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A common coupled fixed point theorem in intuitionistic Menger metric space

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Abstract. We establish a common fixed point theorem for mappings under ϕ-contractive conditions on intuitionistic Menger metric spaces. As an application of our result we study the existence and uniquenes of the solution to a nonlinear Fredholm integral equation. We also give an example to validate our result.

1. Introduction

Many generalizations of the concept of a metric space can be obtained by modifying the requirements placed on the distance function. One such generalization is that of Menger spaces, first introduced by Menger [1] and developed by Schweizer and Sklar [2-4], Chang et al. [5], and others [6-8]. In Menger's theory; the concept of distance $d(x;y)$ between two points x and y is considered as probabilistic, namely, the non-negative number $d(x,y)$ is replaced by a distance distribution function $F_{xy} : \mathbb{R} \to \mathbb{R}^+$. Then, for any real number t, the value $F_{xy}(t)$ is interpreted as the degree of nearness between x and y with respect to t. Modifying the idea of Kramosil and Michalek [9], George and Veeramani [10] introduced fuzzy metric spaces which are very similar that of Menger space [7,8,11]. Recently, Park [22] introduced the notion of intuitionistic fuzzy metric spaces as a generalization of fuzzy metric spaces.

In [19] Bhaskar and Lakshmikantham introduced the notion of coupled fixed point and mixed monotone mappings and gave some coupled fixed point theorems. Bhaskar and Lakshmikantham [19] apply these results to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [20] introduced the concept of coupled coincidence point and proved some common coupled fixed point theorems.Sedghi et al [21] gave a coupled fixed point theorem for contractions in fuzzy metric spaces. On the other hand,integral equations arise in many scientific and engineering problems.A large class of initial and boundary

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value problems can be converted to Volterra or Fredholm integral equations. The potential theory contributed more than any field to give rise to integral equations. Mathematical physics models such as diffraction problems, scattering in quantum mechanics, conformal mapping and water waves also contributed to the creation of integral equations.Many other applications in science and engineering are described by integral equations or integro-differential equations.The Volterra's population growth model, biological species living together, propagation of stocked fish in a new lake,the heat radiation are among many areas that are described by integral equations. Many scientific problems give rise to integral equations with logarithm kernels. Integral equations often arise in electrostatics, low frequency electromagnetic problems, electromagnetic scattering problems and propagation of acoustical and elastical waves.

In this paper,we prove a common fixed point theorem for mappings under ϕ-contractive conditions on intuitionistic Menger metric spaces. As an application of our result we study the existence and uniquenes of the solution to a nonlinear Fredholm integral equation. We also give an example to validate our result.

2. Preliminaries

Definition 2.1 ([1])**.** A binary operation $* : [0;1] \times [0;1] \to [0;1]$ is a continuous t-norm if $*$ satisfies the following conditions

a) $*$ is commutative and associative,
b) $*$ is continuous,
c) $a * 1 = a$ for all $a \in [0;1]$,
d) $a * b \leqslant c * d$ wherever $a \leqslant c$, $b \leqslant d$ and $a, b, c, d \in [0;1]$.

Examples of t-norms are $a * b = \min\{a, b\}$ and $a * b = ab$.

Definition 2.2 ([1])**.** A binary operation $\lozenge : [0;1] \times [0;1] \to [0;1]$ is a continuous t-conorm if $\lozenge$ satisfies the following conditions

(a) $\lozenge$ is an commutative and associative,
(b) $\lozenge$ is continuous,
(c) $a \lozenge 0 = a$ for all $a \in [0;1]$,
(d) $a \lozenge b \geq c \lozenge d$ wherever $a \geq c$, $b \geq d$ and $a, b, c, d \in [0;1]$.

Examples of the t-conorms are $a \lozenge b = \max\{a, b\}$ and $a \lozenge b = \min\{1, a + b\}$.

Remark 2.1. The concept of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic sketlons that we use for characterizing fuzzy intersection and union respectively. These concepts were originally introduced by Menger [19] in his study of statistical metric spaces.

Definition 2.3 ([7])**.** Let $\sup_{0<t<1} \Delta(t, t) = 1$.A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\Delta^1(t) = t\Delta t, \Delta^{m=1}(t) = t\Delta(\Delta^m(t)), \quad m = 1, 2, \ldots, t \in [0,1] \tag{2.1}$$

The t-norm $\Delta_M = \min$ is an example of t-norm of H-type,but there are some other t-norms Δ of H-type [7].

Obviously, Δ is a H-type t-norm if and only if for any $\lambda \in (0,1)$, there exists $\delta(\lambda) \in (0,1)$ such that $\Delta^m(t) > 1 - \lambda$ for all $m \in \mathbb{N}$, when $t > 1 - \delta$.

Definition 2.4 ([23])**.** Let $\inf_{0<t<1} \nabla(t,t) = 0$. A t-conorm ∇ is said to be of H-type if the family of functions $\{\nabla^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 0$, where

$$\nabla^1(t) = t\nabla t, \nabla^{m=1}(t) = t\nabla(\nabla^m(t)), \quad m = 1, 2, \ldots, t \in [0,1] \tag{2.2}$$

The t-conorm $\nabla_M = \max$ is an example of t-conorm of H-type.

Obviously, ∇ is a H-type t-conorm if and only if for any $\lambda \in (0,1)$, there exists $\delta(\lambda) \in (0,1)$ such that $\nabla^m(t) < \lambda$ for all $m \in \mathbb{N}$, when $t < \delta$.

Definition 2.5 ([1])**.** A distance distribution function is a function $F : \mathbb{R} \to \mathbb{R}_+$ which is left continuous on $\mathbb{R}$, non-decreasing and $\inf_{t\in\mathbb{R}} F(t) = 0$, $\sup_{t\in\mathbb{R}} F(t) = 1$.

We will denote by D the family of all distance distribution functions and by H a special of D defined by $H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$

If X is a non-empty set, $F : X \times X \to D$ is called a probabilistic distance on X and $F(x,y)$ is usually denoted by F_{xy}.

Definition 2.6 ([14])**.** A non-distance distribution function is a function $L : \mathbb{R} \to \mathbb{R}_+$ which is right continuous on $\mathbb{R}$, non-increasing and $\inf_{t\in\mathbb{R}} L(t) = 1$, $\sup_{t\in\mathbb{R}} L(t) = 0$. We will denote by E the family of all distance distribution functions and by G a special of E defined by $G(t) = \begin{cases} 1, & \text{if } t \leq 0, \\ 0, & \text{if } t > 0. \end{cases}$

If X is a non-empty set, $L : X \times X \to E$ is called a probabilistic distance on X and $L(x,y)$ is usually denoted by L_{xy}.

Definition 2.7 ([14])**.** A triplet (X, F, L) is said to be an intuitionistic probabilistic metric space if X is an arbitrary set, F is a probabilistic distance and L is a probabilistic non-distance on X satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$,

(1) $F_{xy}(t) + L_{xy}(t) \leqslant 1$,
(2) $F_{xy}(0) = 0$,
(3) $F_{xy}(t) = 1$ if and only if $x = y$,
(4) $F_{xy}(t) = F_{yx}(t)$,
(5) If $F_{xy}(t) = 1$ and $F_{yz}(s) = 1$, then $F_{xz}(t+s) = 1$,
(6) $L_{xy}(0) = 1$,
(7) $L_{xy}(t) = 0$ if and only if $x = y$,

(8) $L_{xy}(t) = L_{yx}(t)$,
(9) If $L_{xy}(t) = 0$ and $L_{yz}(s) = 0$, then $L_{xz}(t+s) = 0$.

Definition 2.8 ([14]). A 5-tuple $(X, F, L, *, \diamondsuit)$ is said to be an intuitionistic Menger metric space If (X, F, L) is an intuitionistic probabilistic metric space and in addition, the following inequalities hold for all $x, y, z \in X$ and $t, s > 0$,

(1) $F_{xy}(t) * F_{yz}(s) \leqslant F_{xz}(t+s)$,
(2) $L_{xy}(t) \diamondsuit L_{yz}(s) \geqslant L_{xz}(t+s)$,

where $*$ is a continuous t-norm and $\diamondsuit$ is a continuous t-conorm.

The functions F_{xy} and L_{xy} denote the degree of nearness and the degree of non-nearness between x and y with respect to t respectively.

Remark 2.2. In intuitionistic Menger space $(X, F, L, *, \diamondsuit)$, F_{xy} is non-decreasing and L_{xy} is non-increasing for all $x, y \in X$.

Remark 2.3 ([14]). Every Menger space $(X, F, *)$ is an intuitionistic Menger space of the form $(X, F, 1-F, *, \diamondsuit)$ such that the t-norm $*$ and the t-conorm $\diamondsuit$ are associated, see [13], that is $x \diamondsuit y = 1-(1-x)*(1-y)$ for any $x, y \in X$.

Remark 2.4. Kutukcu et al. [16] proved that if the t-norm $*$ and the t-conorm of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ satisfy the conditions

$$\sup_{t\in(0,1)} (t*t) = 1 \text{ and } \inf_{t\in(0,1)} ((1-t)\diamondsuit(1-t)) = 0,$$

then $(X, F, L, *, \diamondsuit)$ is a Hausdorff topological space in the (ϵ, λ) topology, i.e., the family of sets

$$\{U_x(\epsilon, \lambda),\ \epsilon > 0,\ \lambda \in (0,1],\ x \in X\}$$

is a basis of neighborhoods of point x for a Hausdorff topology $\tau_{(F,L)}$, or (ϵ, λ) topology on X, where

$$U_x(\epsilon, \lambda) = \{y \in X : F_{xy}(\epsilon) > 1-\lambda \text{ and } L_{xy}(\epsilon) < \lambda\}.$$

Example 2.1 ([14]). Let (X, d) be a metric space. Then the metric d induces a distance distribution function F defined by $F_{xy}(t) = H(t-d(x,y))$ and a non-distance distribution function L defined by $L_{xy}(t) = G(t-d(x,y))$ for all $x, y \in X$ and $t > 0$. Therefore, (X, F, L) is an intuitionistic probabilistic metric space induced by a metric d. If the t-norm $*$ is defined by $a*b = \min\{a, b\}$ and the t-conorm $\diamondsuit$ is defined by $a \diamondsuit b = \min\{1, a+b\}$ for all $a, b \in [0,1]$, then $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space.

