MATHEMATICA MORAVICA
VOL. **20:2** (2016), 15–21

Coefficient problem for certain subclasses of bi-univalent functions defined by convolution

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ABSTRACT. In this paper, we consider a general subclass $H_{\Sigma}^{\lambda}(h,\beta)$ of bi-univalent functions. Bounds on the first two coefficients $|a_2|$ and $|a_3|$ for functions in $H_{\Sigma}^{\lambda}(h,\beta)$ are given. The main results generalize and improve a recent one obtained by Srivastava [18].

1. INTRODUCTION

Let A denote the class of functions f which are analytic in the open unit disk $U = \{z : |z| < 1\}$ with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let S be the subclass of A consisting of the form (1) which are also univalent in U.

For $f(z)$ defined by (1) and $\Phi(z)$ defined by

$$\Phi(z) = z + \sum_{n=2}^{\infty} \Phi_n z^n, \qquad (\Phi_n \geq 0), \tag{2}$$

the Hadamard product $(f * \Phi)(z)$ of the functions $f(z)$ and $\Phi(z)$ defined by

$$(f * \Phi)(z) = z + \sum_{n=2}^{\infty} a_n \Phi_n z^n \tag{3}$$

For $0 \leq \beta < 1$ and $\lambda \in \mathbb{C}$, we let $Q_{\lambda}(h,\beta)$ be the subclass of A consisting of functions $f(z)$ of the form (1) and functions $h(z)$ given by

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \qquad (h_n > 0) \tag{4}$$

2010 *Mathematics Subject Classification.* Primary: 30C45; Secondary: 33C45.
Key words and phrases. Analytic and univalent functions, Bi-univalent functions, convolution.

and satisfying the analytic criterion:

$$Q_\lambda(h,\beta)=\Big\{f\in A:\ \operatorname{Re}\Big((1-\lambda)\tfrac{(f*h)(z)}{z}+\lambda(f*h)'(z)\Big)>\beta \quad 0\le\beta<1,\ z\in U\Big\}. \tag{5}$$

The Koebe one-quarter theorem [8] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z))=z\ ,\ \ (z\in U)$$

and

$$f\left(f^{-1}(w)\right)=w\ ,\ \ \left(|w|<r_0(f)\ ,\ r_0(f)\ge\frac{1}{4}\right),$$

where

$$f^{-1}(w)=w\ -a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots.$$

A function $f(z)\in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ, see [18]. Examples of functions in the class Σ are

$$\frac{z}{1-z},\quad -\log(1-z),\quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in S such as

$$z-\frac{z^2}{2}\ \text{and}\ \frac{z}{1-z^2}$$

are also not members of Σ (see [18]).

In [16] the authors defined the classes of functions $P_m(\beta)$: let $P_m(\beta)$, with $m\ge 2$ and $0\le\beta<1$, denote the class of univalent analytic functions P, normalized $P(0)=1$, and satisfying

$$\int_0^{2\pi}\left|\frac{\operatorname{Re}P(z)-\beta}{1-\beta}\right|\mathrm{d}\theta\le m\pi,$$

where $z=re^{i\theta}\in U$.

For $\beta=0$, we denote $P_m=P_m(0)$, hence the class P_m represents the class of functions p analytic in U, normalized with $p(0)=1$, and having the representation

$$p(z)=\int_0^{2\pi}\frac{1-ze^{it}}{1+ze^{it}}\mathrm{d}\mu(t),$$

where μ is a real valued function with bounded variation, which satisfies

$$\int_0^{2\pi}\mathrm{d}\mu(t)=2\pi\quad\text{and}\quad\int_0^{2\pi}|\mathrm{d}\mu(t)|\le m,\quad m\ge 2.$$

Clearly, $P = P_2$ is the well known class of Caratheodory functions, i.e. the normalized functions with positive real part in U.

Lewin [13] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Netanyahu [15] showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [5] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. $S^\star(\beta)$ and $K(\beta)$ of starlike and convex function of order β $(0 \leq \beta < 1)$ respectively (see [15]). The classes $S^\star_\Sigma(\beta)$ and $K_\Sigma(\beta)$ of bi-starlike functions of order α and bi-convex functions of order β, corresponding to the function classes $S^\star(\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S^\star_\Sigma(\beta)$ and $K_\Sigma(\beta)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1, 3, 9, 10, 14, 17, 18, 19, 20]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2, 7, 11, 12]). The coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N}\setminus\{1,2\}; \mathbb{N} = \{1,2,3,\ldots\}$ is still an open problem.

