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Some congruences related to harmonic numbers and the terms of the second order sequences

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Abstract. In this paper, with helps of some combinatorial identities, we investigate various basic congruences involving harmonic numbers and terms of the second order sequences $\{U_{kn}\}$ and $\{V_{kn}\}$.

1. Introduction

The second order sequence $\{W_n(c, d; r, s)\}$, or briefly $\{W_n\}$, is defined for $n > 0$ by

$$W_{n+1} = rW_n + sW_{n-1}$$

in which $W_0 = c, W_1 = d$, where c, d, r, s are arbitrary integers. As some special cases of $\{W_n\}$, denote $W_n(0, 1; r, 1)$, $W_n(2, r; r, 1)$ by U_n and V_n, respectively.

When $r = 1$, $U_n = F_n$ (the nth Fibonacci number) and $V_n = L_n$ (the nth Lucas number).

If α and β are the roots of the equation $x^2 - rx - 1 = 0$, the Binet formulas of the sequences $\{U_n\}$ and $\{V_n\}$ have the forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

respectively.

From [2, 3], E. Kılıç and P. Stanica derived the following recurrence relations for the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ for $k \geq 0$, $n > 0$. It is clearly that

$$\begin{aligned} U_{k(n+1)} &= V_k U_{kn} + (-1)^{k+1} U_{k(n-1)}, \\ V_{k(n+1)} &= V_k V_{kn} + (-1)^{k+1} V_{k(n-1)}, \end{aligned}$$

where the initial conditions of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ are 0, U_k, and 2, V_k, respectively. Binet formulas of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ are

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given by

$$U_{kn}=\frac{\alpha^{kn}-\beta^{kn}}{\alpha-\beta}\quad \text{and}\quad V_{kn}=\alpha^{kn}+\beta^{kn},$$

respectively. From the Binet formulas, one can see that $U_{-kn}=(-1)^{kn+1}U_{kn}$ and $U_{2kn}=U_{kn}V_{kn}$. Harmonic numbers are those rational numbers given by

$$H_0=0,\quad H_n=\sum_{k=1}^{n}\frac{1}{k},\quad n\in\mathbb{N}=\{1,2,\dots\}.$$

The first few harmonic numbers are $1,\frac{3}{2},\frac{11}{6},\frac{25}{12},\dots$.

For $m\in\mathbb{Z}^+$, harmonic numbers of order m are those rational

$$H_{n,m}=\sum_{k=1}^{n}\frac{1}{k^m},\quad n\in\mathbb{N}.$$

For a prime p and an integer a with $a\nmid p$, we write the Fermat quotient $q_p(a)=\left(a^{p-1}-1\right)/p$. Let $\mathbb{Z}$ be the set of integers. $\mathbb{Z}_p$ denote the set of those rational numbers whose denominator is not divisible by p and is called as the set of p-adic integer numbers. For an integer D, $\sqrt{D}\in\mathbb{Z}_p$ if $\left(\frac{D}{p}\right)=1$ and $\sqrt{D}\notin\mathbb{Z}_p$ if $\left(\frac{D}{p}\right)=-1$ in [6]. It is clearly that x^2-x-1 has two simple roots in $\mathbb{Z}_p$ if and only if $p\equiv\pm1 \pmod{p}$.

In [7], Z.W. Sun and L.L. Zhao established arithmetic properties of harmonic numbers. For example, for any prime $p>3$,

$$\sum_{i=1}^{p-1}\frac{H_i}{i2^i}\equiv\frac{7}{24}pB_{p-3}\pmod{p^2},\quad \sum_{i=1}^{p-1}\frac{H_{i,2}}{i2^i}\equiv-\frac{3}{8}B_{p-3}\pmod{p},$$

where B_0, B_1, B_2, ... Bernoulli numbers.

In [1], A. Granville showed the congruence

$$q(x)\equiv-G(x)\pmod{p},\quad p>3,\tag{1}$$

where $q(x)=\frac{x^p-(x-1)^p-1}{p}$ and $G(x)=\sum\limits_{i=1}^{p-1}\frac{x^i}{i}$.

In [4], H. Pan and Z. W. Sun showed the following lemma and proposition:

Lemma 1.1. *Let $p>3$ be a prime. Then*

$$\left(\frac{x^p+(1-x)^p-1}{p}\right)^2\equiv-2\sum_{i=1}^{p-1}\frac{(1-x)^i}{i^2}-2x^{2p}\sum_{i=1}^{p-1}\frac{\left(1-x^{-1}\right)^i}{i^2}\pmod{p}.\tag{2}$$

Proposition 1.1. *Let r and s be nonzero integers. For an odd prime p such that $p \nmid rs$,*

$$\left(\frac{y_p - r^p}{p}\right)^2 \equiv -2r^2 \sum_{i=1}^{p-1} \frac{\gamma^i}{r^i i^2} - 2\delta^{2p} \sum_{i=1}^{p-1} \frac{\gamma^{2i}}{(-s)^i i^2} \pmod p, \tag{3}$$

$$\left(\frac{y_p - r^p}{p}\right)^2 \equiv -2r\gamma^p \sum_{i=1}^{p-1} \frac{\gamma^i}{r^i i^2} - 2\delta^{2p} \sum_{i=1}^{p-1} \frac{r^i\gamma^i}{s^i i^2} \pmod p, \tag{4}$$

where $y_n = W_n(2, r; r, -s)$ and γ, δ are the two roots of the equation $x^2 - rx + s = 0$.