Remark 2.5 ([14]). Note that the above example holds even with the t-norm $a*b = \min\{a, b\}$ and the t-conorm $a \diamondsuit b = \max\{a, b\}$ and hence $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space with respect to any t-norm and t-conorm. Also note that, in the above example, t-norm $*$ and t-conorm $\diamondsuit$ are not associated.

Remark 2.6. Every an intuitionistic fuzzy metric space $(X, F, L, *, \Diamond)$ is an intuitionistic Menger space by considering $F : X \times X \to D$ and $L : X \times X \to E$ defined by $F_{xy}(t) = M(x, y, t)$ and $L_{xy}(t) = N(x, y, t)$ for all $x, y \in X$.

Throughout this paper, $(X, F, L, *, \Diamond)$ is an intuitionistic Menger space with the following conditions:

$$\lim_{t\to+\infty} F_{xy}(t) = 1 \text{ and } \lim_{t\to+\infty} L_{xy}(t) = 0, \text{ for all } x, y \in X \text{ and } t > 0. \tag{2.3}$$

Definition 2.9 ([14])**.** Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space.

(*a*) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to a point $x \in X$, if for each $t > 0$ and $\epsilon \in (0, 1)$, there exists a positive integer $n_0 = n_0(t, \epsilon)$ such that for all $n \geq n_0$;

(*b*) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called a Cauchy sequence if for all $t > 0$ and $\epsilon \in (0, 1)$, there exists a positive integer $n_0 = n_0(t, \epsilon)$ such that for all $n, m \geq n_0$

$$F_{x_nx_m}(t) > 1 - \epsilon \text{ and } L_{x_nx_m}(t) < \epsilon;$$

(*c*) An intuitionistic Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.7 ([14])**.** An induced intuitionistic Menger space $(X, F, L, *, \Diamond)$ is complete if (X, d) is complete.

Theorem 2.1. *Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space.*

(1) *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to a point $x \in X$ if and only if*

$$\lim_{n\to+\infty} F_{x_nx}(t) = 1 \text{ and } \lim_{n\to+\infty} L_{x_nx}(t) = 0, \text{ for all } t > 0.$$

(2) *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called a Cauchy sequence if and only if*

$$\lim_{n\to+\infty} F_{x_nx_m}(t) = 1 \text{ and } \lim_{n\to+\infty} L_{x_nx_m}(t) = 0, \text{ for all } t > 0.$$

Lemma 2.1. *Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space and $\{x_n\}$, $\{y_n\}$ be two sequences in X with $x_n \to x$ and $y_n \to y$, respectively. Then:*

(*a*) $\liminf_{n\to\infty} F_{x_ny_n}(t) \geq F_{xy}(t)$ *and* $\limsup_{n\to\infty} L_{x_ny_n}(t) \leq L_{xy}(t)$ *for all $t > 0$.*

(*b*) *If $t > 0$ is a continuous point of F_{xy} and L_{xy}, then* $\lim_{n\to\infty} F_{x_ny_n}(t) = F_{xy}(t)$ *and* $\lim_{n\to\infty} L_{x_ny_n}(t) = L_{xy}(t)$.

(*c*) *If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$ is a function such that $\phi(0) = 0$, then ϕ is called a gauge function.If $t \in \mathbb{R}^+$, then $\phi^n(t)$ denotes the nth iteration of $\phi(t)$ and $\phi^{-1}(\{0\}) = \{t \in \mathbb{R}^+ : \phi(t) = 0\}$.*

Definition 2.10 ([19])**.** An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $T : X \times X \to X$ if

$$T(x, y) = x \text{ and } T(y, x) = y.$$

Definition 2.11 ([20]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $T : X \times X \to X$ and $g : X \to X$ if

$$T(x, y) = g(x) \text{ and } T(y, x) = g(y).$$

Definition 2.12 ([20]). An element $(x, y) \in X \times X$ is called a common coupled fixed point of the mappings $T : X \times X \to X$ and $g : X \to X$ if

$$x = T(x, y) = g(x) \text{ and } y = T(y, x) = g(y).$$

Definition 2.13 ([20]). An element $x \in X$ is called a common fixed point of the mappings $T : X \times X \to X$ and $g : X \to X$ if

$$x = T(x, x) = g(x).$$

Definition 2.14 ([20]). The mappings $T : X \times X \to X$ and $g : X \to X$ are said to be commutative if

$$gT(x, y) = T(gx, gy), \text{ for all } (x, y) \in X^2$$

Definition 2.15 ([20]). The mappings $T : X \times X \to X$ and $g : X \to X$ are said to be compatible if

$$\lim_{n\to+\infty} F_{gT(x_n,y_n),T(gx_n,gy_n)}(t) = 1, \lim_{n\to+\infty} L_{gT(x_n,y_n),T(gx_n,gy_n)}(t) = 0$$

and

$$\lim_{n\to+\infty} F_{gT(y_n,x_n),T(gy_n,gx_n)}(t) = 1, \lim_{n\to+\infty} L_{gT(y_n,x_n),T(gy_n,gx_n)}(t) = 0$$

for all $t > 0$, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to+\infty} T(x_n, y_n) = \lim_{n\to+\infty} g(x_n) = x \text{ and } \lim_{n\to+\infty} T(y_n, x_n) = \lim_{n\to+\infty} g(y_n) = y$$

for all $x, y \in X$.

Remark 2.8. It is easy to prove that if T and g are commutative, then they are compatible.

Definition 2.16 ([25]). The mappings $T : X \times X \to X$ and $g : X \to X$ are said to be weakly compatible if

$$T(x, y) = g(x), T(y, x) = g(y)$$

implies that

$$gT(x, y) = T(gx, gy), \quad gT(y, x) = T(gy, gx)$$

for all $x, y \in X$.

Remark 2.9. Two compatible self-mappings are weakly compatible, however the converse is not true in general.

3. Main Results

In this section,the probabilistic distance L and F are assumed to satisfies the conditions $sup_{t>0}F_{x,y}(t) = 1$ and $\inf_{t>0} L_{x,y}(t) = 0$ for all $x, y \in X$.

By using the continuity of $*, \Diamond$ and [26, Lemma 1], we get the following result.

Lemma 3.1. *Let $n \in N$, let $g_n : (0, \infty) \to (0, \infty)$. and let $F_n : \mathbb{R} \to [0, 1]$. Assume that:* $\sup\{F(t) : t > 0\} = 1$ *and*

$$\lim_{n\to+\infty} g_n(t) = 0; \quad F_n(g_n(t)) \geq *^{2n}(F(t)), \quad \forall t > 0$$

If each F_n is nondecreasing, then $\lim_{n\to+\infty} F_n(t) = 1$, for any $t > 0$.

Lemma 3.2. *Let $n \in N$, let $g_n : (0, \infty) \to (0, \infty)$ and let $L_n : \mathbb{R} \to [0, 1]$. Assume that:* $\inf\{L(t) : t > 0\} = 0$ *and*

$$\lim_{n\to+\infty} g_n(t) = 0; \quad L_n(g_n(t)) \leq \Diamond^{2n}(L(t)), \quad \forall t > 0$$

If each L_n is nonincreasing, then $\lim_{n\to+\infty} L_n(t) = 0$, for any $t > 0$.

Proof. Fix $t > 0$ and $\varepsilon > 0$. By hypothesis, there is $t_0 > 0$ such that $L(t_0) < \varepsilon$. Since $g_n(t_0) \to 0$. there is $k \in N$ such that: $g_n(t_0) < t$ for all $n \geq k$. By monotonicity

$$L_n(t) \leq L_n(g_n(t_0)) \leq \Diamond^{2n}(L(t_0)) < \varepsilon, \text{ for } n \geq k$$

Hence we infer that $\lim_{n\to+\infty} L_n(t) = 0$, since $L_n(t) \geq 0$. □

Theorem 3.1. *Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger metric space under a continuous t-norm $*$ of H-type and continuous t-conorm $\Diamond$ of H-type. Let $\phi : (0, \infty) \to (0, \infty)$ be a function satisfying that: $\lim_{n\to+\infty} \phi^n(t) = 0$, for any $t > 0$. Let $T : X \times X \to X$ and $g : X \to X$ be two mappings with $T(X \times X) \subseteq g(X)$ and assume that for any $t > 0$,*

$$\begin{aligned} &F_{T(x,y),T(u,v)}(\phi(t)) \geq F_{g(x),g(u)}(t) * F_{g(y),g(v)}(t) \\ &L_{T(x,y),T(u,v)}(\phi(t)) \leq L_{g(x),g(u)}(t) \Diamond L_{g(y),g(v)}(t) \end{aligned} \tag{3.1}$$

for all $x, y, u, v \in X$. Suppose that $T(X \times X)$ is complete and that g and T are weakly compatible, then g and T have a unique common fixed point x^*, that is $x^* = T(x^*, x^*) = g(x^*)$.