Definition 1.1. A function $f \in \Sigma$ is said to be $H^\lambda_\Sigma(h,\beta)$, if the following conditions are satisfied:

$$(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) \in P_m(\beta); \quad 0 \leq \beta < 1,\ m \geq 2, \quad z \in U$$

and

$$(1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda\left((f*h)^{-1}\right)'(w) \in P_m(\beta);$$
$$0 \leq \beta < 1,\ m \geq 2,\ w \in U,$$

where the function $h(z)$ is given by (4), a number $\lambda \in \mathbb{C}$ and $(f*h)^{-1}(w)$ are defined by:

$$(f*h)^{-1}(w) = w - a_2h_2w^2 + \left(2a_2^2h_2^2 - a_3h_3\right)w^3 - \left(5a_2^3h_2^3 - 5a_2h_2a_3h_3 + a_4h_4\right)w^4 + \cdots.$$

We note that for $\lambda = 1$, $m = 2$ and $h(z) = \frac{z}{1-z}$, the class $H^\lambda_\Sigma(h,\beta)$ reduce to the class $H_\Sigma(\beta)$ studied by Srivastava et al. [18].

The object of the present paper is to find for the first two coefficients $|a_2|$ and $|a_3|$ for functions in $H^\lambda_\Sigma(h,\beta)$. The main results generalize and improve a recent one obtained by Srivastava [18].

In order to derive our main results, we require the following lemma.

Lemma 1.1. [6] *Let the function* $\varphi(z) = 1 + \sum_{n=1}^{\infty} h_nz^n$, $z \in U$, *such that* $\varphi \in P_m(\beta)$. *Then*

$$|h_n| \leq m(1-\beta), \quad n \geq 1.$$

2. MAIN RESULTS

Theorem 2.1. *Let f given by (1) be in the class $H_{\Sigma}^{\lambda}(h,\beta)$, where the function $h(z)$ is given by (4). If $h_2, h_3 \neq 0$ and $\lambda \in \mathbb{C} \setminus \left\{-1; -\frac{1}{2}\right\}$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{|1+2\lambda|\,|h_2|^2}}, \frac{m(1-\beta)}{|1+\lambda|\,|h_2|}\right\},$$

$$|a_3| \leq \min\left\{\frac{m(1-\beta)}{|1+2\lambda|\,|h_3|} + \frac{m^2(1-\beta)^2}{|1+\lambda|^2\,|h_3|}, \frac{m(1-\beta)}{|1+2\lambda|\,|h_3|}\right\}.$$

Proof. Let $f \in H_{\Sigma}^{\lambda}(h,\beta)$. From the Definition 1.1 we have

$$(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) = p(z) \tag{6}$$

$$(1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda\left((f*h)^{-1}\right)'(w) = q(w) \tag{7}$$

where $p, q \in P_m(\beta)$. Using the fact that the functions p and q have the following Taylor expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots,$$
$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots,$$

and it follows from (6) and (7) that

$$(1+\lambda)\,a_2 h_2 = p_1, \tag{8}$$

$$(1+2\lambda)a_3 h_3 = p_2, \tag{9}$$

$$(1+2\lambda)\left(2a_2^2 h_2^2 - a_3 h_3\right) = q_2. \tag{10}$$

Since $p, q \in P_m(\beta)$, according to Lemma 1.1, the next inequalities hold:

$$|p_k| \leq m(1-\beta), \quad k \geq 1, \tag{11}$$

$$|q_k| \leq m(1-\beta), \quad k \geq 1, \tag{12}$$

and thus, from (9) and (10), by using the inequalities (11) and (12)

$$|a_2|^2 \leq \frac{|p_2| + |q_2|}{2\,|1+2\lambda|\,|h_2|^2} \leq \frac{m(1-\beta)}{|1+2\lambda|\,|h_2|^2}, \quad \text{for } \lambda \in \mathbb{C} \setminus \left\{-\frac{1}{2}\right\}. \tag{13}$$

