Mathematica Moravica
Vol. **20:2** (2016), 87–97

$\mathcal{I}$ - Fréchet-Urysohn spaces*

V. Renukadevi and B. Prakash

Abstract. In this paper, we introduce the concept ss-sequentially quotient mapping. Using this concept, we characterize s-Fréchet-Urysohn spaces and s-sequential spaces.

Finally, we develop the properties of $\mathcal{I}$-Fréchet-Urysohn spaces which is the generalized form of s-Fréchet-Urysohn spaces. Also, we give an example that product of two $\mathcal{I}$-Fréchet-Urysohn spaces need not be an $\mathcal{I}$-Fréchet-Urysohn space for any $\mathcal{I}$.

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [7] and Schoenberg [29]. If $K \subset \mathbb{N}$, then K_n will denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ stands for the cardinality of K_n. The natural density of K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

if the limit exists [12, 23]. A sequence $\{x_n\}$ in a topological space X is said to converge statistically [20](or shortly s-converge) to $x \in X$, if for every neighborhood U of x, $d(\{n \in \mathbb{N} : x_n \in U\}) = 1$. Any convergent sequence is statistically convergent but the converse is not true [27]. But in general, s-convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces. It has been discussed and developed by many authors [3, 5, 6, 9, 10, 11, 21, 22, 25, 26].

The concept of $\mathcal{I}$-convergence of real sequences [13, 14] is a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of natural numbers. In the recent literature, several works on $\mathcal{I}$-convergence including remarkable contributions by Šalát et al have occured [2, 4, 13, 14, 16, 19, 28]. The idea of $\mathcal{I}$-convergence has been extended from real number space to topological space [17] and to a normed

2000 *Mathematics Subject Classification.* Primary 54C10, 54D30; Secondary 54D55, 97I30, 54E40, 54A20.

Key words and phrases. sequentially quotient, sequential space, Fréchet-Urysohn space, s-convergence, $\mathcal{I}$-convergence.

*The research of the second author is supported by the Council of Scientific & Industrial Research Fellowship in Sciences (CSIR, New Delhi) for Meritorious Students, India.

linear space [28]. $\mathcal{I}$-convergence coincides with the ordinary convergence if $\mathcal{I}$ is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if $\mathcal{I}$ is the ideal of subsets of $\mathbb{N}$ of natural density zero.

We recall the following definition ([15], p.34).

If X is a nonvoid set, then a family of sets $\mathcal{I} \subset 2^X$ is an *ideal* if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$. The ideal is called *nontrivial* if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ is called *admissible* if it contains all the singleton sets. Several examples of nontrivial admissible ideals may be seen in [13]. $x_n \to x$ denotes a sequence $\{x_n\}$ converging to x. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n / n > k\} \cup \{x\} \subset P$ for some $k \in \mathbb{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a family of subsets of X. Then $\cup\mathcal{P}$ and $\cap\mathcal{P}$ denote the union $\cup\{P/\, P \in \mathcal{P}\}$ and the intersection $\cap\{P/\, P \in \mathcal{P}\}$, respectively.

Throughout this paper, (X, τ) will stand for a topological space and $\mathcal{I}$ for a nontrivial admissible ideal of $\mathbb{N}$, the set of all positive integers and all functions $f : X \to Y$ are continuous and onto.

Definition 1.1. *Let $\mathcal{P} = \cup\{\mathcal{P}_x \mid x \in X\}$ be a cover of a space X. Assume that $\mathcal{P}$ satisfies the following conditions* (a) *and* (b) *for each $x \in X$.*

- *(a) $\mathcal{P}_x$ is a* network *at x in X, i.e., $x \in \cap\mathcal{P}_x$ and for each neighborhood U of x in $X, P \subset U$ for some $P \in \mathcal{P}_x$.*
- *(b) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.*

$\mathcal{P}$ is called a weak base [1] *of X if whenever $G \subset X$, G is open in X if and only if for each $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$. The space X is weakly first-countable* [1] *if X has a weak base $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable for each $x \in X$.*

Definition 1.2. *(a) f is called pseudo-open* [1] *if for each $y \in Y$ and each neighborhood U of $f^{-1}(y)$ in X, $y \in int(f(U))$.*

(b) Let $f : X \to Y$ be a mapping. f is sequentially quotient [18] *if for every convergent sequence S in Y, there is a convergent sequence L in X such that $f(L)$ is an infinite subsequence of S. Equivalently, if whenever $\{y_n\}$ is a convergent sequence in Y, there is a convergent sequence $\{x_k\}$ in X with each $x_k \in f^{-1}(y_{n_k})$* [30].