Proof. Since $T(X \times X) \subseteq g(X)$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = T(x_n, y_n) \quad \text{and} \quad gy_{n+1} = T(y_n, x_n) \text{ for all } n \in \mathbb{N} \cup \{0\} \tag{3.2}$$

From (3.1) and (3.2) we have

$$\begin{aligned} F_{gx_n,gx_{n+1}}(\phi(t)) &= F_{T(x_{n-1},y_{n-1}),T(x_n,y_n)}(\phi(t)) \\ &\geq F_{g(x_{n-1}),g(x_n)}(t) * F_{g(y_{n-1}),g(y_n)}(t) \\ L_{gx_n,gx_{n+1}}(\phi(t)) &= L_{T(x_{n-1},y_{n-1}),T(x_n,y_n)}(\phi(t)) \\ &\leq L_{g(x_{n-1}),g(x_n)}(t) \Diamond L_{g(y_{n-1}),g(y_n)}(t) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} F_{gy_n,gy_{n+1}}(\phi(t)) &= F_{T(y_{n-1},x_{n-1}),T(y_n,x_n)}(\phi(t)) \\ &\geq F_{g(x_{n-1}),g(x_n)}(t) * F_{g(y_{n-1}),g(y_n)}(t) \\ L_{gy_n,gy_{n+1}}(\phi(t)) &= L_{T(y_{n-1},x_{n-1}),T(y_n,x_n)}(\phi(t)) \\ &\leq L_{g(x_{n-1}),g(x_n)}(t) \diamondsuit L_{g(y_{n-1}),g(y_n)}(t) \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned} & F_{gx_n,gx_{n+1}}(\phi^n(t)) * F_{gy_n,gy_{n+1}}(\phi^n(t)) \\ \geq\ & *^2\left(F_{gx_{n-1},gx_n}\left(\phi^{n-1}(t)\right) * F_{gy_{n-1},gy_n}\left(\phi^{n-1}(t)\right)\right) \\ \geq\ & \cdots \geq *^{2n}\left(F_{gx_0,gx_1}(t) * F_{gy_0,gy_1}(t)\right) \end{aligned}$$

and

$$\begin{aligned} & L_{gx_n,gx_{n+1}}(\phi^n(t)) \diamondsuit L_{gy_n,gy_{n+1}}(\phi^n(t)) \\ \leq\ & \diamondsuit^2\left(L_{gx_{n-1},gx_n}\left(\phi^{n-1}(t)\right) \diamondsuit L_{gy_{n-1},gy_n}\left(\phi^{n-1}(t)\right)\right) \\ \leq\ & \cdots \leq \diamondsuit^{2n}\left(L_{gx_0,gx_1}(t) \diamondsuit L_{gy_0,gy_1}(t)\right) \end{aligned}$$

Let $E_n(t) = F_{gx_n,gx_{n+1}}(t) * F_{gy_n,gy_{n+1}}(t)$ and $P_n(t) = L_{gx_n,gx_{n+1}}(t) \diamondsuit L_{gy_n,gy_{n+1}}(t)$. Then

$$\begin{aligned} E_n(\phi^n(t)) &\geq *^2\left(E_{n-1}\left(\phi^{n-1}(t)\right)\right) \geq \cdots \geq *^{2n}(E_0(t)). \\ P_n(\phi^n(t)) &\leq \diamondsuit^2\left(P_{n-1}\left(\phi^{n-1}(t)\right)\right) \leq \cdots \leq \diamondsuit^{2n}(P_0(t)) \end{aligned}$$

Since $\phi^n(t) \to 0$ and $\sup_{t>0} E_0(t) = 1, \inf_{t>0} P_n(t) = 0$, by lemma (3.1) and (3.2) we have

$$\lim_{n\to+\infty} E_n(t) = 1 \quad \text{and} \quad \lim_{n\to+\infty} P_n(t) = 0$$

Noting that

$$\min\left\{F_{gx_n,gx_{n+1}}(t), F_{gy_n,gy_{n+1}}(t)\right\} \geq E_n(t),$$

and

$$\max\left\{L_{gx_n,gx_{n+1}}(t), L_{gy_n,gy_{n+1}}(t)\right\} \leq P_n(t),$$

we get that

$$\begin{aligned} \lim_{n\to+\infty} F_{gx_n,gx_{n+1}}(t) &= \lim_{n\to+\infty} F_{gy_n,gy_{n+1}}(t) = 1, \quad \forall t > 0 \\ \lim_{n\to+\infty} L_{gx_n,gx_{n+1}}(t) &= \lim_{n\to+\infty} L_{gy_n,gy_{n+1}}(t) = 0, \quad \forall t > 0 \end{aligned} \tag{3.5}$$

For any fixed $t > 0$, since $\lim_{n\to+\infty} \phi^n(t) = 0$, there exists $n_0 = n_0(t) \in \mathbb{N}$ such that $\phi^{n_0+1}(t) < \phi^{n_0}(t) < t$. Next we show by induction that for any $k \in \mathbb{N} \cup \{0\}$, there exists $b_k \in \mathbb{N}$ such that

(3.6)

$$\begin{aligned} & F_{gx_n,gx_{n+k}}(\phi^{n_0}(t)) * F_{gy_n,gy_{n+k}}(\phi^{n_0}(t)) \\ & \geq *^{b_k}\left(F_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t) - \phi^{n_0+1}(t)\right) * F_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t) - \phi^{n_0+1}(t)\right)\right) \end{aligned}$$

and

(3.7)
$$L_{gx_n,gx_{n+k}}\left(\phi^{n_0}(t)\right)\lozenge L_{gy_n,gy_{n+k}}\left(\phi^{n_0}(t)\right)$$
$$\leq \lozenge^{b_k}\left(L_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\lozenge L_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\right)$$

It is obvious for $k=0$ since $F_{gx_n,gx_n}\left(\phi^{n_0}(t)\right)=F_{gy_n,gy_n}\left(\phi^{n_0}(t)\right)=1$ and $L_{gx_n,gx_n}\left(\phi^{n_0}(t)\right)=L_{gy_n,gy_n}\left(\phi^{n_0}(t)\right)=0$. Assume that (3.6) and (3.7) holds for some $k\in\mathbb{N}$. Since $\phi^{n_0}(t)-\phi^{n_0+1}(t)>0$, by (IM5) we have

$$\begin{aligned} F_{gx_n,gx_{n+k+1}}\left(\phi^{n_0}(t)\right) &= F_{gx_n,gx_{n+k+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)+\phi^{n_0+1}(t)\right) \\ &\geq F_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\ &\quad * F_{gx_{n+1},gx_{n+k+1}}\left(\phi^{n_0+1}(t)\right). \end{aligned} \tag{3.8}$$

and

(3.9)
$$\begin{aligned} L_{gx_n,gx_{n+k+1}}\left(\phi^{n_0}(t)\right) &= L_{gx_n,gx_{n+k+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)+\phi^{n_0+1}(t)\right) \\ &\leq L_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\ &\quad \lozenge L_{gx_{n+1},gx_{n+k+1}}\left(\phi^{n_0+1}(t)\right) \end{aligned}$$

It follows from (3.1), (3.6) and (3.7) that

$$\begin{aligned} & F_{gx_{n+1},gx_{n+k+1}}\left(\phi^{n_0+1}(t)\right) \\ &= F_{T(x_n,y_n),T(x_{n+k},y_{n+k})}\left(\phi^{n_0+1}(t)\right) \\ &\geq F_{gx_n,gx_{n+k}}\left(\phi^{n_0}(t)\right) * F_{gy_n,gy_{n+k}}\left(\phi^{n_0}(t)\right) \\ &\geq *^{b_k}\left(F_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\right. \\ &\quad \left.* F_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\right) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} & L_{gx_{n+1},gx_{n+k+1}}\left(\phi^{n_0+1}(t)\right) \\ &= L_{T(x_n,y_n),T(x_{n+k},y_{n+k})}\left(\phi^{n_0+1}(t)\right) \\ &\leq L_{gx_n,gx_{n+k}}\left(\phi^{n_0}(t)\right)\lozenge L_{gy_n,gy_{n+k}}\left(\phi^{n_0}(t)\right) \\ &\leq \lozenge^{b_k}\left(L_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\right. \\ &\quad \left.\lozenge L_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\right) \end{aligned} \tag{3.11}$$

Now from (3.8), (3.9), (3.10) and (3.11) we get

$$\begin{aligned} & F_{gx_n,gx_{n+k+1}}\left(\phi^{n_0+1}(t)\right) \\ &\geq F_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\ &\quad * \left(*^{b_k}\left(\begin{array}{c} F_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\ *F_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \end{array}\right)\right) \end{aligned} \tag{3.12}$$

and

$$
\begin{aligned}
& L_{gx_n,gx_{n+k+1}}\left(\phi^{n_0+1}(t)\right) \\
& \leq L_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\
& \quad \Diamond\left(\Diamond^{b_k}\left(\begin{array}{c} L_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\ \Diamond L_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \end{array}\right)\right)
\end{aligned} \tag{3.13}
$$

Similarly, we have

$$
\begin{aligned}
& F_{gy_n,gy_{n+k+1}}\left(\phi^{n_0+1}(t)\right) \\
& \geq F_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\
& \quad *\left(*^{b_k}\left(\begin{array}{c} F_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\ *F_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \end{array}\right)\right)
\end{aligned} \tag{3.14}
$$

and

$$
\begin{aligned}
& L_{gy_n,gy_{n+k+1}}\left(\phi^{n_0+1}(t)\right) \\
& \leq L_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\
& \quad \Diamond\left(\Diamond^{b_k}\left(\begin{array}{c} L_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\ \Diamond L_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \end{array}\right)\right)
\end{aligned} \tag{3.15}
$$

From (3.12), (3.13), (3.14) and (3.15) we conclude that

$$
\begin{aligned}
& F_{gx_n,gx_{n+k+1}}\left(\phi^{n_0}(t)\right) * F_{gy_n,gy_{n+k+1}}\left(\phi^{n_0}(t)\right) \\
& \geq F_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) * F_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\
& \quad *\left[*^{2b_k}\left(F_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) * F_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\right)\right] \\
& = *^{2b_k+1}\left(F_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) * F_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\right).
\end{aligned}
$$

and

$$
\begin{aligned}
& L_{gx_n,gx_{n+k+1}}\left(\phi^{n_0}(t)\right) \Diamond L_{gy_n,gy_{n+k+1}}\left(\phi^{n_0}(t)\right) \\
& \leq L_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \Diamond L_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \\
& \quad \Diamond\left[\Diamond^{2b_k}\left(L_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \Diamond L_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\right)\right] \\
& = \Diamond^{2b_k+1}\left(L_{gx_n,gx_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right) \Diamond L_{gy_n,gy_{n+1}}\left(\phi^{n_0}(t)-\phi^{n_0+1}(t)\right)\right).
\end{aligned}
$$

Since $b_{k+1} = 2b_k + 1 \in \mathbb{N}$, this implies that (3.6) and (3.7) holds for $k+1$. Therefore, there exists $b_k \in \mathbb{N}$ such that (3.6) holds for each $k \in \mathbb{N} \cup \{0\}$.

Now we prove that $\{T(x_n, y_n)\}$ and $\{T(y_n, x_n)\}$ are Cauchy sequences in X. Let $t > 0$ and $\varepsilon > 0$. Since $\lim_{n\to+\infty} \phi^n(t) = 0$, there exists $n_1 = n_1(t) \in \mathbb{N}$ such that $\phi^{n_1+1}(t) < \phi^{n_1}(t) < t$.