In this paper, we investigate the congruences involving harmonic numbers and terms of second order sequences $\{U_{kn}\}$ and $\{V_{kn}\}$. For example, for $\left(\frac{\Delta}{p}\right) = 1$,

$$\Delta V_k \sum_{i=1}^{(p-1)/2} U_{2k(2i+1)} H_i \equiv \frac{1}{p}\left((-1)^k \left(V_k^p V_{kp} - \Delta^{(p+1)/2} U_{kp}\right) - 4\right) -2q_p(2)\, V_{2k(p+1)} \pmod p,$$

and

$$\sum_{i=1}^{p-1} \frac{V_{k(p+i-1)}}{V_k^i} H_{i,2} \equiv -\frac{(-1)^k}{2}\left(\frac{V_{kp} - V_k^p}{p}\right)^2 \pmod p,$$

where $\Delta = V_k^2 + 4(-1)^{k+1}$, a prime number $p > 3$, and an integer k with $p \nmid V_k$.

2. Some Lemmas

In this section, we need the following lemmas for further use.

Lemma 2.1. *For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have the following sums:*

$$\sum_{j=1}^{n-1} x^j H_j = \frac{1}{1-x} \sum_{i=1}^{n-1} \frac{x^i}{i} - \frac{x^n}{1-x} H_{n-1}, \tag{5}$$

$$\sum_{j=1}^{n-1} x^j H_{j,2} = \frac{1}{1-x} \sum_{i=1}^{n-1} \frac{x^i}{i^2} - \frac{x^n}{1-x} H_{n-1,2}. \tag{6}$$

Proof. For the proof of (5), from the sum $\sum\limits_{i=0}^{n-1} x^i y^{n-i-1} = \frac{x^n - y^n}{x-y}$, we have

$$\sum_{j=1}^{n-1} x^j H_j = \sum_{j=1}^{n-1} x^j \sum_{i=1}^{j} \frac{1}{i} = \sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=i}^{n-1} x^j$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1}\frac{1}{i}\left(\sum_{j=0}^{n-1}x^j-\sum_{j=0}^{i-1}x^j\right)\\
&= \sum_{i=1}^{n-1}\frac{1}{i}\left(\frac{1-x^n}{1-x}-\frac{1-x^i}{1-x}\right)\\
&= \sum_{i=1}^{n-1}\frac{1}{i}\left(\frac{x^i-x^n}{1-x}\right)=\frac{1}{1-x}\sum_{i=1}^{n-1}\frac{x^i}{i}-\frac{x^n}{1-x}H_{n-1},
\end{aligned}
$$

as claimed. Similarly, the other result is proven. Thus, this ends the proof. □

Lemma 2.2. *For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have the following sums:*

$$
\sum_{j=1}^{n-1} jx^jH_j = \frac{nx^n(x-1)-x(x^n-1)-x}{(x-1)^2}H_{n-1} - \frac{x^n-x}{(x-1)^2}+\frac{x}{(x-1)^2}\sum_{i=1}^{n-1}\frac{x^i}{i}, \tag{7}
$$

$$
\sum_{j=1}^{n-1} jx^jH_{j,2} = \frac{nx^n(x-1)-x(x^n-1)-x}{(x-1)^2}H_{n-1,2} - \frac{1}{x-1}\sum_{i=1}^{n-1}\frac{x^i}{i}+\frac{x}{(x-1)^2}\sum_{i=1}^{n-1}\frac{x^i}{i^2}. \tag{8}
$$

Proof. For the first claim, from the sums

$$
\sum_{i=0}^{n-1} x^iy^{n-i-1} = \frac{x^n-y^n}{x-y}
$$

and

$$
\sum_{i=0}^{n-1} ix^iy^{n-i-1} = \frac{nx^n(x-y)-x(x^n-y^n)}{(x-y)^2},
$$

we write

$$
\begin{aligned}
\sum_{j=1}^{n-1} jx^jH_j &= \sum_{j=1}^{n-1}jx^j\sum_{i=1}^{j}\frac{1}{i}=\sum_{i=1}^{n-1}\frac{1}{i}\sum_{j=i}^{n-1}jx^j\\
&= \sum_{i=1}^{n-1}\frac{1}{i}\left(\sum_{j=0}^{n-1}jx^j-\sum_{j=0}^{i-1}jx^j\right)\\
&= \sum_{i=1}^{n-1}\frac{1}{i}\left(\frac{nx^n(x-1)-x(x^n-1)}{(x-1)^2}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left. -\frac{ix^i(x-1)-x\left(x^i-1\right)}{(x-1)^2}\right) \\
= \quad & \frac{nx^n(x-1)-x(x^n-1)-x}{(x-1)^2}H_{n-1} \\
& -\frac{x^n-x}{(x-1)^2}+\frac{x}{(x-1)^2}\sum_{i=1}^{n-1}\frac{x^i}{i},
\end{aligned}
$$

as claimed. The other claim is similarly obtained. Thus, the proof is completed. □

Lemma 2.3. *For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have the following sums:*

$$
\begin{aligned}
\sum_{k=1}^{n-1} k^2 x^k H_k = & \frac{x^n\left((nx-x-n)^2+x\right)}{(x-1)^3}H_{n-1} \\
& -\frac{nx^n(x-1)-3x^{n+1}+x+2x^2}{(x-1)^3}-\frac{x(x+1)}{(x-1)^3}\sum_{i=1}^{n-1}\frac{x^i}{i}, \qquad (9)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{k=1}^{n-1} k^2 x^k H_{k,2} = & \frac{x^n\left((nx-x-n)^2+x\right)}{(x-1)^3}H_{n-1,2} \\
& -\frac{x^n-x}{(x-1)^2}+\frac{2x}{(x-1)^2}\sum_{i=1}^{n-1}\frac{x^i}{i}-\frac{x(x+1)}{(x-1)^3}\sum_{i=1}^{n-1}\frac{x^i}{i^2}.
\end{aligned}
$$