Definition 1.3. *Let X be a space. $P \subset X$ is called a sequential neighborhood of x in X, if each sequence convergence to $x \in X$ is eventually in P. A subset U of X is called sequentially open if U is a sequential neighborhood of each of its points. X is called a sequential space* [8] *if each sequentially open subset of X is open. X is called a Fréchet-Urysohn space* [8] *if for each $x \in cl(A) \subset X$, there exists a sequence $\{x_n\}$ such that $\{x_n\}$ converges to x and $\{x_n / n \in \mathbb{N}\} \subset A$.*

Definition 1.4. [17] *A sequence* $\{x_n\}$ *in* X *is said to be* $\mathcal{I}$*-convergent to* $x_0 \in X$ *if for any nonvoid open set* U *containing* x_0, $\{n \in \mathbb{N}/x_n \notin U\} \in \mathcal{I}$. *We call* x_0 *as the* $\mathcal{I}$*-limit of the sequence* $\{x_n\}$.

Definition 1.5. [24] O *is* $\mathcal{I}$*-sequentially open if and only if no sequence in* $X \setminus O$ *has an* $\mathcal{I}$*-limit in* O.

Definition 1.6. [24] *A subset* A *of a space* X *is said to be an* $\mathcal{I}$*-sequentially closed set if for every sequence* $\{x_n\}$ *in* A *with* $\{x_n\}$ $\mathcal{I}$*-converges to* x*, then* $x \in A$.

Definition 1.7. [24] *A topological space is* $\mathcal{I}$*-sequential when any set* O *is open if and only if it is* $\mathcal{I}$*-sequentially open.*

Even though we mainly deal with $\mathcal{I}$-sequential and $\mathcal{I}$-Fréchet-Urysohn spaces, we see the basic definitions for s-sequential and s-Fréchet-Urysohn spaces which will be useful for the theorems which deal s-sequential and s-Fréchet-Urysohn spaces. An $\mathcal{I}$-sequential space X is statistically sequential if $\mathcal{I} = \{A \subset X/d(A) = 0\}$.

Definition 1.8. *A subset* K *of the set* $\mathbb{N}$ *is called statistically dense* [20] *if* $d(K) = 1$.

Definition 1.9. *A space* X *is called statistically sequential(or shortly, s-sequential) space* [20] *if for each non-closed subset* $A \subset X$*, there is a point* $x \in X \setminus A$ *and a sequence* $\{x_n\}$ *in* A *statistically converging to* x.

There is another way to define s-sequential space.

Definition 1.10. *A subset* A *of a space* X *is said to be a statistically sequentially open set (s-sequentially open)* [31] *if for any sequence* $\{x_n\}$ *statistically converge to* x *and* $x \in A$*, then* $|\{n/x_n \in A\}| = \omega$.
A topological space is s-sequential when any set O *is open if and only if it is s-sequentially open.*
A topological space X *is statistically Fréchet-Urysohn* [20]*(or shortly, s-Fréchet-Urysohn), if for each* $A \subset X$ *and each* $x \in cl(A)$*, there is a sequence in* A *statistically converging to* x.

Definition 1.11. *A subsequence* S *of the sequence* L *is called statistically dense in* L [11] *if the set of all indices of elements from* S *is statistically dense.*

Definition 1.12. *A subsequence* $\{x_{n_k}\}$ *of the sequence* $\{x_n\}$ *is called a thin subsequence of* $\{x_n\}$ [31] *if* $d(K) = 0$ *where* $K = \{n_k/k \in \mathbb{N}\}$.

Remark 1.13. [17, 20]

(a) *The limit of an* $\mathcal{I}$*-convergent sequence is uniquely determined in Hausdorff spaces.*
(b) *If a sequence* $\{x_n\}$ *converges to* x *in the usual sense, then it statistically converges to* x*. But the converse is not true in general.*

(c) *A sequence $\{x_n\}$ is statistically convergent if and only if each of its statistically dense subsequence is statistically convergent.*
(d) *If a sequence $\{x_n\}$ $\mathcal{I}$-converges to x, then every subsequence $\{x_{n_k}\}_{n_k \in \mathbb{N} \setminus I}$ is $\mathcal{I}$-convergent for every $I \in \mathcal{I}$.*

Lemma 1.14. [24] *Let X be a topological space and $A \subset X$. Then the following hold.*
(a) *A is $\mathcal{I}$-sequentially open.*
(b) *$X \setminus A$ is $\mathcal{I}$-sequentially closed.*

2. $\mathcal{I}$-Fréchet-Urysohn Space

In this section, we introduce $\mathcal{I}$-Fréchet-Urysohn spaces and study their properties. A space X is called an $\mathcal{I}$-Fréchet-Urysohn space if for each $A \subset X$ and each $x \in cl(A)$, there is a sequence in A $\mathcal{I}$-converges to x.