Since $\{*^n : n \in \mathbb{N}\}$ is equicontinuous at 1 and $*(1) = 1$, there is $\delta > 0$ such that

$$
\text{if } s \in (1-\delta, 1], \quad \text{then } *^n(s) > 1-\varepsilon \quad \text{for all } n \in \mathbb{N} \tag{3.16}
$$

and $\{\Diamond^n : n \in \mathbb{N}\}$ is equicontinuous at 0 and $\Diamond(0) = 0$, there is $\delta > 0$ such that

$$\text{if } s \in [0, \delta), \quad \text{then } \Diamond^n(s) < \varepsilon \ \text{ for all } n \in \mathbb{N} \tag{3.17}$$

By (3.5), one has

$$\lim_{n\to+\infty} F_{gx_n,gx_{n+1}}\left(\phi^{n_1}(t) - \phi^{n_1+1}(t)\right) =$$
$$\lim_{n\to+\infty} F_{gy_n,gy_{n+1}}\left(\phi^{n_1}(t) - \phi^{n_1+1}(t)\right) = 1$$

and

$$\lim_{n\to+\infty} L_{gx_n,gx_{n+1}}\left(\phi^{n_1}(t) - \phi^{n_1+1}(t)\right) =$$
$$\lim_{n\to+\infty} L_{gy_n,gy_{n+1}}\left(\phi^{n_1}(t) - \phi^{n_1+1}(t)\right) = 0.$$

Since $*$ is continuous, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$F_{gx_n,gx_{n+1}}\left(\phi^{n_1}(t) - \phi^{n_1+1}(t)\right) * F_{gy_n,gy_{n+1}}\left(\phi^{n_1}(t) - \phi^{n_1+1}(t)\right) > 1 - \delta$$
$$L_{gx_n,gx_{n+1}}\left(\phi^{n_1}(t) - \phi^{n_1+1}(t)\right) \Diamond L_{gy_n,gy_{n+1}}\left(\phi^{n_1}(t) - \phi^{n_1+1}(t)\right) < \delta$$

Hence, by (3.6), (3.7) (replacing n_0 with n_1) and (3.16), (3.17), we get

$$F_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right) * F_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right) > 1 - \varepsilon$$
$$L_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right) \Diamond L_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right) < \varepsilon$$

for any $k \in \mathbb{N} \cup \{0\}$. Since

$$\min\left\{F_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right), F_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right)\right\} >$$
$$F_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right) * F_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right)$$

and

$$\max\left\{L_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right), L_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right)\right\} <$$
$$L_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right) \Diamond L_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right.$$

one has

$$\min\left\{F_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right), F_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right)\right\} > 1 - \varepsilon$$
$$\max\left\{L_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right), L_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right)\right\} < \varepsilon$$

By monotonicity of F and L,we have, for any $k \in \mathbb{N} \cup \{0\}$,

$$\min\left\{F_{gx_n,gx_{n+k}}(t), F_{gy_n,gy_{n+k}}(t)\right\} \geq$$
$$\min\left\{F_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right), F_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right)\right\} > 1 - \varepsilon$$

and

$$\max\left\{L_{gx_n,gx_{n+k}}(t), L_{gy_n,gy_{n+k}}(t)\right\} \leq$$
$$\max\left\{L_{gx_n,gx_{n+k}}\left(\phi^{n_1}(t)\right), L_{gy_n,gy_{n+k}}\left(\phi^{n_1}(t)\right)\right\} < \varepsilon.$$

Thus $\{gx_n\}$ and $\{gy_n\}$, i.e., $\{T(x_n, y_n)\}$ and $\{T(y_n, x_n)\}$ are Cauchy sequences in X. Since $T(X \times X)$ is complete and $T(X \times X) \subseteq g(X)$, there exist $a, b \in X$ such that $\{T(x_n, y_n)\}$ converges to ga and $\{T(y_n, x_n)\}$ converges to gb.

Next we prove that $ga = T(a, b)$ and $gb = T(b, a)$. Let $t > 0$; since $\lim_{n\to+\infty} \phi^n(t) = 0$, there exists $n_2 = n_2(t) \in \mathbb{N}$ such that $\phi^{n_2}(\phi(t)) < \phi(t)$. By (IM5) and (3.1), we have

$$\begin{aligned}
&F_{T(a,b),ga}(\phi(t)) \\
&\geq F_{T(a,b),T\left(x_{n+n_2},y_{n+n_2}\right)}\left(\phi^{n_2+1}(t)\right) \\
&\qquad * F_{T\left(x_{n+n_2},y_{n+n_2}\right),ga}\left(\phi(t) - \phi^{n_2+1}(t)\right) \\
&\geq F_{ga,gx_{n+n_2}}(\phi^{n_2}(t)) * F_{gb,gy_{n+n_2}}(\phi^{n_2}(t)) \\
&\qquad * F_{T\left(x_{n+n_2},y_{n+n_2}\right),ga}\left(\phi(t) - \phi^{n_2+1}(t)\right).
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&L_{T(a,b),ga}(\phi(t)) \\
&\leq L_{T(a,b),T\left(x_{n+n_2},y_{n+n_2}\right)}\left(\phi^{n_2+1}(t)\right) \\
&\qquad \Diamond L_{T\left(x_{n+n_2},y_{n+n_2}\right),ga}\left(\phi(t) - \phi^{n_2+1}(t)\right) \\
&\leq L_{ga,gx_{n+n_2}}(\phi^{n_2}(t)) \Diamond L_{gb,gy_{n+n_2}}(\phi^{n_2}(t)) \\
&\qquad \Diamond L_{T\left(x_{n+n_2},y_{n+n_2}\right),ga}\left(\phi(t) - \phi^{n_2+1}(t)\right).
\end{aligned} \tag{3.19}$$

Note that $\{gx_n\} \to ga, \{gy_n\} \to gb$ and $\{T(x_{n+n_2}, y_{n+n_2})\} \to ga$. Thus, letting $n \to +\infty$ in (3.18) and (3.19), we have

$$F_{T(a,b),ga}(\phi(t)) \geq 1 * 1 = 1$$
$$L_{T(a,b),ga}(\phi(t)) \leq 0\Diamond 0 = 0$$

By induction we can get

$$F_{T(a,b),ga}(\phi^n(t)) \geq 1$$
$$L_{T(a,b),ga}(\phi^n(t)) \leq 0$$

By (IM2) one has $ga = T(a, b)$. Similarly,we can prove that $gb = T(b, a)$.

Next we prove that if $(a^*, b^*) \in X \times X$ is another coupled coincidence point of g and T, then $ga = ga^*$ and $gb = gb^*$. In fact, by (3.1) we have

$$\begin{aligned}
F_{ga,ga^*}(\phi(t)) &= F_{T(a,b),T(a^*,b^*)}(\phi(t)) \geq F_{ga,ga^*}(t) * F_{gb,gb^*}(t) \\
F_{gb,gb^*}(\phi(t)) &= F_{T(b,a),T(b^*,a^*)}(\phi(t)) \geq F_{ga,ga^*}(t) * F_{gb,gb^*}(t)
\end{aligned}$$

and

$$\begin{aligned}
L_{ga,ga^*}(\phi(t)) &= L_{T(a,b),T(a^*,b^*)}(\phi(t)) \leq L_{ga,ga^*}(t) \Diamond L_{gb,gb^*}(t) \\
L_{gb,gb^*}(\phi(t)) &= L_{T(b,a),T(b^*,a^*)}(\phi(t)) \leq L_{ga,ga^*}(t) \Diamond L_{gb,gb^*}(t)
\end{aligned}$$

It follows that

$$F_{ga,ga^*}(\phi(t)) * F_{gb,gb^*}(\phi(t)) \geq *^2\left(F_{ga,ga^*}(t) * F_{gb,gb^*}(t)\right)$$

and

$$L_{ga,ga^*}(\phi(t)) * L_{gb,gb^*}(\phi(t)) \geq *^2 (L_{ga,ga^*}(t) * L_{gb,gb^*}(t)).$$

By induction we get

$$\begin{aligned}&\min\{F_{ga,ga^*}(\phi^n(t)),\ F_{gb,gb^*}(\phi^n(t))\}\\ &\geq F_{ga,ga^*}(\phi^n(t)) *\ F_{gb,gb^*}(\phi^n(t)) \geq *^{2n}(F_{ga,ga^*}(t) * F_{gb,gb^*}(t))\end{aligned}$$

and

$$\begin{aligned}&\max\{L_{ga,ga^*}(\phi^n(t)), L_{gb,gb^*}(\phi^n(t))\}\\ &\leq L_{ga,ga^*}(\phi^n(t)) \Diamond L_{gb,gb^*}(\phi^n(t)) \leq \Diamond^{2n}(L_{ga,ga^*}(t) \Diamond\ L_{gb,gb^*}(t)).\end{aligned}$$

It follows from lemma 3.1 and (IM2) that $ga = ga^*$ and $gb = gb^*$. This shows that g and T have the unique coupled point of coincidence.

Now we show that $ga = gb$. In fact, from (3.1) we get

$$\begin{aligned} F_{ga,gy_n}(\phi(t)) &= F_{T(a,b),T(y_{n-1},x_{n-1})}(\phi(t))\\ &\geq F_{ga,gy_{n-1}}(t) * F_{gb,gx_{n-1}}(t)\\ F_{gb,gx_n}(\phi(t)) &= F_{T(b,a),T(x_{n-1},y_{n-1})}(\phi(t))\\ &\geq F_{gb,gx_{n-1}}(t) * F_{ga,gy_{n-1}}(t). \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} L_{ga,gy_n}(\phi(t)) &= L_{T(a,b),T(y_{n-1},x_{n-1})}(\phi(t))\\ &\leq L_{ga,gy_{n-1}}(t) \Diamond L_{gb,gx_{n-1}}(t)\\ L_{gb,gx_n}(\phi(t)) &= L_{T(b,a),T(x_{n-1},y_{n-1})}(\phi(t))\\ &\leq L_{gb,gx_{n-1}}(t) \Diamond L_{ga,gy_{n-1}}(t) \end{aligned} \tag{3.21}$$

Let

$$F_n(t) = F_{gb,gx_n}(t) * F_{ga,gy_n}(t) \text{ and } L_n(t) = L_{gb,gx_n}(t) \Diamond L_{ga,gy_n}(t).$$

From (3.20) and (3.21) it follows that

$$\begin{aligned} F_n(\phi^n(t)) &\geq *^2\left(F_{n-1}\left(\phi^{n-1}(t)\right)\right) \geq \cdots \geq *^{2n}(F_0(t)),\\ L_n(\phi^n(t)) &\leq \Diamond^2\left(L_{n-1}\left(\phi^{n-1}(t)\right)\right) \leq \cdots \leq \Diamond^{2n}(L_0(t)) \end{aligned}$$

By Lemma 3.1 we get $\lim_{n\to+\infty} F_n(t) = 1$, and $\lim_{n\to+\infty} L_n(t) = 0$, which implies that

$$\begin{aligned} \lim_{n\to+\infty} F_{gb,gx_n}(t) &= \lim_{n\to+\infty} F_{ga,gy_n}(t) = 1\\ \lim_{n\to+\infty} L_{gb,gx_n}(t) &= \lim_{n\to+\infty} L_{ga,gy_n}(t) = 0 \end{aligned}$$

Since $\{gx_n\}$ converges to ga and $\{gy_n\}$ converges to gb, we see that $gb = ga$.

