpha \in A\}$ and $N(\tau_2) = \{O_\beta(V_1, V_2, \ldots, V_k)\langle V'_1, V'_2, \ldots, V'_{k'}\rangle : k, k' \in N; \beta \in B\}$ are two topology on $NX$ and satisfies conditions (i) and (ii). Consider the frame $N(\tau_1)$ and $N(\tau_2)$ on $X$, i.e. $\tau_1 = \{U_1, U_2, \ldots, U_n, U'_1, U'_2, \ldots, U'_{n'} : n, n' \in N; \alpha \in A\}$, $\tau_2 = \{V_1, V_2, \ldots, V_k, V'_1, V'_2, \ldots, V'_{k'} : k, k' \in N; \beta \in B\}$. We prove condition (ii) i.e. we will show that the topology $\tau_1$ is a $\pi$-base for the topology $\tau_2$. Let $V_i$ be an arbitrary element of $\tau_2$ on $X$. Then there exist open set $O(V_1, V_2, \ldots, V_k)\langle V'_1, V'_2, \ldots, V'_{k'}\rangle$ on $NX$, which contains $V_i$. Since $N(\tau_1)$ is a $\pi$- base for the topology $N(\tau_2)$, then there exists an element $O(U_1, U_2, \ldots, U_n)\langle U'_1, U'_2, \ldots, U'_{n'}\rangle$ such that $O(U_1, U_2, \ldots, U_n)$ $\langle U'_1, U'_2, \ldots, U'_{n'}\rangle \subset O(V_1, V_2, \ldots, V_k)\langle V'_1, V'_2, \ldots, V'_{k'}\rangle$. We shall prove that for each sets $V_i, i = 1, 2, \ldots, k$ and $V'_i, i = 1, 2, \ldots, k'$ there exists $U_s, U_{s'} \in \tau_1$ such that $U_s \subset V_i, U_{s'} \subset V_{i'}, s = 1, 2, \ldots, n, s' = 1, 2, \ldots, n'$.

Suppose opposite, i.e. there exists $V_i \in \tau_2$ such that $U_i \not\subset V_i$, $U_{i'} \not\subset V_i$, $i = 1, 2, \ldots, n$, $i' = 1, 2, \ldots, n'$. Take points $x_i \in U_i \setminus V_i$, $x_{i'} \in U_{i'} \setminus V_i$ for each $i = 1, 2, \ldots, n$, $i' = 1, 2, \ldots, n'$. Since sets $U_s$, $U_{s'}$ are linked, we can take points $x_{ss'} \in U_s \cap U_{s'}$, $s = 1, 2, \ldots, n$, $s' = 1, 2, \ldots, n'$, $s \neq s'$ and $y_{sl} \in U_s \cap U_l$, $s = 1, 2, \ldots, n$, $l = 1, 2, \ldots, n'$. Put $F_1 = \{x_1, x_{12}, \ldots, x_{1n}, y_{11}, y_{12}, \ldots, y_{1n'}\}$, $F_2 = \{x_2, x_{21}, x_{23}, \ldots, x_{2n}, y_{21}, y_{22}, \ldots, y_{2n'}\}$, $\ldots$, $F_n = \{x_n, x_{n1}, x_{n2}, \ldots, x_{nn}, y_{n1}, y_{n2}, \ldots, y_{nn'}\}$, $\ldots$, $F_{n+n'} = \{x_1, x_2, x_3, \ldots, x_n, y_1, y_2, \ldots, y_{n'}\}$.

It is clear that $\mu = \{F_1, F_2, \ldots, F_n, F_{n+1}, F_{n+2}, \ldots, F_{n+n'}\}$ is a linked system of closed sets. Fill $\mu$ to a CLS $\xi$. For each $s = 1, 2, \ldots, n$ we have $F_s \subset U_s$ and for each $s' = 1, 2, \ldots, n'$ we have $F_s \cap U'_s \neq \emptyset$, where $F_s \in \xi$. Therefore $\xi \in O(U_1, U_2, \ldots, U_n)\langle U'_1, U'_2, \ldots, U'_{n'}\rangle$. Let's show $\xi \notin O(V_1, V_2, \ldots, V_k)$ $\langle V'_1, V'_2, \ldots, V'_{k'}\rangle$.