Proof. Considering the sums

$$
\sum_{i=0}^{n-1} ix^i y^{n-i-1} = \frac{nx^n(x-y)-x(x^n-y^n)}{(x-y)^2}, \quad \sum_{i=0}^{n-1} x^i y^{n-i-1} = \frac{x^n-y^n}{x-y}
$$

and

$$
\sum_{k=0}^{n-1} k^2 x^k y^{n-k-1} = \frac{x^n\left((nx-ny-x)^2+xy\right)-xy^n(x+y)}{(x-y)^3},
$$

the proof is clearly given. □

Lemma 2.4. *Let p be an odd prime. For $\left(\frac{\Delta}{p}\right)=1$,*

$$
\sum_{i=1}^{(p-1)/2} \frac{U_{4ki}}{i} \equiv \frac{(-1)^k}{p}\left(-V_k^p U_{kp}+\left(\sqrt{\Delta}\right)^{p-1}V_{kp}\right) \pmod p, \qquad (10)
$$

$$
\sum_{i=1}^{(p-1)/2} \frac{V_{4ki}}{i} \equiv \frac{4}{p}-\frac{(-1)^k}{p}\left(V_k^p V_{kp}-\left(\sqrt{\Delta}\right)^{p+1}U_{kp}\right) \pmod p, \qquad (11)
$$

where $\Delta = V_k^2 + 4\left(-1\right)^{k+1}$ *and Legendre symbol* $\left(\frac{\cdot}{p}\right)$.

Proof. For the proof of (11), using the Binet formula of the sequence $\{V_{kn}\}$ and taking $\frac{\alpha^{2k}}{\beta^{2k}}, \frac{\beta^{2k}}{\alpha^{2k}}$ instead of x in $\sum_{i=1}^{(p-1)/2} \frac{x^i}{i} \equiv \frac{2}{p} - \frac{\left(\sqrt{x}+1\right)^p - \left(\sqrt{x}-1\right)^p}{p} \pmod p$[5], where any p-adic integer x. We get

$$\begin{aligned}
\sum_{i=1}^{(p-1)/2} \frac{V_{4ki}}{i} &= \sum_{i=1}^{(p-1)/2} \frac{\alpha^{4ki}}{i} + \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i} \\
&= \sum_{i=1}^{(p-1)/2} \frac{1}{i}\left(\frac{\alpha^{2ki}}{\beta^{2ki}}\right) + \sum_{i=1}^{(p-1)/2} \frac{1}{i}\left(\frac{\beta^{2ki}}{\alpha^{2ki}}\right) \\
&\equiv \frac{4}{p} - \frac{V_k^p - \left(\sqrt{\Delta}\right)^p}{p\beta^{kp}} - \frac{V_k^p - \left(-\sqrt{\Delta}\right)^p}{p\alpha^{kp}} \\
&= \frac{4}{p} - \left(-1\right)^k \alpha^{kp} \frac{V_k^p - \left(\sqrt{\Delta}\right)^p}{p} - \left(-1\right)^k \beta^{kp} \frac{V_k^p - \left(-\sqrt{\Delta}\right)^p}{p} \\
&= \frac{4}{p} - \frac{V_k^p}{p}\left(-1\right)^k\left(\alpha^{kp} + \beta^{kp}\right) + \frac{\left(\sqrt{\Delta}\right)^p}{p}\left(-1\right)^k\left(\alpha^{kp} - \beta^{kp}\right) \\
&= \frac{4}{p} - \frac{\left(-1\right)^k}{p}\left(V_k^p V_{kp} - \left(\sqrt{\Delta}\right)^{p+1} U_{kp}\right) \pmod p.
\end{aligned}$$

Similarly, using Binet formula of the sequence $\{U_{kn}\}$, the proof of the congruence in (10) is given. □

Lemma 2.5. *Let* $p > 3$ *be a prime. For an integer* k *with* $p \nmid V_k$ *and* $\left(\frac{\Delta}{p}\right) = 1$,

$$\sum_{i=1}^{p-1} \frac{V_{k(i+p)}}{V_k^{i-1} i} \equiv \left(-1\right)^k \frac{V_k^p V_{k(p-2)} - V_{2k(p-1)} - \left(-1\right)^k V_{2k}}{p V_k^{p-1}} \pmod p.$$

Proof. Consider

$$\begin{aligned}
\sum_{i=1}^{p-1} \frac{V_{k(i+p)}}{V_k^{i-1} i} &= \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p-2)}}{V_k^{i-1} i} + \sum_{i=1}^{p-1} \frac{\beta^{k(i+p-2)}}{V_k^{i-1} i} \\
&= V_k \alpha^{k(p-2)} \sum_{i=1}^{p-1} \left(\frac{\alpha^k}{V_k}\right)^i \frac{1}{i} + V_k \beta^{k(p-2)} \sum_{i=1}^{p-1} \left(\frac{\beta^k}{V_k}\right)^i \frac{1}{i}.
\end{aligned}$$