Now let $u = ga$. Then we have $u = gb$ since $ga = gb$. Since T and g are w-compatible, we have

$$gu = g(ga) = g(T(a,b)) = T(ga, gb) = T(u,u).$$

which implies that (u,u) is a coupled coincidence point of T and g. Since T and g have a unique coupled point of coincidence, we can conclude that $gu = ga$, i.e., Therefor, we have $u = gu = T(u,u)$.

Finally, we prove the uniqueness of a common fixed point of T and g.

Let $v \in X$, be such that $v = gv = T(v,v)$. By (3.1) we have

$$F_{u,v}(\phi(t)) = F_{T(u,u),T(v,v)}(\phi(t)) \geq F_{gu,gv}(t) * F_{gu,gv}(t) = *^2(F_{u,v}(t)).$$

and

$$L_{u,v}(\phi(t)) = L_{T(u,u),T(v,v)}(\phi(t)) \leq L_{gu,gv}(t) \Diamond L_{gu,gv}(t) = \Diamond^2(L_{u,v}(t)).$$

which implies that

$$F_{u,v}(\phi^n(t)) \geq *^{2n}(F_{u,v}(t))$$
$$L_{u,v}(\phi^n(t)) \leq \Diamond^{2n}(L_{u,v}(t))$$

By Lemma 3.1 and (IM-2), we see that $u = v$. This completes the proof. □

Theorem 3.2. *Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger metric space under a continuous t-norm of H-type and continuous t-conorm of H-type. Let $\phi : (0,\infty) \to (0,\infty)$ be a function satisfying that: $\lim_{n\to+\infty} \phi^n(t) = \infty$, for any $t > 0$. Suppose that $T : X \times X \to X$ and $g : X \to X$ are two mappings such that $T(X \times X) \subseteq g(X)$ and assume that for any $t > 0$,*

$$\begin{aligned} F_{T(x,y),T(u,v)}(t) &\geq F_{g(x),g(u)}(\phi(t)) * F_{g(y),g(v)}(\phi(t)) \\ L_{T(x,y),T(u,v)}(t) &\leq L_{g(x),g(u)}(\phi(t)) \Diamond L_{g(y),g(v)}(\phi(t)) \end{aligned} \tag{3.22}$$

for all $x, y, u, v \in X$. Suppose that $T(X \times X)$ is complete and that g and T are weakly compatible, then g and T have a unique common fixed point $x^* \in X$, that is $x^* = T(x^*, x^*) = g(x^*)$.

Proof. Since $T(X \times X) \subseteq g(X)$,we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = T(x_n, y_n) \text{ and } gy_{n+1} = T(y_n, x_n) \text{ for all } n \in \mathbb{N} \cup \{0\} \tag{3.23}$$

From (3.22) and (3.23) we have

$$\begin{aligned} F_{gx_n,gx_{n+1}}(t) &= F_{T(x_{n-1},y_{n-1}),T(x_n,y_n)}(t) \\ &\geq F_{g(x_{n-1}),g(x_n)}(\phi(t)) * F_{g(y_{n-1}),g(y_n)}(\phi(t)) \\ L_{gx_n,gx_{n+1}}(t) &= L_{T(x_{n-1},y_{n-1}),T(x_n,y_n)}(t) \\ &\leq L_{g(x_{n-1}),g(x_n)}(\phi(t)) \Diamond L_{g(y_{n-1}),g(y_n)}(\phi(t)) \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} F_{gy_n,gy_{n+1}}(t) &= F_{T(y_{n-1},x_{n-1}),T(y_n,x_n)}(t) \\ &\geq F_{g(x_{n-1}),g(x_n)}(\phi(t)) * F_{g(y_{n-1}),g(y_n)}(\phi(t)) \\ L_{gy_n,gy_{n+1}}(t) &= L_{T(y_{n-1},x_{n-1}),T(y_n,x_n)}(t) \\ &\leq L_{g(x_{n-1}),g(x_n)}(\phi(t)) \Diamond L_{g(y_{n-1}),g(y_n)}(\phi(t)) \end{aligned} \tag{3.25}$$

Now, let

$$E_n(t) = F_{gx_n,gx_{n+1}}(t) * F_{gy_n,gy_{n+1}}(t)$$

and

$$P_n(t) = L_{gx_n,gx_{n+1}}(t) \Diamond L_{gy_n,gy_{n+1}}(t).$$

From (3.24) and (3.25) we get

$$E_{n+1}(t) \geq E_n(\phi(t)) \text{ and } P_{n+1}(t) \leq P_n(\phi(t)).$$

It follows that

$$\begin{aligned} E_{n+1}(t) &\geq *^2(E_n(\phi(t))) \geq \cdots \geq *^{2n}(E_1(\phi^n(t))) \\ P_{n+1}(t) &\leq \Diamond^2(P_n(\phi(t))) \leq \cdots \leq \Diamond^{2n}(P_1(\phi^n(t))) \end{aligned} \tag{3.26}$$

Since

$$\begin{aligned} &\lim_{t\to+\infty} E_1(t) = \lim_{t\to+\infty} F_{gx_0,gx_1}(t) * F_{gy_0,gy_1}(t) = 1, \\ &\lim_{t\to+\infty} P_1(t) = \lim_{t\to+\infty} L_{gx_0,gx_1}(t) * L_{gy_0,gy_1}(t) = 0, \\ &\lim_{n\to+\infty} \phi^n(t) = \infty \end{aligned}$$

for each $t > 0$, we have

$$\lim_{n\to+\infty} E_1(\phi^n(t)) = 1 \text{ and } \lim_{n\to+\infty} P_1(\phi^n(t)) = 0.$$

By Lemma 3.1 we have

$$\begin{aligned} &\lim_{n\to+\infty} E_n(t) = 1, \quad \text{for all } t > 0, \\ &\lim_{n\to+\infty} P_n(t) = 0, \quad \text{for all } t > 0. \end{aligned} \tag{3.27}$$

For any fixed $t > 0$, since $\lim_{n\to+\infty} \phi^n(t) = \infty$, there exists $n_0 = n_0(t) \in \mathbb{N}$ such that $\phi^{n_0+1}(t) < \phi^{n_0}(t) < t$.

Similarly, since $\lim_{n\to+\infty} \phi^n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) = \infty$, there exists $m_0 = m_0(t) \in \mathbb{N}$ such that $\phi^{m_0}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) > \phi^{n_0+1}(t) - \phi^{n_0}(t)$. By (3.24) we have

(3.28)

$$\begin{aligned} &F_{gx_{n+m_0},gx_{n+m_0+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) \\ &\geq E_{n+m_0}\left(\phi\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right) \\ &\geq \cdots \\ &\geq *^{2m_0}\left(E_n\left(\phi^{m_0}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right)\right) \\ &\geq *^{2m_0}\left(E_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right) \\ &L_{gx_{n+m_0},gx_{n+m_0+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) \leq P_{n+m_0}\left(\phi\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right) \\ &\leq \cdots \leq \Diamond^{2m_0}\left(P_n\left(\phi^{m_0}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right)\right) \\ &\leq \Diamond^{2m_0}\left(P_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right). \end{aligned}$$

Next we show by induction that for any $k \in \mathbb{N} \cup \{0\}$, there exists $b_k \in \mathbb{N}$ such that

$$\begin{aligned} &F_{gx_{n+m_0}, gx_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right) \geq *^{b_k}\left(E_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right) \\ &L_{gx_{n+m_0}, gx_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right) \leq \Diamond^{b_k}\left(P_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right) \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} &F_{gy_{n+m_0}, gy_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right) \geq *^{b_k}\left(E_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right) \\ &L_{gy_{n+m_0}, gy_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right) \leq \Diamond^{b_k}\left(P_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right) \end{aligned} \tag{3.30}$$

This is obvious for $k = 0$ since

$$F_{gx_{n+m_0}, gx_{n+m_0}}\left(\phi^{n_0+1}(t)\right) = F_{gy_{n+m_0}, gy_{n+m_0}}\left(\phi^{n_0+1}(t)\right) = 1$$

and

$$L_{gx_{n+m_0}, gx_{n+m_0}}\left(\phi^{n_0+1}(t)\right) = L_{gy_{n+m_0}, gy_{n+m_0}}\left(\phi^{n_0+1}(t)\right) = 0.$$