It is easy to see that, if $\mathcal{I}$ is an admissible ideal, then the following implications hold.

$$\begin{array}{ccc} \text{Fréchet-Urysohn space} & \rightarrow & \mathcal{I}\text{-Fréchet-Urysohn space} \\ \downarrow & & \downarrow \\ \text{Sequential space} & \rightarrow & \mathcal{I}\text{-sequential space} \end{array}$$

If $\mathcal{I} = \{A \subset \mathbb{N}/d(A) = 0\}$, then $\mathcal{I}$-Fréchet-Urysohn space becomes s-Fréchet-Urysohn space.

Proposition 2.1. *Subspace of an $\mathcal{I}$-Fréchet-Urysohn space is an $\mathcal{I}$-Fréchet-Urysohn space.*

Proof. Let Y be a nonempty subspace of X and $x \in cl_Y(A)$ where $A \subset Y$. Then $cl_Y(A) = Y \cap cl_X(A)$ which implies $x \in cl_X(A)$. Since X is an $\mathcal{I}$-Fréchet-Urysohn space, there exists a sequence in A $\mathcal{I}$-converges to x. Therefore, Y is an $\mathcal{I}$-Frechet-Urysohn space. □

Proposition 2.2. *The disjoint topological sum of any family of $\mathcal{I}$-Fréchet-Urysohn spaces is an $\mathcal{I}$-Fréchet-Urysohn space.*

Proposition 2.3. *If $f : X \to Y$ is a quotient map, when X is an $\mathcal{I}$-Fréchet-Urysohn space, then Y is an $\mathcal{I}$-Fréchet-Urysohn space $\Longleftrightarrow$ f is pseudo open.*

Proof. Suppose that Y is an $\mathcal{I}$-Fréchet-Urysohn space. Let $y \in Y$ and U be an open neighborhood of $f^{-1}(y)$. If $y \notin int f(U)$, then $y \in cl(Y \setminus f(U))$. Since Y is an $\mathcal{I}$-Fréchet-Urysohn space, there is a sequence $\{y_n\} \subset Y \setminus f(U)$ $\mathcal{I}$-converges to y. Since f is quotient, $cl(f^{-1}(\{y_n\})) \subset f^{-1}(cl\{y_n\}) = f^{-1}(\{y_n\}) \cup f^{-1}(y)$. Since U is an open neighborhood of $f^{-1}(y)$ and $U \cap f^{-1}(\{y_n\}) = \emptyset$, $f^{-1}(y) \cap cl(f^{-1}(\{y_n\})) = \emptyset$ and thus, $f^{-1}(\{y_n\})$ is closed. This implies $X \setminus f^{-1}(\{y_n\}) = f^{-1}(Y \setminus \{y_n\})$ is open. Since f is quotient, $Y \setminus \{y_n\}$ is open which is a contradiction to $\{y_n\}$ $\mathcal{I}$-converges to y. Therefore, $y \in int f(U)$ and hence f is pseudo open.

Conversely, let $y \in cl(A)$ with $A \subset Y$. Suppose $f^{-1}(y) \cap cl(f^{-1}(A)) = \emptyset$. Let $U = X \setminus cl(f^{-1}(A))$. Then $f^{-1}(y) \subset U$ and f is pseudo open implies that

$$\begin{aligned} y \in int(f(U)) &\subset int(f(int(X \setminus f^{-1}(A)))) \\ &\subset int(intf(X \setminus f^{-1}(A))) \\ &= intf(X \setminus f^{-1}(A)) \\ &= int(Y \setminus A) \\ &= Y \setminus cl(A) \end{aligned}$$

Therefore, $y \in Y \setminus cl(A)$ which is a contradiction. There exists $x \in f^{-1}(y) \cap cl(f^{-1}(A))$. Since X is $\mathcal{I}$-Fréchet-Urysohn, there exists a sequence $\{x_n\} \subset f^{-1}(A)$ such that $\{x_n\}$ $\mathcal{I}$-converges to x so that $\{f(x_n)\} \subset A$ and $\{f(x_n)\}$ $\mathcal{I}$-converges to y. Therefore, Y is an $\mathcal{I}$-Fréchet-Urysohn space. □

Since Cartesian product of two Fréchet-Urysohn spaces is not a Fréchet-Urysohn space, naturally, one can arise a question that "Is Cartesian product of two $\mathcal{I}$-Fréchet-Urysohn spaces is $\mathcal{I}$-Fréchet-Urysohn space?" The answer is not for all $\mathcal{I}$ as shown by the following Example 2.4.