For $\left(\frac{\Delta}{p}\right) = 1$, taking $\frac{\alpha^k}{V_k}, \frac{\beta^k}{V_k}$ place of x in (1), respectively, we write

$$\sum_{i=1}^{p-1} \frac{V_{k(i+p)}}{V_k^{i-1} i}$$

$$\begin{aligned}
&\equiv V_k\alpha^{k(p-2)}\frac{1-\left(\frac{\alpha^k}{V_k}\right)^p+\left(-\frac{\beta^k}{V_k}\right)^p}{p}+V_k\beta^{k(p-2)}\frac{1-\left(\frac{\beta^k}{V_k}\right)^p+\left(-\frac{\alpha^k}{V_k}\right)^p}{p}\\
&= \frac{V_k^p\left(\alpha^{k(p-2)}+\beta^{k(p-2)}\right)-\left(\alpha^{2k(p-1)}+\beta^{2k(p-1)}\right)-(-1)^k\left(\alpha^{-2k}+\beta^{-2k}\right)}{pV_k^{p-1}}\\
&= (-1)^k\frac{V_k^pV_{k(p-2)}-V_{2k(p-1)}-(-1)^kV_{2k}}{pV_k^{p-1}} \pmod p,
\end{aligned}$$

as claimed. □

Lemma 2.6. *Let $p>3$ be a prime. For an integer k with $p\nmid V_k$ and $\left(\frac{\Delta}{p}\right)=1$,*

$$\sum_{i=1}^{p-1}\frac{V_{k(i+p)}}{V_k^{i-1}i^2}\equiv-\frac{1}{2}\left(\frac{V_{kp}-V_k^p}{p}\right)^2 \pmod p.$$

Proof. Consider that

$$\begin{aligned}
\sum_{i=1}^{p-1}\frac{V_{k(i+p)}}{V_k^{i-1}i^2} &= \sum_{i=1}^{p-1}\frac{\alpha^{k(i+p)}}{V_k^{i-1}i^2}+\sum_{i=1}^{p-1}\frac{\beta^{k(i+p)}}{V_k^{i-1}i^2}\\
&= V_k\sum_{i=1}^{p-1}\frac{\alpha^{k(i+p)}}{V_k^{i}i^2}+V_k\sum_{i=1}^{p-1}\frac{\beta^{k(i+p)}}{V_k^{i}i^2}.
\end{aligned}$$

For $\left(\frac{\Delta}{p}\right)=1$, by taking V_k, $(-1)^k$ instead of r, s in (4), respectively, we have

(12)
$$\sum_{i=1}^{p-1}\frac{V_{k(i+p)}}{V_k^{i-1}i^2}\equiv\frac{-1}{2}\left(\frac{V_{kp}-V_k^p}{p}\right)^2-\beta^{2kp}\sum_{i=1}^{p-1}\frac{V_k^i\alpha^{ki}}{(-1)^{ki}i^2}+V_k\sum_{i=1}^{p-1}\frac{\beta^{k(i+p)}}{V_k^{i}i^2} \pmod p,$$

and from Fermat's little theorem, the congruence $\frac{1}{(p-k)^2}\equiv\frac{1}{k^2} \pmod p$ for $k\nmid p$ and $\alpha^k\beta^k=(-1)^k$, we get

$$\beta^{2kp}\sum_{i=1}^{p-1}\frac{\left(V_k\alpha^k\right)^i}{(-1)^{ki}i^2}=\beta^{2kp}\sum_{i=1}^{p-1}\frac{V_k^i\alpha^{ki}}{\alpha^{ki}\beta^{ki}i^2}=\beta^{2kp}\sum_{i=1}^{p-1}\frac{V_k^{p-i}}{\beta^{k(p-i)}(p-i)^2}$$

(13)
$$\equiv \beta^{2kp}\frac{V_k^p}{\beta^{kp}}\sum_{i=1}^{p-1}\frac{\beta^{ki}}{V_k^ii^2}\equiv V_k\sum_{i=1}^{p-1}\frac{\beta^{k(i+p)}}{V_k^ii^2} \pmod p.$$

By (12) and (13), we obtain the desired result. □

3. The Results involving the terms of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$

In this section, we give congruences for the terms of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$. Now we start with our first result.

Theorem 3.1. *Let p be an odd prime. For $\left(\frac{\Delta}{p}\right) = 1$,*

$$\Delta V_k \sum_{i=1}^{(p-1)/2} U_{2k(2i+1)} H_i$$

$$(14) \equiv \quad -\frac{4}{p} + \frac{(-1)^k}{p}\left(V_k^p V_{kp} - \Delta^{(p+1)/2} U_{kp}\right) - 2q_p(2)\, V_{2k(p+1)} \pmod{p},$$

and

$$V_k \sum_{i=1}^{(p-1)/2} V_{2k(2i+1)} H_i$$

$$(15) \quad \equiv \quad \frac{(-1)^k}{p}\left(V_k^p U_{kp} - \Delta^{(p-1)/2} V_{kp}\right) - 2q_p(2)\, U_{2k(p+1)} \pmod{p},$$

where the Fermat quotient $q_p(2) = (2^{p-1} - 1)/p$.

Proof. For the proof of (14), by the Binet formula of the sequence $\{U_{kn}\}$, we have

$$\Delta V_k \sum_{i=1}^{(p-1)/2} U_{2k(2i+1)} H_i = V_k\sqrt{\Delta} \sum_{i=1}^{(p-1)/2} \alpha^{2k(2i+1)} H_i - V_k\sqrt{\Delta} \sum_{i=1}^{(p-1)/2} \beta^{2k(2i+1)} H_i.$$

Writing $(p+1)/2$ place of n and α^{4k}, β^{4k} place of x in (5), respectively, we write

$$\left(1 - \alpha^{4k}\right) \sum_{i=1}^{(p-1)/2} \alpha^{4ki} H_i = \sum_{i=1}^{(p-1)/2} \frac{\alpha^{4ki}}{i} - \alpha^{2k(p+1)} H_{(p-1)/2},$$

$$\left(1 - \beta^{4k}\right) \sum_{i=1}^{(p-1)/2} \beta^{4ki} H_i = \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i} - \beta^{2k(p+1)} H_{(p-1)/2}.$$