Assume that (3.29) and (3.30) holds for some $k \in \mathbb{N}$. By (3.22), (3.29), (3.28) and (IM-5), we have

$$\begin{aligned} &* F_{gx_{n+m_0}, gx_{n+m_0+k+1}}\left(\phi^{n_0+1}(t)\right) \\ &= F_{gx_{n+m_0}, gx_{n+m_0+k+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t) + \phi^{n_0}(t)\right) \\ &\geq F_{gx_{n+m_0}, gx_{n+m_0+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) * F_{gx_{n+m_0+1}, gx_{n+m_0+k+1}}\left(\phi^{n_0}(t)\right) \\ &= F_{gx_{n+m_0}, gx_{n+m_0+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) \\ &\qquad * F_{T\left(x_{n+m_0}, y_{n+m_0}\right), T\left(x_{n+m_0+k}, y_{n+m_0+k}\right)}\left(\phi^{n_0}(t)\right) \\ &\geq F_{gx_{n+m_0}, gx_{n+m_0+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) \\ &\qquad * \left(F_{gx_{n+m_0}, gx_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right) * F_{gy_{n+m_0}, gy_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right)\right) \\ &\geq F_{gx_{n+m_0}, gx_{n+m_0+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) * \left(*^{2b_k}\left(E_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right)\right) \\ &\geq *^{2m_0}\left(E_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right) * \left(*^{2b_k}\left(E_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right)\right) \\ &= *^{2(m_0+b_k)}\left(E_n\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right)\right) \end{aligned}$$

and

$$\begin{aligned} &L_{gx_{n+m_0}, gx_{n+m_0+k+1}}\left(\phi^{n_0+1}(t)\right) \\ &= L_{gx_{n+m_0}, gx_{n+m_0+k+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t) + \phi^{n_0}(t)\right) \\ &\leq L_{gx_{n+m_0}, gx_{n+m_0+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) \Diamond L_{gx_{n+m_0+1}, gx_{n+m_0+k+1}}\left(\phi^{n_0}(t)\right) \\ &= L_{gx_{n+m_0}, gx_{n+m_0+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) \\ &\qquad \Diamond L_{T\left(x_{n+m_0}, y_{n+m_0}\right), T\left(x_{n+m_0+k}, y_{n+m_0+k}\right)}\left(\phi^{n_0}(t)\right) \\ &\leq L_{gx_{n+m_0}, gx_{n+m_0+1}}\left(\phi^{n_0+1}(t) - \phi^{n_0}(t)\right) \Diamond \Big(L_{gx_{n+m_0}, gx_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right) \\ &\qquad \Diamond L_{gy_{n+m_0}, gy_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right)\Big) \end{aligned}$$

$$\leq L_{gx_{n+m_0},gx_{n+m_0+1}}\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)\lozenge\left(\lozenge^{2b_k}\left(E_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)\right)\right)$$
$$\leq \lozenge^{2m_0}\left(E_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)\right)\lozenge\left(\lozenge^{2b_k}\left(E_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)\right)\right)$$
$$=\lozenge^{2(m_0+b_k)}\left(E_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)\right).$$

Similarly, we can prove that

$$F_{gy_{n+m_0},gy_{n+m_0+k+1}}\left(\phi^{n_0+1}(t)\right)\geq *^{2(m_0+b_k)}\left(E_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)\right),$$
$$L_{gy_{n+m_0},gy_{n+m_0+k+1}}\left(\phi^{n_0+1}(t)\right)\leq \lozenge^{2(m_0+b_k)}\left(E_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)\right).$$

Since $b_{k+1}=2(m_0+b_k)\in\mathbb{N}$, (3.29) holds for $k+1$. Therefor, there exists $b_k\in\mathbb{N}$ such that (3.29) holds for all $k\in\mathbb{N}\cup\{0\}$.

Let $t>0$ and $\varepsilon>0$. By hypothesis, $\{*^n : n\in\mathbb{N}\}$ is equicontinuous at 1 and $*(1)=1$, there is $\delta>0$ such that

$$\text{if } s\in(1-\delta,1], \text{ then } *^n(s)>1-\varepsilon \text{ for all } n\in\mathbb{N} \tag{3.31}$$

and $\{\lozenge^n : n\in\mathbb{N}\}$ is equicontinuous at 0 and $\lozenge(0)=0$, there is $\delta>0$ such that

$$\text{if } s\in[0,\delta), \text{ then } \lozenge^n(s)<\varepsilon \text{ for all } n\in\mathbb{N} \tag{3.32}$$

Since by (3.20)

$$\lim_{n\to+\infty}E_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)=1,$$
$$\lim_{n\to+\infty}P_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)=0,$$

there is $N_0\in\mathbb{N}$ such that for all $n>N_0$,

$$E_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)\in(1-\delta,1],$$
$$P_n\left(\phi^{n_0+1}(t)-\phi^{n_0}(t)\right)\in[0,\delta).$$

Hence, it follows from (3.29),(3.30),(3.31) and (3.32) that

$$F_{gx_{n+m_0},gx_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right)*F_{gy_{n+m_0},gy_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right) > 1-\varepsilon$$
$$L_{gx_{n+m_0},gx_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right)\lozenge L_{gy_{n+m_0},gy_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right) < \varepsilon$$

for all $n>N_0$ and any $k\in\mathbb{N}\cup\{0\}$. Noting that (3.17) and (3.18), we have

$$\min\left\{F_{gx_{n+m_0+n_0+1},gx_{n+m_0+n_0+1+k}}(t),F_{gy_{n+m_0+n_0+1},gy_{n+m_0+n_0+1+k}}(t)\right\}$$
$$\geq *^{2n_0+1}\left(F_{gx_{n+m_0},gx_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right)*F_{gy_{n+m_0},gy_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right)\right)$$
$$>1-\varepsilon$$
$$\max\left\{L_{gx_{n+m_0+n_0+1},gx_{n+m_0+n_0+1+k}}(t),L_{gy_{n+m_0+n_0+1},gy_{n+m_0+n_0+1+k}}(t)\right\}$$
$$\leq \lozenge^{2n_0+1}\left(L_{gx_{n+m_0},gx_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right)\lozenge L_{gy_{n+m_0},gy_{n+m_0+k}}\left(\phi^{n_0+1}(t)\right)\right)$$
$$<\varepsilon.$$

This implies that for all $k \in \mathbb{N}$,

$$\begin{aligned} F_{gx_m, gx_{m+k}}(t) &> 1-\varepsilon \text{ and } F_{gy_m, gy_{m+k}}(t) > 1-\varepsilon \\ L_{gx_m, gx_{m+k}}(t) &< \varepsilon \text{ and } L_{gy_m, gy_{m+k}}(t) < \varepsilon \end{aligned}$$

where $m > N_0 + n_0 + m_0 + 1$. Thus $\{gx_n\}$ and $\{gy_n\}$, i.e, $\{T(x_n, y_n)\}$ and $\{T(y_n, x_n)\}$ are the Cauchy sequences. Since $T(X \times X)$ is complete and $T(X \times X) \subseteq g(X)$, there exists $(a, b) \in X \times X$ such that $\{T(x_n, y_n)\}$ converges to ga and $\{T(y_n, x_n)\}$ converges to gb.

Next we prove that $ga = T(a, b)$ and $gb = T(b, a)$. By (IM-5) and (3.22), we have for any $t > 0$,

$$\begin{aligned} F_{T(a,b),T(x_n,y_n)}(t) &\geq F_{ga,gx_n}(\phi(t)) * F_{gb,gy_n}(\phi(t)) \\ L_{T(a,b),T(x_n,y_n)}(t) &\leq L_{ga,gx_n}(\phi(t)) \lozenge L_{gb,gy_n}(\phi(t)) \end{aligned} \tag{3.33}$$

Since $\lim_{n\to+\infty} gx_n = ga$ and $\lim_{n\to+\infty} gy_n = gb$, letting $n \to +\infty$ in (3.33), we have $\lim_{n\to+\infty} T(x_n, y_n) = T(a, b)$. Noting that $\lim_{n\to+\infty} T(x_n, y_n) = ga$, we have $T(a, b) = ga$. Similarly, we can prove that $T(a, b) = gb$.

Let $u = ga$ and $v = gb$. Since g and T are weakly compatible, we have

$$\begin{aligned} gu &= g(ga) = g(T(a,b)) = T(ga, gb) = T(u, v) \\ gv &= g(gb) = g(T(b,a)) = T(gb, ga) = T(v, u). \end{aligned} \tag{3.34}$$

This shows that (u, v) is a coupled coincidence point of g and T. Now we prove that $gu = ga$ and $gv = gb$. In fact, from (3.22) we have

$$\begin{aligned} F_{gu,gx_n}(t) &= F_{T(u,v),T(x_{n-1},y_{n-1})}(t) \geq F_{gu,gx_{n-1}}(\phi(t)) * F_{gv,gy_{n-1}}(\phi(t)) \\ L_{gu,gx_n}(t) &= L_{T(u,v),T(x_{n-1},y_{n-1})}(t) \leq L_{gu,gx_{n-1}}(\phi(t)) \lozenge L_{gv,gy_{n-1}}(\phi(t)) \end{aligned}$$

and

$$\begin{aligned} F_{gv,gy_n}(t) &= F_{T(v,u),T(y_{n-1},x_{n-1})}(t) \geq F_{gv,gy_{n-1}}(\phi(t)) * F_{gu,gx_{n-1}}(\phi(t)) \\ L_{gv,gy_n}(t) &= L_{T(v,u),T(y_{n-1},x_{n-1})}(t) \leq L_{gv,gy_{n-1}}(\phi(t)) \lozenge L_{gu,gx_{n-1}}(\phi(t)). \end{aligned}$$

Let $F_n(t) = F_{gu,gx_n}(t) * F_{gv,gy_n}(t)$, and $L_n(t) = L_{gu,gx_n}(t) \lozenge L_{gv,gy_n}(t)$. Then we have

$$\begin{aligned} F_n(t) &\geq *^2(F_{n-1}(\phi(t))) \geq \cdots \geq *^{2n}(F_0(\phi^n(t))) \\ L_n(t) &\leq \lozenge^2(L_{n-1}(\phi(t))) \leq \cdots \leq \lozenge^{2n}(L_0(\phi^n(t))) \end{aligned}$$

Since $\lim_{n\to+\infty} \phi^n(t) = \infty$ and $*, \lozenge$ are continuous, we have

$$*^{2n}(F_0(\phi^n(t))) = *^{2n}(F_{gv,gx_0}(\phi^n(t)) * F_{gu,gy_0}(\phi^n(t))) \to 1 \text{ as } n \to +\infty$$

and

$$\lozenge^{2n}(L_0(\phi^n(t))) = \lozenge^{2n}(L_{gv,gx_0}(\phi^n(t)) \lozenge L_{gu,gy_0}(\phi^n(t))) \to 0 \text{ as } n \to +\infty.$$

This shows that $F_n(t) \to 1$ as $n \to \infty$, $L_n(t) \to 0$ as $n \to \infty$, and so we have $gu = ga$ and $gv = gb$. Therefore, we have $gu = u$ and $gv = v$. Now, from (3.34) it follows that $u = gu = T(u, v)$ and $v = gv = T(v, u)$.