Example 2.4. *Let $S_m = \{x_{m,n}/n \in \mathbb{N}\} \bigcup \{x_m\}$ be a space with a topology defined as follows:*
Each $\{x_{m,n}\}$is open and U is a neighborhood of x_m, then $\{n/x_{m,n} \notin U\} \in \mathcal{I}$. Clearly, each S_m is an $\mathcal{I}$-Fréchet-Urysohn space and X' be the disjoint topological sum of S_m for $m \in \mathbb{N}$. By Proposition 2.2, X' is an $\mathcal{I}$- Fréchet-Urysohn space. Now form X from X' by identifying all x_m to x_1. Then the natural map $f : X' \to X$ is a psuedo open map, since for a neighborhood U of $f^{-1}(x)$, $f(U)$ is a neighborhood of x. By Proposition 2.3, X is an $\mathcal{I}$-Fréchet-Urysohn space.
Let $Y = \{x_n/n \in \mathbb{N}\} \bigcup \{x\}$ be a space with a topology as defined for S_m and hence Y is an $\mathcal{I}$-Fréchet-Urysohn space.
But $X \times Y$ is not an $\mathcal{I}$-Fréchet-Urysohn space.
For $A = \bigcup_{m \in \mathbb{N}} (S_m \times \{x_m\})$, $z = (x_1, x) \in cl(A)$.

Suppose there exists a sequence $\{(x'_n, x_n)\}_{n \in \mathbb{N}}$ $\mathcal{I}$-converges to (x_1, x). Then $\{\pi_1(x'_n, x_n)\}_{n \in \mathbb{N}}$ $\mathcal{I}$-converges to x_1 and $\{\pi_2(x'_n, x_n)\}_{n \in \mathbb{N}}$ $\mathcal{I}$-converges to x, by Proposition 2.1 in [24]. *$\{\pi_1(x'_n, x_n)\}_{n \in \mathbb{N}} = \{x'_n\}_{n \in \mathbb{N}}$ $\mathcal{I}$-converges to x_1 implies for some m, $x'_n \in S_m$ for $n \in N' \notin \mathcal{I}$. This implies that $\{\pi_2(x'_n, x_n)\}_{n \in N'} = \{x_m\}_{n \in N'}$ is a constant sequence $\mathcal{I}$-converges to x_m. Since Y is Hausdorff and the subsequence $\{x_{n_k}\}_{n_k \in N''}$ of an $\mathcal{I}$-convergent sequence $\{x_n\}_{n \in \mathbb{N}}$ $\mathcal{I}$-converges to x if $N'' \notin \mathcal{I}$, $\{x_n\}_{n \in N'}$ $\mathcal{I}$-converges to x that is, $\{x_n\}_{n \in N'}$ $\mathcal{I}$-converges to two different limits which is a contradiction.*
Therefore, there is no sequence in A $\mathcal{I}$-converges to x.
Hence $X \times Y$ is not an $\mathcal{I}$-Fréchet-Urysohn space.

Theorem 2.5. *Every $\mathcal{I}$-Fréchet-Urysohn space is an $\mathcal{I}$-sequential space.*

Proof. Let U be an $\mathcal{I}$-sequential open set. Let $x \in cl(X \setminus U)$. Then there exists a sequence $\{x_n\}$ in $X \setminus U$ $\mathcal{I}$-converges to x. Now $X \setminus U$ is $\mathcal{I}$-sequentially closed implies $x \in X \setminus U$. Therefore, $X \setminus U$ is closed and hence U is open. Therefore, X is an $\mathcal{I}$-sequential space. □

Converse of the above Theorem 2.5 need not be true as shown by Example 3.1 [31].