Since $\alpha^{2k} = \beta^{2k}\alpha^{4k}$ and $\beta^{2k} = \alpha^{2k}\beta^{4k}$, we can rewrite

$$(16) \quad -V_k\sqrt{\Delta} \sum_{i=1}^{(p-1)/2} \alpha^{4ki+2k} H_i = \sum_{i=1}^{(p-1)/2} \frac{\alpha^{4ki}}{i} - \alpha^{2k(p+1)} H_{(p-1)/2},$$

$$(17) \quad V_k\sqrt{\Delta} \sum_{i=1}^{(p-1)/2} \beta^{4ki+2k} H_i = \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i} - \beta^{2k(p+1)} H_{(p-1)/2}.$$

By (16) and (17), we get

$$\begin{aligned}
&\Delta V_k \sum_{i=1}^{(p-1)/2} U_{2k(2i+1)} H_i \\
&= -\sum_{i=1}^{(p-1)/2} \frac{\alpha^{4ki}}{i} + \alpha^{2k(p+1)} H_{(p-1)/2} - \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i} + \beta^{2k(p+1)} H_{(p-1)/2} \\
&= -\sum_{i=1}^{(p-1)/2} \frac{V_{4ki}}{i} + V_{2k(p+1)} H_{(p-1)/2},
\end{aligned}$$

which, by (11) and the congruence $H_{(p-1)/2} \equiv -2q_p(2) \pmod p$, equivalents

$$\frac{1}{p}\left((-1)^k \left(V_k^p V_{kp} - \left(\sqrt{\Delta}\right)^{p+1} U_{kp}\right) - 4\right) - 2q_p(2)\, V_{2k(p+1)} \pmod p.$$

Similarly, using the Binet formula of the sequence $\{V_{kn}\}$, (16), (17), (10) and the congruence $H_{(p-1)/2} \equiv -2q_p(2) \pmod p$, the other claim is obtained. □

For example, by taking $k = 1$ in Teorem 3.1, for $\left(\frac{r^2+4}{p}\right) = 1$,

$$\begin{aligned}
&r\left(r^2+4\right) \sum_{i=1}^{(p-1)/2} U_{4i+2} H_i \\
\equiv\ & -\frac{1}{p}\left(r^p V_p - \left(r^2+4\right)^{(p+1)/2} U_p + 4\right) - 2q_p(2)\, V_{2p+2} \pmod p,
\end{aligned}$$

and

$$\begin{aligned}
&r \sum_{i=1}^{(p-1)/2} V_{4i+2} H_i \\
\equiv\ & -\frac{1}{p}\left(r^p U_p - \left(r^2+4\right)^{(p-1)/2} V_p\right) - 2q_p(2)\, U_{2p+2} \pmod p.
\end{aligned}$$

Theorem 3.2. *Let p be an odd prime. For $\left(\frac{\Delta}{p}\right) = 1$,*

$$\begin{aligned}
\Delta V_k^2 \sum_{i=1}^{(p-1)/2} i U_{4ki} H_i \equiv\ & \left(2U_{2k(p+1)} - V_k V_{2kp}\right) q_p(2) - U_{2k(p-1)} \\
& -\frac{(-1)^k}{p}\left(V_k^p U_{kp} - \Delta^{(p-1)/2} V_{kp}\right) \pmod p, \qquad (18)
\end{aligned}$$

and

$$\Delta V_k^2 \sum_{i=1}^{(p-1)/2} i V_{4ki} H_i \equiv \left(2V_{2k(p+1)} - \Delta V_k U_{2kp}\right) q_p(2) - V_{2k(p-1)} + 2$$

$$(19) \qquad +\frac{1}{p}\left(4-(-1)^{k}\left(V_k^{p}V_{kp}-\Delta^{(p+1)/2}U_{kp}\right)\right)(\mod p),$$

where $q_p(2)$ *as before.*

Proof. For the proof of (19), using the Binet formula of the sequence $\{V_{kn}\}$, we have

$$\Delta V_k^2\sum_{i=1}^{(p-1)/2} iV_{4ki}H_i=\Delta V_k^2\sum_{i=1}^{(p-1)/2} i\alpha^{4ki}H_i+\Delta V_k^2\sum_{i=1}^{(p-1)/2} i\beta^{4ki}H_i.$$

Putting $(p+1)/2$ instead of n and α^{4k},β^{4k} instead of x in (7), respectively, we write

$$\begin{aligned}&\left(\alpha^{4k}-1\right)^2\sum_{i=1}^{(p-1)/2} i\alpha^{4k(i-1)}H_i\\ =\;&\left(\frac{p+1}{2}\alpha^{2k(p-1)}\left(\alpha^{4k}-1\right)-\alpha^{2k(p+1)}\right)H_{(p-1)/2}\\ &-\left(\alpha^{2k(p-1)}-1\right)+\sum_{i=1}^{(p-1)/2}\frac{\alpha^{4ki}}{i}\end{aligned}$$

and

$$\begin{aligned}&\left(\beta^{4k}-1\right)^2\sum_{i=1}^{(p-1)/2} i\beta^{4k(i-1)}H_i\\ =\;&\left(\frac{p+1}{2}\beta^{2k(p-1)}\left(\beta^{4k}-1\right)-\beta^{2k(p+1)}\right)H_{(p-1)/2}\\ &-\left(\beta^{2k(p-1)}-1\right)+\sum_{i=1}^{(p-1)/2}\frac{\beta^{4ki}}{i}.\end{aligned}$$