Finally, we prove that $u = v$. In fact, by (3.22) we have, for any $t > 0$,

$$\begin{aligned} F_{u,v}(t) &= F_{T(u,v),T(v,u)}(t) \geq F_{gu,gv}(\phi(t)) * F_{gv,gu}(\phi(t)) \\ &= *^2 (F_{u,v}(\phi(t))), \\ L_{u,v}(t) &= L_{T(u,v),T(v,u)}(t) \leq L_{gu,gv}(\phi(t)) \lozenge L_{gv,gu}(\phi(t)) \\ &= \lozenge^2 (L_{u,v}(\phi(t))). \end{aligned}$$

By induction we can get $F_{u,v}(t) \geq *^{2n}(F_{u,v}(\phi^n(t)))$ and $L_{u,v}(t) \leq \lozenge^{2n}(L_{u,v}(\phi^n(t)))$. Letting $n \to +\infty$ and noting that $\phi^n(t) \to \infty$ as $n \to +\infty$, we have $F_{u,v}(t) = 1$ and $L_{u,v}(t) = 0$ for any $t > 0$, i.e., $u = v$.

Therefor, u is a common fixed point of g and T.

The uniqueness of u is similar to the final proof line of Theorem 3.1. This completes the proof. □

In Theorem 3.1 and Theorem 3.2, if we let $gx = x$ for all $x \in X$, we get the following result.

Corollary 3.1. *Let $(X, F, L, *, \lozenge)$ be an intuitionistic Menger metric space under a continuous t-norm of H-type and continuous t-conorm of H-type. Let $\phi : (0, \infty) \to (0, \infty)$ be a function satisfying that: $\lim_{n\to+\infty} \phi^n(t) = 0$ for any $t > 0$, $T : X \times X \to X$ be a mapping, and assume that for any $t > 0$,*

$$\begin{aligned} F_{T(x,y),T(u,v)}(\phi(t)) &\geq F_{x,u}(t) * F_{y,v}(t), \\ L_{T(x,y),T(u,v)}(\phi(t)) &\leq L_{x,u}(t) \lozenge L_{y,v}(t) \end{aligned}$$

for all $x, y, u, v \in X$. Suppose that $T(X \times X)$ is complete. Then T has a unique fixed point $x^* \in X$, that is $x^* = T(x^*, x^*)$.

Corollary 3.2. *Let $(X, F, L, *, \lozenge)$ be an intuitionistic Menger metric space under a continuous t-norm of H-type and continuous t-conorm of H-type. Let $\phi : (0, \infty) \to (0, \infty)$ be a function satisfying that: $\lim_{n\to+\infty} \phi^n(t) = \infty$ for any $t > 0$, $T : X \times X \to X$ be a mapping, and assume that for any $t > 0$,*

$$\begin{aligned} F_{T(x,y),T(u,v)}(t) &\geq F_{x,u}(\phi(t)) * F_{y,v}(\phi(t)) \\ L_{T(x,y),T(u,v)}(t) &\leq L_{x,u}(\phi(t)) \lozenge L_{y,v}(\phi(t)) \end{aligned}$$

for all $x, y, u, v \in X$. Suppose that $T(X \times X)$ is complete. Then T has a unique fixed point $x^* \in X$, that is $x^* = T(x^*, x^*)$.

Now, we illustrate Theorem 3.1 by the following example.

Example 3.1. Let $X = [0, \frac{1}{8}) \cup \left\{\frac{1}{4}\right\}$ and $x * y = \min(x, y)$, $x \lozenge y = \max(x, y)$ for all $x, y \in X$. Define $F_{x,y}(t) = \frac{t}{t+|x-y|}$ and $L_{x,y}(t) = \frac{|x-y|}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$.Then $(X, F, L, *, \lozenge)$ be an intuitionistic Menger metric space,but it is not complete.Obviously, $(X, F, L, *, \lozenge)$ is not complete. Define two mappings $g : X \to X$ and $T : X \times X \to X$ by

$$g(x) = \begin{cases} \frac{x}{4}, & \text{if } x \in \left[0, \frac{1}{16}\right], \\ x, & \text{if } x \in \left(\frac{1}{16}, \frac{1}{8}\right), \\ \frac{1}{4}, & \text{if } x = \frac{1}{4}, \end{cases}$$

and

$$T(x,y) = \begin{cases} \frac{x}{16}, & \text{if } x \in [0, \frac{1}{4}), \\ \frac{1}{128}, & \text{if } x = \frac{1}{4}. \end{cases}$$

It is easy to see that g and T are not commuting since $g\left(T\left(\frac{1}{4}, \frac{1}{4}\right)\right) \neq T\left(g\left(\frac{1}{4}\right), g\left(\frac{1}{4}\right)\right), T(X \times X) \subseteq g(X)$, and $T(X \times X)$ is complete.

Let $\phi : (0, +\infty) \to (0, +\infty)$ by

$$\phi(t) = \begin{cases} \frac{3}{2}, & t = 1, \\ \frac{t}{2}, & t \neq 1. \end{cases}$$

Then $\lim_{n \to +\infty} \phi^n(t) = 0$ for any $t > 0$.

From $T(x,y) = g(x)$ and $T(y,x) = g(y)$, we can get $(x,y) = (0,0)$ and we have $gT(0,0) = T(g0, g0)$, which implies that T and g are weakly compatible.The following results is easy to be verified:

$$\left.\begin{array}{l} \dfrac{t}{X+t} \geq \min\left\{\dfrac{t}{Y+t}, \dfrac{t}{Z+t}\right\} \\ \dfrac{X}{X+t} \leq \max\left\{\dfrac{Y}{Y+t}, \dfrac{Z}{Z+t}\right\} \end{array}\right\} \quad \Leftrightarrow \quad \begin{array}{l} X \leq \max\{Y, Z\}, \\ \forall X, Y, Z \geq 0, \ t > 0. \end{array}$$

By the definition of F, L, ϕ and result above,we can get inequality (3.1)

$$F_{T(x,y),T(u,v)}(\phi(t)) \geq F_{g(x),g(u)}(t) * F_{g(y),g(v)}(t)$$

and

$$L_{T(x,y),T(u,v)}(\phi(t)) \leq L_{g(x),g(u)}(t) \Diamond L_{g(y),g(v)}(t)$$

Which is equivalent to the following

$$(**) \qquad 2|T(x,y) - T(u,v)| \leq \max\{|gx - gu|, |gy - gv|\}$$

Now, we verify inequality (*).for $t \neq 1$; we shall consider the following four cases:

Case 1: Let $x \neq \frac{1}{4}$ and $u \neq \frac{1}{4}$. In this case there are four possibilities.

Case 1.1: Let $x \in \left[0, \frac{1}{16}\right]$. Then we have

$$\begin{aligned} 2|T(x,y) - T(u,v)| &= 2\left|\frac{x}{16} - \frac{u}{16}\right| = \frac{1}{2}\left|\frac{x}{4} - \frac{u}{4}\right| < \left|\frac{x}{4} - \frac{u}{4}\right| \\ &\leq \max\left\{\left|\frac{x}{4} - \frac{u}{4}\right|, \left|\frac{y}{4} - \frac{v}{4}\right|\right\} \\ &\leq \max\{|gx - gu|, |gy - gv|\} \quad \text{for all } y, v \in X \end{aligned}$$

So, (*) holds.

Case 1.2: Let $x \in \left[0, \frac{1}{16}\right]$ and $u \in \left(\frac{1}{16}, \frac{1}{8}\right)$. Then

$$\begin{aligned} 2\left|T(x,y) - T(u,v)\right| &= 2\left|\frac{x}{16} - \frac{u}{16}\right| = \frac{1}{2}\left|\frac{x}{4} - \frac{u}{4}\right| \leq \frac{1}{2}\left|x - \frac{u}{4}\right| \\ &\leq \max\left\{\left|x - \frac{u}{4}\right|, \left|\frac{y}{4} - \frac{v}{4}\right|\right\} \\ &\leq \max\left\{\left|gx - gu\right|, \left|gy - gv\right|\right\} \text{ for all } y, v \in X \end{aligned}$$

Case 1.3: Let $x \in \left(\frac{1}{16}, \frac{1}{8}\right)$ and $u \in \left[0, \frac{1}{16}\right]$. This case is similar to case 1.2.
Case 1.4: Let $x \in \left(\frac{1}{16}, \frac{1}{8}\right)$ and $u \in \left(\frac{1}{16}, \frac{1}{8}\right)$. Then

$$\begin{aligned} 2\left|T(x,y) - T(u,v)\right| &= 2\left|\frac{x}{16} - \frac{u}{16}\right| = \frac{1}{2}\left|\frac{x}{4} - \frac{u}{4}\right| \leq \left|\frac{x}{4} - \frac{u}{4}\right| \\ &\leq \max\left\{\left|x - u\right|, \left|\frac{y}{2} - \frac{v}{2}\right|\right\} \\ &\leq \max\left\{\left|gx - gu\right|, \left|gy - gv\right|\right\} \text{ for all } y, v \in X \end{aligned}$$

Case 2: Let $x = \frac{1}{4}$ and $u = \frac{1}{4}$. Then we have

$$\begin{aligned} 2\left|T(x,y) - T(u,v)\right| &= 2\left|\frac{x}{128} - \frac{u}{128}\right| = 0 \\ &\leq \max\left\{\left|g\left(\frac{1}{4}\right) - g\left(\frac{1}{4}\right)\right|, \left|gy - gv\right|\right\} \text{ for all } y, v \in X \end{aligned}$$

Case 3: Let $x = \frac{1}{4}$ and $u \neq \frac{1}{4}$. Then we have
Case 3.1: If $u \in \left[0, \frac{1}{8}\right]$, then

$$\begin{aligned} 2\left|T(x,y) - T(u,v)\right| &= 2\left|T\left(\frac{1}{2}, y\right) - T(u,v)\right| = 2\left|\frac{1}{128} - \frac{u}{16}\right| \\ &= \left|\frac{1}{64} - \frac{u}{8}\right| \leq \left|\frac{1}{8} - u\right| \leq \left|\frac{1}{4} - \frac{u}{4}\right| \\ &\leq \max\left\{\left|g\left(\frac{1}{4}\right) - gu\right|, \left|gy - gv\right|\right\} \text{ for all } y, v \in X \end{aligned}$$

Case 3.2: If $u \left(\frac{1}{16}, \frac{1}{8}\right)$, then

$$\begin{aligned} 2\left|T(x,y) - T(u,v)\right| &= 2\left|\frac{1}{128} - \frac{u}{16}\right| \leq \left|\frac{1}{64} - \frac{u}{8}\right| \leq \left|\frac{1}{8} - u\right| \leq \left|\frac{1}{4} - u\right| \\ &\leq \max\left\{\left|g\left(\frac{1}{4}\right) - gu\right|, \left|gy - gv\right|\right\} \text{ for all } y, v \in X \end{aligned}$$

Case 4: $x \neq \frac{1}{4}$ and $u = \frac{1}{4}$. This case is similar to case 3.