Proposition 2.6. *If every subspace of a space X is $\mathcal{I}$-sequential, then X is an $\mathcal{I}$-Fréchet-Urysohn space.*

Proof. Let $x \in cl(A)$. If $x \in A$, then the proof is obvious. If $x \notin A$, then A is not closed in X. Now let $Y = A \cup \{x\}$, then A is not closed in Y. But by our assumption, Y is an $\mathcal{I}$-sequential space. Therefore, there exists a sequence $\{x_n\} \subset A$ such that $\{x_n\}$ $\mathcal{I}$-converges to x. □

Theorem 2.7. *Let X be an $\mathcal{I}$-Fréchet-Urysohn space. If W is a weak neighborhood of $x \in X$, then $x \in int(W)$.*

Proof. Suppose $x \notin int(W)$. Then $x \in cl(X \setminus W)$. Since X is an $\mathcal{I}$- Fréchet-Urysohn space, there exists a sequence $\{x_n\}$ in $X \setminus W$ $\mathcal{I}$-converges to $x \in W$. This implies that W is not an $\mathcal{I}$-sequential neighborhood of x in X which is a contradiction. Therefore, $x \in int(W)$. □

Corollary 2.8. *Let X be an $\mathcal{I}$-Fréchet-Urysohn space. If X is weakly first countable, then X is first countable.*

Lemma 2.9. *Every $\mathcal{I}$-Fréchet-Urysohn space is $\mathcal{J}$-Fréchet-Urysohn space $\Longleftrightarrow \mathcal{I} \subset \mathcal{J}$.*

Proof. Suppose $\mathcal{I} \not\subseteq \mathcal{J}$ that is, there exists $I \in \mathcal{I}$ and $I \notin \mathcal{J}$. Now form a space $X = \{x_n\}_{n\in\mathbb{N}} \bigcup \{x\}$ and its topology is defined as follows :
Each $\{x_n\}$ is open and each neighborhood U of x is such that $\{n/x_n \notin U\} \in \mathcal{I}$.
Then clearly X is an $\mathcal{I}$-Fréchet-Urysohn space.
Now let $A = \{x_n/n \notin I\}$.
Then $x \in cl(A)$ and there is no sequence in A which is $\mathcal{J}$-convergent to x.
Suppose $\{x_n\}_{n\in\mathbb{N}} \subset A$ $\mathcal{J}$-converges to x.
Form $U = \{x_n/n \notin I\} \bigcup \{x\}$ which is an open neighborhood of x.
$\{n/x_n \notin U\} = I \notin \mathcal{J}$ which is a contradiction. Therefore, X is not a $\mathcal{J}$-Fréchet-Urysohn space.
Conversely, suppose $\mathcal{I} \subset \mathcal{J}$
Let X be an $\mathcal{I}$-Fréchet-Urysohn Space and $x \in cl(A)$.
Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $\{x_n\}$ $\mathcal{I}$-converges to x, that is, $\{n/x_n \notin U\} \in \mathcal{I}$ for all neighborhood U of x.
Since $\mathcal{I} \subset \mathcal{J}$, $\{n/x_n \notin U\} \in \mathcal{J}$ for all neighborhood U of x.
Then the sequence $\{x_n\}$ in A $\mathcal{J}$-converges to x.
Therefore, X is a $\mathcal{J}$-Fréchet-Urysohn space. □

3. ss-Sequentially Quotient Maps

In this section, we introduce a map namely, ss-sequentially quotient map and using this we characterize s-sequential spaces and s-Fréchet-Urysohn spaces. Also, we give their properties. A mapping $f : X \to Y$ is said to be an *ss-sequentially quotient* map if for given $\{y_n\}$ s-converges to y in Y, there exist $\{x_n\}$ s-converges to x, $x \in f^{-1}(y)$ and $x_n \in f^{-1}(y_n)$. In Proposition 3.1, s-σX denote the set X topologized by the statistical sequential closure of the relative topology from X that is, all statistically sequentially open sets are open. Therefore, X and s-σX have same s-convergent sequences and hence it is easy to prove Proposition 3.1.

Proposition 3.1. *Let $f : X \to Y$ be a mapping and $g = f|_{s-\sigma X}$: s-$\sigma X \to$ s-σY. Then f is an ss-sequentially quotient if and only if g is ss-sequentially quotient.*

By Proposition 2.1 in [24] and the definition of ss-sequentially quotient mapping, the proof of the following Proposition 3.2 is clear.