From the equalities $\alpha^{2k}=\beta^{2k}\alpha^{4k}$ and $\beta^{2k}=\alpha^{2k}\beta^{4k}$, we have

$$\begin{aligned}\Delta V_k^2\sum_{i=1}^{(p-1)/2} i\alpha^{4ki}H_i &= \left(\frac{p+1}{2}\sqrt{\Delta}V_k\alpha^{2kp}-\alpha^{2k(p+1)}\right)H_{(p-1)/2}\\ &\quad-\left(\alpha^{2k(p-1)}-1\right)+\sum_{i=1}^{(p-1)/2}\frac{\alpha^{4ki}}{i}, \qquad (20)\end{aligned}$$

$$\begin{aligned}\Delta V_k^2\sum_{i=1}^{(p-1)/2} i\beta^{4ki}H_i &= \left(-\frac{p+1}{2}\sqrt{\Delta}V_k\beta^{2kp}-\beta^{2k(p+1)}\right)H_{(p-1)/2}\\ &\quad-\left(\beta^{2k(p-1)}-1\right)+\sum_{i=1}^{(p-1)/2}\frac{\beta^{4ki}}{i}. \qquad (21)\end{aligned}$$

Using the Binet formulas of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$, by (20) and (21), we rewrite

$$\begin{aligned}
&\Delta V_k^2 \sum_{i=1}^{(p-1)/2} iV_{4ki}H_i \\
=\ &\left(\frac{p+1}{2}\sqrt{\Delta}V_k\alpha^{2kp}-\alpha^{2k(p+1)}\right)H_{(p-1)/2}-\left(\alpha^{2k(p-1)}-1\right) \\
&+\left(-\frac{p+1}{2}\sqrt{\Delta}V_k\beta^{2kp}-\beta^{2k(p+1)}\right)H_{(p-1)/2}-\left(\beta^{2k(p-1)}-1\right) \\
&+\sum_{i=1}^{(p-1)/2}\frac{\alpha^{4ki}}{i}+\sum_{i=1}^{(p-1)/2}\frac{\beta^{4ki}}{i} \\
=\ &\left(\frac{p+1}{2}\Delta V_kU_{2kp}-V_{2k(p+1)}\right)H_{(p-1)/2}-V_{2k(p-1)}+2+\sum_{i=1}^{(p-1)/2}\frac{V_{4ki}}{i}.
\end{aligned}$$

From (11) and the congruence $H_{(p-1)/2}\equiv -2q_p(2)\ (\bmod\ p)$, we have

$$\begin{aligned}
\Delta V_k^2 \sum_{i=1}^{(p-1)/2} iV_{4ki}H_i \equiv\ &\left(2V_{2k(p+1)}-\Delta V_k(p+1)U_{2kp}\right)q_p(2)-V_{2k(p-1)}+2 \\
&+\frac{4}{p}-\frac{(-1)^k}{p}\left(V_k^pV_{kp}-\left(\sqrt{\Delta}\right)^{p+1}U_{kp}\right)(\bmod\ p),
\end{aligned}$$

as claimed. Similarly, the other congruence is given. Thus, the proof is completed. □

For example, when $k=r=1$ in Teorem 3.2, we have the congruences as follows: For $\left(\frac{5}{p}\right)=1$,

$$\begin{aligned}
5\sum_{i=1}^{(p-1)/2} iF_{4i}H_i \equiv\ &(2F_{2p+2}-L_{2p})\,q_p(2)-F_{2p-2} \\
&+\frac{F_p-5^{(p-1)/2}L_p}{p}(\bmod\ p),
\end{aligned}$$

and

$$\begin{aligned}
5\sum_{i=1}^{(p-1)/2} iL_{4i}H_i \equiv\ &(2L_{2p+2}-5F_{2p})\,q_p(2)-L_{2p-2}+2 \\
&+\frac{L_p-5^{(p+1)/2}F_p+4}{p}(\bmod\ p).
\end{aligned}$$

Theorem 3.3. *Let p be an odd prime. For $\left(\frac{\Delta}{p}\right)=1$,*

$$\begin{aligned}
&\Delta^2 V_k^3 \sum_{i=1}^{(p-1)/2} i^2 U_{4ki} H_i \\
\equiv\ & -V_{2k}\left(3+\frac{4}{p}-\frac{(-1)^k}{p}\left(V_k^p V_{kp}-\left(\sqrt{\Delta}\right)^{p+1} U_{kp}\right)\right) \\
&+V_{2kp}\left(3-q_p(2)\left(\frac{V_{4k}}{2}+3\right)\right)-\frac{\Delta}{2}U_{2k(p-1)}U_{2k}(\mod p),
\end{aligned}$$

and

$$\begin{aligned}
&\Delta V_k^3 \sum_{i=1}^{(p-1)/2} i^2 V_{4ki} H_i \\
\equiv\ & U_{2kp}\left(3-q_p(2)\left(\frac{V_{4k}}{2}+3\right)\right)-U_{2k}\left(\frac{1}{2}V_{2k(p-1)}+1\right) \\
&+\frac{(-1)^k}{p}V_{2k}\left(V_k^p U_{kp}-\left(\sqrt{\Delta}\right)^{p-1} V_{kp}\right)(\mod p),
\end{aligned}$$

Proof. Using the Binet formulas of the sequences $\{U_{kn}\}$, $\{V_{kn}\}$, by (9), (10) and the congruence $H_{(p-1)/2} \equiv -2q_p(2)(\mod p)$, we obtained the desired result. □

Now, we will give the congruences with harmonic numbers of order 2, $H_{n,2}$.