It is easy to see that $(0,0)$ is a coupled coincidence point of g and T. Also, g and T are weakly compatible at $(0,0)$. By Theorem 3.1. we conclude that g and T have a unique common fixed point in X. Obviously in this example, 0 is the unique common fixed point of g and T in X.

4. Application to Integral Equations

As an application of the coupled fixed point theorems established in section 3 of our paper, we study the existence and uniqueness of the solution to a Fredholm nonlinear integral equation.

We shall consider the following integral equation,

$$x(p) = \int_a^b (K_1(p,q) + K_2(p,q))\,[f(q,x(q)) + g(q,x(q))]\,dq + h(q), \tag{4.1}$$

for all $p \in I = [a,b]$.

Let Θ denote the set of all functions $\theta : [0,1] \to [0,1]$ satisfying

(i_θ) θ is non-decreasing,

(ii_θ) $\theta(p) \leq p$.

We assume that the functions K_1, K_2, f, g fulfill the following conditions:

Assumption 4.1.

(i) $K_1(p,q) \geq 0$ and $K_2(p,q) \leq 0$ for all $p, q \in I$,

(ii) There exists $\theta \in \Theta$ such that for all $x, y \in \mathbb{R}$ with $x \geq y$, the following conditions hold:

$$0 \leq f(q,x) - f(q,y) \leq \lambda\theta(x-y) \tag{4.2}$$

and

$$-\mu\theta(x-y) \leq g(q,x) - g(q,y) \leq 0, \tag{4.3}$$

(iii)

$$\max\{\lambda,\mu\} \sup_{p\in I} \int_a^b [K_1(p,q) - K_2(p,q)]\,dq \leq \frac{1}{8} \tag{4.4}$$

Consider the integral equation (4.1) with $K_1, K_2 \in C(I \times I, \mathbb{R})$ and $h \in C(I,\mathbb{R})$. Suppose that Assumption 4.1 is satisfied. Then the integral equation (4.1) has a unique solution in $C(I,\mathbb{R})$.

Proof. Consider $X = C(I,\mathbb{R})$. It is easy to check that $(X, F, L, *, \Diamond)$ is a complete intuitionistic Menger metric space with respect to the distribution distance

$$F_{x,y}(t) = \frac{t}{t + |x-y|}, \quad L_{x,y}(t) = \frac{|x-y|}{t + |x-y|}, \quad \forall x, y \in X,\ t > 0$$

with

$$x * y = \min(x,y),\, x\Diamond y = \max(x,y)\,\forall x, y \in X.$$

Define now the mapping $T : X \times X \to X$ by

$$T(x,y)(p) = \int_a^b K_1(p,q)\left[f(q,x(q)) + g(q,y(q))\right]dq + \int_a^b K_2(p,q)\left[f(q,y(q)) + g(q,x(q))\right]dq + h(p) \tag{4.5}$$

for all $p \in I$ and $\phi(t) = \frac{t}{2}$ for all $t > 0$. Now, for all $x, y, u, v \in X$, using (4.2) and (4.3), we have

$$\begin{aligned} & T(x,y)(p) - T(u,v)(p) \\ &= \int_a^b K_1(p,q)\left[f(q,x(q)) + g(q,y(q))\right]dq \\ &\quad + \int_a^b K_2(p,q)\left[f(q,y(q)) + g(q,x(q))\right]dq \\ &\quad - \int_a^b K_1(p,q)\left[f(q,u(q)) + g(q,v(q))\right]dq \\ &\quad - \int_a^b K_2(p,q)\left[f(q,v(q)) + g(q,u(q))\right]dq \\ &= \int_a^b K_1(p,q)\big[f(q,x(q)) - f(q,u(q)) \\ &\qquad + g(q,y(q)) - g(q,v(q))\big]dq \\ &\quad + \int_a^b K_2(p,q)\big[f(q,y(q)) - f(q,v(q)) \\ &\qquad + g(q,x(q)) - g(q,u(q))\big]dq \\ &= \int_a^b K_1(p,q)\big[(f(q,x(q)) - f(q,u(q))) \\ &\qquad - (g(q,v(q)) - g(q,y(q)))\big]dq \\ &\quad - \int_a^b K_2(p,q)\big[(f(q,v(q)) - f(q,y(q))) \\ &\qquad - (g(q,x(q)) - g(q,u(q)))\big]dq \\ &\leq \int_a^b K_1(p,q)\left[\lambda\theta(x(q) - u(q)) + \mu\theta(v(q) - y(q))\right]dq \\ &\quad - \int_a^b K_2(p,q)\left[\lambda\theta(v(q) - y(q)) + \mu\theta(x(q) - u(q))\right]dq \end{aligned} \tag{4.6}$$

Since the function θ is non-decreasing and so we have

$$\theta(x(q) - u(q)) \leq \theta(|x(q) - u(q)|)$$

and

$$\theta(v(q) - y(q)) \leq \theta(|v(q) - y(q)|),$$

hence by (4.6), in view of the fact $K_2(p,q) \leq 0$, we get

$$\begin{aligned} &|T(x,y)(p) - T(u,v)(p)| \\ &\leq \int_a^b K_1(p,q)\left[\lambda\theta(|x(q) - u(q)|) + \mu\theta(|v(q) - y(q)|)\right] dq \\ &\quad - \int_a^b K_2(p,q)\left[\lambda\theta(|v(q) - y(q)|) + \mu\theta(|x(q) - u(q)|)\right] dq \\ &\leq \int_a^b K_1(p,q)\big[\max\{\lambda,\mu\}\theta(|x(q) - u(q)|) \\ &\qquad + \max\{\lambda,\mu\}\theta(|v(q) - y(q)|)\big] dq \\ &\quad - \int_a^b K_2(p,q)\big[\max\{\lambda,\mu\}\theta(|v(q) - y(q)|) \\ &\qquad + \max\{\lambda,\mu\}\theta(|x(q) - u(q)|)\big] dq \end{aligned} \tag{4.7}$$

as all the quantities on the right hand side of (4.6) are non-negative. Now by using (4.5), we get

$$\begin{aligned} &|T(x,y) - T(u,v)| \\ &\leq \max\{\lambda,\mu\}\int_a^b \left[K_1(p,q) - K_2(p,q)\right] dq \\ &\qquad \cdot \left[\theta(|x(q) - u(q)|) + \theta(|v(q) - y(q)|)\right] \\ &\leq \max\{\lambda,\mu\}\sup_{p\in I}\int_a^b \left[K_1(p,q) - K_2(p,q)\right] dq \\ &\qquad \cdot \left[\theta(|x(q) - u(q)|) + \theta(|v(q) - y(q)|)\right] \\ &\leq \frac{\theta(|x-u|) + \theta(|v-y|)}{8}, \end{aligned} \tag{4.8}$$

Thus

$$2|T(x,y) - T(u,v)| \leq \frac{\theta(|x-u|) + \theta(|v-y|)}{4} \tag{4.9}$$

Now, since θ is nondecreasing, we have

$$\begin{aligned} \theta(|x-u|) &\leq \theta(|x-u|) + \theta(|y-v|), \\ \theta(|y-v|) &\leq \theta(|x-u|) + \theta(|y-v|), \end{aligned} \tag{4.10}$$

which, by using (ii_θ), this implies

$$\begin{aligned} \frac{\theta(|x-u|) + \theta(|y-v|)}{2} &\leq \theta(|x-u| + |y-v|) \\ &\leq |x-u| + |y-v| \\ &\leq 2\max\{|x-u|, |y-v|\}, \end{aligned} \tag{4.11}$$

and so

$$\frac{\theta(|x-u|)+\theta(|y-v|)}{4} \leq \max\{|x-u|,|y-v|\} \tag{4.12}$$

Thus, by (4.9) and (4.12), we get

$$2|T(x,y)-T(u,v)| \leq \max\{|x-u|,|y-v|\} \tag{4.13}$$

Now, by (4.13) and $(**)$, it follows that

$$\begin{aligned}
F_{T(x,y),T(u,v)}(\phi(t)) &= F_{T(x,y),T(u,v)}\left(\frac{t}{2}\right) \\
&= \frac{\frac{t}{2}}{\frac{t}{2}+|T(x,y)-T(u,v)|} \\
&= \frac{t}{t+2|T(x,y)-T(u,v)|} \\
&\geq \frac{t}{t+\max\{|x-u|,|y-v|\}} \\
&\geq \min\left\{\frac{t}{t+|x-u|},\frac{t}{t+|y-v|}\right\} \\
&\geq \min\{F_{x,u}(t),F_{y,v}(t)\},
\end{aligned}$$

and

$$\begin{aligned}
L_{T(x,y),T(u,v)}(\phi(t)) &= L_{T(x,y),T(u,v)}\left(\frac{t}{2}\right) \\
&= \frac{|T(x,y)-T(u,v)|}{\frac{t}{2}+|T(x,y)-T(u,v)|} \\
&= \frac{2|T(x,y)-T(u,v)|}{t+2|T(x,y)-T(u,v)|} \\
&\leq \frac{\max\{|x-u|,|y-v|\}}{t+\max\{|x-u|,|y-v|\}} \\
&\leq \max\left\{\frac{|x-u|}{t+|x-u|},\frac{|y-v|}{t+|y-v|}\right\} \\
&\leq \max\{L_{x,u}(t),L_{y,v}(t)\}
\end{aligned}$$

Thus

$$F_{T(x,y),T(u,v)}(\phi(t)) \geq F_{x,u}(t) * F_{y,v}(t)$$

and

$$L_{T(x,y),T(u,v)}(\phi(t)) \leq L_{x,u}(t) \lozenge L_{y,v}(t)$$

which are the conditions in (3.1), show that all hypotheses of corollary 3.1 are satisfied.

This proves that T has a unique fixed point $a \in X$, that is, $a = T(a,a)$ and therefore $a \in C(I,\mathbb{R})$ is the unique solution of the integral equation (4.1). □

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