Proposition 3.2. *Let $f : X \to Y$ and $g : Y \to Z$ be any two mappings. Then the following hold.*
(a) *If f and g are ss-sequentially quotient, then $g \circ f$ is ss-sequentially quotient.*
(b) *If $g \circ f$ is ss-sequentially quotient, then g is ss-sequentially quotient.*

Proposition 3.3. *For any topological space, the following hold.*
(a) *Finite product of ss-sequentially quotient mappings is ss-sequentially quotient.*
(b) *ss-sequentially quotient mappings are hereditarily ss-sequentially quotient mappings.*

Proof. (a) Let $\prod_{i=1}^{N} f_i : \prod_{i=1}^{N} X_i \to \prod_{i=1}^{N} Y_i$ be a map where each $f_i : X_i \to Y_i$ is ss- sequentially quotient map for $i = 1, 2, 3, ...N$. Let $\{(y_{i,n})\}_{n\in N}$ be s-converges to (y_i) in $\prod_{i=1}^{N} Y_i$. By Proposition 2.1 in [24], each $\{y_{i,n}\}$ is a sequence s-converges to y_i in Y_i. Since each f_i is an ss-sequentially quotient map, there exists a sequence $\{x_{i,n}\}$ s-converges to x_i such that $f_i(x_{i,n}) = y_{i,n}$.
Take $(x_i) \in \prod_{i=1}^{N} X_i$. Then $\{(x_{i,n})\}$ s-converges to (x_i), since for neighborhood U of (x_i), there exists a thin subsequence N_i of $\mathbb{N}$ for each $i = 1, 2, 3, ...N$ such that $\{n \in \mathbb{N}/(x_{i,n}) \notin U\} \in \bigcup N_i$ which is a thin subsequence of $\mathbb{N}$ as set of all thin subsequence form an ideal. Therefore, $\prod_{i=1}^{N} f_i$ is an ss-sequentially quotient map.
(b) Let $f : X \to Y$ be an ss-sequentially quotient map and H be a subspace of Y. Take $g = f|_{f^{-1}(H)}$ such that $g : f^{-1}(H) \to H$ be a map. Given a sequence $\{y_n\}$ s-convergence to y in H, there exists a sequence $x_n \in f^{-1}(y_n) \in f^{-1}(H)$ such that (x_n) s-converges to $x \in f^{-1}(y) \in f^{-1}(H)$, since f is ss-sequentially quotient map and $\{y_n\}$

s-converges to y in Y. Therefore, g is an ss-sequentially quotient map. □

The following examples shows that sequentially quotient and ss-sequentially quotient mappings are independent.

Example 3.4. *Let $X = S_1 \bigoplus S_2$ and $Y = S_1$ be a topological space as defined in Example 2.4. Let $f : X \to Y$ be a mapping defined by*

$$f(x_{i,n}) = \begin{cases} x_{1,2n}, & \text{if } i = 1 \\ x_{1,2n-1}, & \text{if } i = 2 \end{cases}$$

and $f(x_1) = f(x_2) = x_1$.
Then clearly, f is sequentially quotient but not ss-sequentially quotient since for an s-convergent sequence $\{x_{1,n}\}$ in Y, there is no s-convergent sequence $\{x_n\}$ in X such that $x_n \in f^{-1}(x_{1,n})$.

Example 3.5. *Let $X = \{x_n/n \in \mathbb{N}\} \bigcup \{x\}$ be a topological space such that $\{x_n\}$ converges to x. Take $X' = \bigoplus_{L\in\wedge} L$, where $\wedge$ is the set of all subsequences of X with x and L s-converges to x. Let $f : X' \to X$ be an identity mapping. Then clearly, f is ss-sequentially quotient but not sequentially quotient, since there is no convergent sequence in X'.*

We observe that the following implication is true when X and Y are first countable, by Theorem 2.2 in [20].

ss-sequentially quotient map $\Longrightarrow$ sequentially quotient map

In [31], author raised a question: "How to characterize s-sequential spaces as the images of metric spaces under some continuous mappings?". Also, for s-Fréchet-Urysohn spaces. So, we characterize s-sequential spaces and s-Fréchet-Urysohn spaces in terms of mappings.

Theorem 3.6. *Y is an s-sequential space $\Leftrightarrow$ every ss-sequentially quotient mapping onto Y is quotient.*

Proof. Let Y be an s-sequential space and $f : X \to Y$ be an ss-sequentially quotient mapping onto Y. Suppose that $f^{-1}(U)$ is open in X and U is not open in Y. Then $Y \setminus U$ is not closed in Y. Therefore, by hypothesis, there exists $y \in U$ such that $\{y_n\}$ s-converges to y such that $y_n \in X \setminus U$. Since f is ss-sequentially quotient, there exists a sequence $\{x_n\}$ s-converges to x such that $x \in f^{-1}(y) \subset f^{-1}(U)$ and $x_n \in f^{-1}(y_n) \subset f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$. Therefore, $f^{-1}(U)$ is not open in X, a contradiction.