Theorem 3.4. *Let $p>3$ be a prime. For $\left(\frac{\Delta}{p}\right)=1$,*

$$\sum_{i=1}^{p-1} \frac{V_{k(i+p-1)}}{V_k^i} H_{i,2} \equiv -\frac{(-1)^k}{2}\left(\frac{V_{kp}-V_k^p}{p}\right)^2(\mod p).$$

Proof. From Binet formula of the sequence $\{V_{kn}\}$, we consider

$$\begin{aligned}
&(-1)^k \sum_{i=1}^{p-1} \frac{V_{k(i+p-1)}}{V_k^i} H_{i,2} \\
=\ & (-1)^k \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p-1)}}{V_k^i} H_{i,2} + \sum_{i=1}^{p-1} \frac{\beta^{k(i+p-1)}}{V_k^i} H_{i,2} \\
=\ & \frac{(-1)^k}{\alpha^k V_k} \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V_k^{i-1}} H_{i,2} + \frac{(-1)^k}{\beta^k V_k} \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V_k^{i-1}} H_{i,2} \\
=\ & \frac{\beta^k}{V_k} \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V_k^{i-1}} H_{i,2} + \frac{\alpha^k}{V_k} \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V_k^{i-1}} H_{i,2}.
\end{aligned}$$

By taking p instead of n and $\frac{\alpha^k}{V_k}, \frac{\beta^k}{V_k}$ instead of x in (6), respectively, we have

$$(22)\left(\frac{V_k-\alpha^k}{V_k}\right)\sum_{i=1}^{p-1}\left(\frac{\alpha^k}{V_k}\right)^{i+p}H_{i,2} = \sum_{i=1}^{p-1}\frac{\left(\frac{\alpha^k}{V_k}\right)^{i+p}}{i^2}-\left(\frac{\alpha^k}{V_k}\right)^{2p}H_{p-1,2},$$

$$(23)\left(\frac{V_k-\beta^k}{V_k}\right)\sum_{i=1}^{p-1}\left(\frac{\beta^k}{V_k}\right)^{i+p}H_{i,2} = \sum_{i=1}^{p-1}\frac{\left(\frac{\beta^k}{V_k}\right)^{i+p}}{i^2}-\left(\frac{\beta^k}{V_k}\right)^{2p}H_{p-1,2}.$$

From (22), (23) and the congruence $H_{p-1,2}\equiv 0 \pmod p$, we get

$$(-1)^k\sum_{i=1}^{p-1}\frac{V_{k(i+p-1)}}{V_k^i}H_{i,2}\equiv\sum_{i=1}^{p-1}\frac{\alpha^{k(i+p)}}{V_k^{i-1}i^2}+\sum_{i=1}^{p-1}\frac{\alpha^{k(i+p)}}{V_k^{i-1}i^2} \pmod p.$$

Using Binet formula of the sequence $\{V_{kn}\}$ and Lemma 2.6, we have

$$\begin{aligned}(-1)^k\sum_{i=1}^{p-1}\frac{V_{k(i+p-1)}}{V_k^i}H_{i,2} &\equiv \sum_{i=1}^{p-1}\frac{V_{k(i+p)}}{V_k^{i-1}i^2}\\ &\equiv -\frac{1}{2}\left(\frac{V_{kp}-V_k^p}{p}\right)^2 \pmod p.\end{aligned}$$

which settles the proof. □

Theorem 3.5. *Let $p>3$ be a prime. For an integer k with $p\nmid V_k$ and $\left(\frac{\Delta}{p}\right)=1$,*

$$\begin{aligned}\sum_{i=1}^{p-1}i\frac{V_{k(i+p-3)}}{V_k^i}H_{i,2} &\equiv (-1)^k\frac{V_k^pV_{k(p-2)}-V_{2k(p-1)}-(-1)^kV_{2k}}{pV_k^{p-1}}\\ &\quad -\frac{1}{2}\left(\frac{V_{kp}-V_k^p}{p}\right)^2 \pmod p.\end{aligned}$$

Proof. From Binet formula of the sequence $\{V_{kn}\}$ and $\alpha^{2k}\beta^{2k}=1$, we have

$$\begin{aligned}&\sum_{i=1}^{p-1}i\frac{V_{k(i+p-3)}}{V_k^i}H_{i,2}\\ =\;& \sum_{i=1}^{p-1}i\frac{\alpha^{k(i+p-3)}}{V_k^i}H_{i,2}+\sum_{i=1}^{p-1}i\frac{\beta^{k(i+p-3)}}{V_k^i}H_{i,2}\\ =\;& \frac{1}{\alpha^{2k}V_k^2}\sum_{i=1}^{p-1}i\frac{\alpha^{k(p+i-1)}}{V_k^{i-2}}H_{i,2}+\frac{1}{\beta^{2k}V_k}\sum_{i=1}^{p-1}i\frac{\beta^{k(i+p-1)}}{V_k^{i-2}}H_{i,2}\\ =\;& \frac{\beta^{2k}}{V_k^2}\sum_{i=1}^{p-1}i\frac{\alpha^{k(i+p-1)}}{V_k^{i-2}}H_{i,2}+\frac{\alpha^{2k}}{V_k}\sum_{i=1}^{p-1}i\frac{\beta^{k(i+p-1)}}{V_k^{i-2}}H_{i,2}.\end{aligned}$$