Assume $\xi \in O(V_1, V_2, \ldots, V_k)\langle V'_1, V'_2, \ldots, V'_{k'}\rangle$. Then for each $i = 1, 2, \ldots, k$ there exist closed sets $M_i \in \xi$, $i = 1, 2, \ldots, k$ such that $M_i \subset V_i$ and $M_i \cap V'_s \neq \emptyset$, $s = 1, 2, \ldots, k'$, $i = 1, 2, \ldots, k$.

The set $F_{n+n'} = \{x_1, x_2, x_3, \ldots, x_n, y_1, y_2, \ldots, y_{n'}\}$ consists of finite points $x_s, y_{s'} \in U_s \setminus V_i$, $s = 1, 2, \ldots, n$, $s' = 1, 2, \ldots, n'$ . For any set $M_i \in \xi$, $i = 1, 2, \ldots, k$ we have $M_i \cap F_{n+n'} = \emptyset$. So, $\xi \notin O(V_1, V_2, \ldots, V_k)\langle V'_1, V'_2, \ldots, V'_{k'}\rangle$.

This contradiction proves that for each $V_i$, $i = 1, 2, \ldots, k$ there exists at least one element $U_s$, $s = 1, 2, \ldots, n$ such that $U_s \subset V_i$. Therefore, the topology $\tau_1$ is a $\pi$-base of the topology $\tau_2$. (ii) is proved.

Now we prove condition (i). Let $U_s$ be an arbitrary element of the topology $\tau_1$. Then there exists an element $O(U_1, U_2, \ldots, U_s, \ldots, U_n)$ $\langle U'_1, U'_2, \ldots, U'_s, \ldots, U'_k\rangle \in N(\tau_1)$ from the system $N(\tau_1)$, which contains $U_s$. Since the topology $N(\tau_2)$ is an admissible extension of the topology $N(\tau_1)$ on $NX$, from condition (i) we have $O(U_1, U_2, \dots, U_s, \dots, U_n)\ \langle U_1', U_2', \dots, U_s', \dots, U_k'\rangle \in N(\tau_2)$.

Consider the frame $K(O(U_1, U_2, \dots, U_s, \dots, U_n)\ \langle U_1', U_2', \dots, U_s', \dots, U_k'\rangle) = \{U_1, U_2, \dots, U_s, \dots, U_n, U_1', U_2', \dots, U_s', \dots, U_k'\} \in \tau_2$. Then we have $U_s \in \tau_2$. The element $U_s \in \tau_1$ being arbitrary, we have $\tau_1 \subseteq \tau_2$. Condition (i) holds. Theorem 2.8 is proved. □

Uniting Theorems 2.7 and 2.8 we obtain the following theorem

**Theorem 2.9.** *Let $\tau_1$ and $\tau_2$ be two topologies on $T_1$ - spaces $X$. Topologies $\tau_1$ and $\tau_2$ satisfy the conditions (i) and (ii) in Theorem 2.1, iff the topologies $N(\tau_1)$ and $N(\tau_2)$ also satisfies conditions (i) and (ii) on $NX$.*

T. Radul [5] proved that the space of closed sets $expX$ and superextension $\lambda X$ are subsets of the space $O(X)$ of weakly additive functionals. In the work [5] he proved that the functor of probability measures $P$ is a functor subfunctor $O$.

**Question 2.1.** *Suppose a topological space $X$ satisfies conditions (i) and (ii) in Theorem 2.1. Do spaces $P(X)$ and $O(X)$ satisfy conditions (i) and (ii) too?*

Or more common

**Question 2.2.** *Suppose a topological space $X$ satisfies conditions (i) and (ii) in Theorem 2.1. Then for what covariant functors $F$ the space $F(X)$ satisfies conditions (i) and (ii) or inversely?*

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