Conversely, let every ss-sequentially quotient mapping onto Y be quotient. For each $y \in Y$, and for each sequence $\{s_n\}$ in Y, s-converges to y, let $\mathcal{SC}(S, y) = \{s_n/n = 1, 2, 3, \dots\} \cup \{y\}$ be a topological space, where each s_n is a discrete point and neighborhood U of y is such that $\{n \in \mathbb{N}/s_n \notin U\}$ is a thin subsequence of $\mathbb{N}$. Let $Y^* = \bigoplus_{S\in\mathscr{S}} \mathcal{SC}(S, y) \times \{S\}$ where $\mathscr{S}$ be the set of all s-convergent sequences. Now we consider a mapping $f : Y^* \to Y$ by $f((y_m, S)) = y_m$.

(1) f is onto.
For each point $y \in Y$, there is a constant sequence S in Y such that $\mathcal{SC}(S, y) = \{s_n = y/n = 1, 2, 3, ...\} \cup \{y\}$ that is, there exists $\mathcal{SC}(S, x) \times \{S\} \subset Y^*$ and $f((y, S)) = y$. Therefore, f is onto.

(2) f is continuous.
Let U be an open set in Y and $(y', S) \in f^{-1}(U)$. Then there is a sequence S in Y such that $y' \in \mathcal{SC}(S, y) = \{s_n/n = 1, 2, 3, ...\} \cup \{y\}$ and $f((y', S)) = y'$. If (y', S) is an isolated point, then there is nothing to prove. If $(y', S) = (y, S)$, then there exists a thin subsequence N' of $\mathbb{N}$ such that $s_n \in U$ for $n \in \mathbb{N} \setminus N'$ and hence $\{(s_n, S)/n \in \mathbb{N} \setminus N'\} \subset f^{-1}(U)$ which is open in $\mathcal{SC}(S, y)$ and hence open in Y^*. Therefore, $f^{-1}(U)$ is open in Y^*. Hence f is continuous.

(3) It is clear from the definition of Y^* that f is ss-sequentially quotient.

By our assumption f is quotient. Since Y^* is an s-sequential space and f is quotient, Y is an s-sequential space, by Theorem 2.4 in [31]. □

Theorem 3.7. *Y is an s-Fréchet-Urysohn space $\Leftrightarrow$ every ss-sequentially quotient mapping onto Y is psuedo open.*

Proof. Let Y be an s-Fréchet-Urysohn space and $f : X \to Y$ be an ss-sequentially quotient mapping onto Y. Let y be a point in Y and U an open neighborhood of $f^{-1}(y)$ such that $y \notin int f(U)$. Then $y \in cl(Y \setminus f(U))$. Since Y is s-Fréchet-Urysohn space, there exists a sequence $\{y_n\}$ in $Y \setminus f(U)$ s-converges to y. Thus, there exists a sequence $\{x_n\}$ in X s-converges to x where $x_n \in f^{-1}(y_n)$ for all n and $x \in f^{-1}(y)$, that is, $x_n \in f^{-1}(y_n) \subset f^{-1}(Y \setminus f(U)) \subset X \setminus U$ and $\{x_n\}$ s-converges to $x \in U$ which is a contradiction to U is open. Therefore, f is pseudo open.
Conversely, let every ss-sequentially quotient mapping onto Y is pseudo open.
Let Y^* be a space defined in Theorem 3.6 which is an s-Fréchet-Urysohn space, by Proposition 2.2, and $f : Y^* \to Y$ mapping defined in the previous Theorem 3.6. Then f is ss-sequentially quotient mapping and hence pseudo open. Since Y^* is an s-Fréchet-Urysohn space and f is pseudo open, Y is an s-Fréchet-Urysohn space, by Proposition 2.3. □

References

[1] A. V. Arhangel′skii, *Mappings and spaces,* Russian Math. Surveys, 21 (1966), 115 - 162.

[2] V. Baláž, J. Červeňanský, P. Kostyrko and T. Šalát, *$\mathcal{I}$-convergence and $\mathcal{I}$-continuity of real functions,* Acta Math., 5 (2002), 43-50.

[3] J. Connor and J. Kline, *On statistical limit points and the consistency of statistical convergence,* J. Math. Anal. Appl., 197 (1996), 392-399.