If we take p instead of n and $\frac{\alpha^k}{V_k}$, $\frac{\beta^k}{V_k}$ instead of x in (8), respectively, we get

$$\begin{aligned}
&\frac{\beta^{2k}}{V_k^2}\sum_{i=1}^{p-1} i\frac{\alpha^{k(i+p-1)}}{V_k^{i-2}}H_{i,2}\\
&= V_k^2\alpha^{k(p-1)}\left(p\left(\frac{\alpha^k}{V_k}\right)^p\left(\frac{\alpha^k}{V_k}-1\right)-\left(\frac{\alpha^k}{V_k}\right)^{p+1}\right)H_{p-1,2}\\
&\quad+\frac{\beta^k}{\alpha^k}\sum_{i=1}^{p-1}\frac{\alpha^{k(i+p)}}{V_k^{i-1}i}+\sum_{i=1}^{p-1}\frac{\alpha^{k(i+p)}}{V_k^{i-1}i^2}, \qquad (24)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\alpha^{2k}}{V_k^2}\sum_{i=1}^{p-1} i\frac{\beta^{k(i+p-1)}}{V_k^{i-2}}H_{i,2}\\
&= V_k^2\beta^{k(p-1)}\left(p\left(\frac{\beta^k}{V_k}\right)^p\left(\frac{\beta^k}{V_k}-1\right)-\left(\frac{\beta^k}{V_k}\right)^{p+1}\right)H_{p-1,2}\\
&\quad+\frac{\alpha^k}{\beta^k}\sum_{i=1}^{p-1}\frac{\beta^{k(i+p)}}{V_k^{i-1}i}+\sum_{i=1}^{p-1}\frac{\beta^{k(i+p)}}{V_k^{i-1}i^2}. \qquad (25)
\end{aligned}$$

From (24), (25) and the congruence $H_{p-1,2}\equiv 0(\mod p)$, we have

$$\begin{aligned}
\sum_{i=1}^{p-1} i\frac{V_{k(i+p-3)}}{V_k^i}H_{i,2} \equiv\ & \frac{\beta^k}{\alpha^k}\sum_{i=1}^{p-1}\frac{\alpha^{k(i+p)}}{V_k^{i-1}i}+\sum_{i=1}^{p-1}\frac{\alpha^{k(i+p)}}{V_k^{i-1}i^2}\\
&+\frac{\alpha^k}{\beta^k}\sum_{i=1}^{p-1}\frac{\beta^{k(i+p)}}{V_k^{i-1}i}+\sum_{i=1}^{p-1}\frac{\beta^{k(i+p)}}{V_k^{i-1}i^2}(\mod p).
\end{aligned}$$

By $\alpha^k\beta^k=(-1)^k$, we rewrite

$$\begin{aligned}
\sum_{i=1}^{p-1} i\frac{V_{k(i+p-3)}}{V_k^i}H_{i,2} \equiv\ & (-1)^k\sum_{i=1}^{p-1}\frac{\alpha^{k(i+p-2)}}{V_k^{i-1}i}+\sum_{i=1}^{p-1}\frac{\alpha^{k(i+p)}}{V_k^{i-1}i^2}\\
&+(-1)^k\sum_{i=1}^{p-1}\frac{\beta^{k(i+p-2)}}{V_k^{i-1}i}+\sum_{i=1}^{p-1}\frac{\beta^{k(i+p)}}{V_k^{i-1}i^2}\\
=\ & (-1)^k\sum_{i=1}^{p-1}\frac{V_{k(i+p-2)}}{V_k^{i-1}i}+\sum_{i=1}^{p-1}\frac{V_{k(i+p)}}{V_k^{i-1}i^2}(\mod p),
\end{aligned}$$

which, by Lemma2.5 and Lemma2.6, equivalents

$$(-1)^k\frac{V_k^pV_{k(p-2)}-V_{2k(p-1)}-(-1)^kV_{2k}}{pV_k^{p-1}}-\frac{1}{2}\left(\frac{V_{kp}-V_k^p}{p}\right)^2(\mod p).$$

□

As a result of Teorem 3.5, by taking 1 instead of k, we have the following corollary:

Corollary 3.1. *Let $p>3$ be a prime. For $p\nmid r$,and $\left(\frac{\Delta}{p}\right)=1$,*

$$\sum_{i=1}^{p-1} i\frac{V_{i+p-3}}{r^i}H_{i,2}\equiv -\frac{r^pV_{p-2}-V_{2p-2}+r^2+2}{pr^{p-1}}-\frac{1}{2}\left(\frac{V_p-r^p}{p}\right)^2 \pmod p.$$

For example, when $r=1$ in Corollary 3.1, we have the congruence as follows:For $\left(\frac{1}{p}\right)=1$,

$$\sum_{i=1}^{p-1} iL_{i+p-3}H_{i,2}\equiv -\frac{L_{p-2}-L_{2p-2}+3}{p}-\frac{1}{2}\left(\frac{L_p-1}{p}\right)^2 \pmod p.$$

REFERENCES

[1] A. Granville, *The square of the Fermat quotient*, Integers:Electron. J. Combin. Number Theory, **4** (2004) , A22.

[2] E. Kılıc, P.Stanica, *Factorizations and representations of second linear recurrences with indices in arithmetic progressions*, Bulletin of the Mexican Mathematical Society, Vol. **15**, No. **1** (2009) , pp. 23–36.

[3] E. Kılıc, P.Stanica, *Factorizations of binary polynomial recurrences by matrix methods*, Rocky Mountain Journal of Mathematics, Vol. **41**, No. **4** (2011) , pp. 1247–1264.

[4] H. Pan, Z. W. Sun, *Proof of three conjectures on congruences*, Sci. China Math., Vol. **57**, No. **10** (2014) , pp. 2091–2102.

[5] S. Koparal and N. Ömür,*On congruences related to Central binomial coefficients, Harmonic and Lucas Numbers*, Turkish Journal of Mathematics, accepted.

[6] P.T. Young,*p-adic congruences for the generalized Fibonacci sequences*, The Fibonacci Quart., Vol. **32**, No. **1** (1994) , pp. 2-10.

[7] Z. W. Sun, L. L. Zhao, *Arithmetic theory of harmonic numbers II*, Colloq. Math., Vol. **130**, No. **1** (2013) , pp. 67–78.

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