[4] K. Demirci, *$\mathcal{I}$-limit superior and limit inferior,* Math. Commun., 6(2001), 165-172.

[5] G. Di Maio, D. Djurčić, Lj. D. R . Kočinac and M. R. Žižović, *Statistical convergence, selection principles and asymptotic analysis,* Chaos, Solitons & Fractals 42 (2009), 2815-2821.

[6] P. Erdös and G. Tenenbaum, *Sur les densities de certaines suites dâĂŹentiers,* Proc. London Math. Soc. (Ser. 3) 59 (1989), 417-438.

[7] H. Fast, *Sur la convergence statistique,* Colloq. Math., 2 (1951) 241-244.

[8] S. P. Franklin, *Spaces in which sequences suffice,* Fund. Math., 57(1) (1965), 107-115.

[9] J. A. Fridy, *On statistical convergence,* Analysis, 5 (1985), 301-313.

[10] J. A. Fridy and H. I. Miller, *A matrix characterization of statistical convergence,* Analysis, 11 (1991), 59-66.

[11] J. A. Fridy, *Statistical limit points,* Proc. Amer. Math. Soc., 118 (1993), 1187-1192.

[12] H. Halberste and K. F. Roth, *Sequences,* Springer, New York, 1983.

[13] P. Kostyrko, T. Šalát and W. Wilczyński, *$\mathcal{I}$-convergence,* Real Analysis Exch., 26 (2000/2001), 669-685.

[14] P. Kostyrko, M. Mačaj, T. Šalát and M. Sleziak, *$\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points,* Math. Slovaca, 55 (2005), 443-464.

[15] C. Kuratowski, *Topologie ,* Vol-I, PWN Warszawa, 1958.

[16] B. K. Lahiri and P. Das, *Further results on $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior,* Math. Commun., 8 (2003), 151-156.

[17] B. K. Lahiri and P. Das, *$\mathcal{I}$ and $\mathcal{I}^*$-Convergence in topological spaces,* Math. Bohem., 130 (2005), 153-160.

[18] S. Lin, *Point-countable covers and sequence-covering mappings,* (in Chinese): Science Press, Beijing, 2002.

[19] M. Mačaj and T. Šalát, *Statistical convergence of subsequences of a given sequence,* Math. Bohem., 126(2001), 191-208.

[20] G. Di Maio and Lj. D. R. Kočinac, *Statistical convergence in topology,* Topology Appl., 156 (2008), 28-45.

[21] H. I. Miller, *A measure theoretical subsequence characterization of statistical convergence,* Trans. Amer. Math. Soc., 347 (1995), 1811-1819.

[22] H. I. Miller and C. Orhan, *On almost convergent and statistically convergent subsequences,* Acta Math. Hungar., 93(1-2) (2001), 135-151.

[23] I. Niven, H. S. Zuckerman and H. Montgomery, *An introduction to the theory of numbers,* Fifth ed., Wiley, New york, 1991.

[24] S. K. Pal, *$\mathcal{I}$-Sequential topological spaces,* Appl. Math. E-Notes, 14 (2014), 236-241.

[25] S. Pehlivan and M. A. Mamedov, *Statistical cluster points and turnpike,* Optimization, 48(2000), 93-106.

[26] S. Pehlivan, A. Güncan and M. A. Mamedov, *Statistical cluster points of sequences in finite dimensional spaces,* Czechoslovak J. Math., 54 (2004), 95-102.

[27] T. Šalát, *On statistically convergent sequences of real numbers,* Math. Slovaca, 30 (1980), 139-150.

[28] T. Šalát, B. C. Tripathy and M. Ziman, *On $\mathcal{I}$-convergence field,* Italian J. Pure Appl. Math., 17(2005), 45-54.

[29] I. J. Schoenberg, *The Integrability of certain function and related summability methods,* Amer. Math. Monthly, 66 (1959), 361-375.

[30] F. Siwiec, *Sequence-covering and countably bi-quotient maps,* General Topology Appl., 1 (1971), 143-154.

[31] T. Zhongbao and L. Fucai, *Statistical versions of sequential and Fréchet-Urysohn spaces,* Adv. Math. (China), 44 (2015), 945-954.

V. Renukadevi
Department of Mathematics
ANJA College(Autonomous)
Sivakasi 626 124, Tamil Nadu
India
E-mail address: renu_siva2003@yahoo.com

B. Prakash
Department of Mathematics
ANJA College(Autonomous)
Sivakasi 626 124, Tamil Nadu
India
E-mail address: prakashphd101@gmail.com