From (8), by using (11) we have

$$|a_2| \leq \frac{m(1-\beta)}{|1+\lambda|\,|h_2|}, \quad \text{for } \lambda \in \mathbb{C} \setminus \{-1\}.$$

From (9), by using (11) we have

$$|a_3| \leq \frac{m(1-\beta)}{|1+2\lambda|\,|h_3|}, \quad \text{for } \lambda \in \mathbb{C} \setminus \left\{-\frac{1}{2}\right\}.$$

Also, subtracting (10) from (9), we have

$$2(1+2\lambda)(a_3h_3 - a_2^2h_2^2) = p_2 - q_2,$$

and using (8), (11) and (12), we finally obtain

$$|a_3| \leq \frac{m(1-\beta)}{|1+2\lambda|\,|h_3|} + \frac{m^2(1-\beta)^2}{|1+\lambda|^2\,|h_3|}, \quad \text{for } \lambda \in \mathbb{C}\backslash\left\{-1,-\frac{1}{2}\right\}$$

which completes our proof. □

Taking $\lambda = 0$ and $\lambda = 1$ in Theorem 2.1 we get following special cases, respectively.

Corollary 2.1. *Let f given by (1) be in the class $H_\Sigma(h,\beta)$, where the function $h(z)$ is given by (4). If $h_2, h_3 \neq 0$, then*

$$|a_2| \leq \frac{\sqrt{m(1-\beta)}}{|h_2|},$$

$$|a_3| \leq \frac{m(1-\beta)}{|h_3|}.$$

Corollary 2.2. *Let f given by (1) be in the class $H_\Sigma^1(h,\beta)$, where the function $h(z)$ is given by (4). If $h_2, h_3 \neq 0$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{3\,|h_2|^2}}, \frac{m(1-\beta)}{2\,|h_2|}\right\},$$

$$|a_3| \leq \frac{m(1-\beta)}{3\,|h_3|}.$$

If we put $\lambda = 1$, $m = 2$ and $h(z) = \frac{z}{1-z}$ in Theorem 2.1, we deduce next corollary.

Corollary 2.3. *Let f given by (1) be in the class $H_\Sigma(\beta)$, then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & \text{if } 0 \leq \beta \leq \frac{1}{3}, \\ (1-\beta), & \frac{1}{3} < \beta < 1, \end{cases}$$

$$|a_3| \leq \frac{2(1-\beta)}{3},$$

$$\left|2a_2^2 - a_3\right| \leq \frac{2(1-\beta)}{3}.$$

Remark 2.1. For the special case $\frac{1}{3} < \beta < 1$, the above first inequality, and the second one for all $0 \leq \beta < 1$, improve the estimates given by Srivastava et al. in ([18], Theorem 2).

REFERENCES

[1] Ş. Altınkaya and S. Yalçın, *Initial coefficient bounds for a general class of bi-univalent functions*, International Journal of Analysis, Article ID 867871, (2014), 4 pp.

[2] Ş. Altınkaya and S. Yalçın, *Coefficient bounds for a subclass of bi-univalent functions*, TWMS Journal of Pure and Applied Mathematics, Vol. **6**, No. **2** (2015), 180-185.

[3] Ş. Altınkaya and S. Yalçın, *Coefficient Estimates for Two New Subclasses of Bi-univalent Functions with respect to Symmetric Points*, Journal of Function Spaces, Article ID 145242, (2015), 5 pp.

[4] D. A. Brannan and J. Clunie, *Aspects of contemporary complex analysis*, Proceedings of the NATO Advanced Study Instute Held at University of Durham, New York: Academic Press, 1979.

[5] D. A. Brannan and T. S. Taha, *On some classes of bi-univalent functions*, Studia Universitatis Babeş-Bolyai. Mathematica, Vol. **31**, No. **2** (1986), pp. 70–77.

[6] P. Goswami, B. S. Alkahtani and T. Bulboacă, *Estimate for initial Maclaurin Coefficients of Certain Sublasses of Bi-univalent Functions*, http://arxiv.org/abs/1503.04644v1, 2015.

[7] S. Bulut, *Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions*, C. R. Acad. Sci. Paris, Ser. I, Vol. **352**, No. **6** (2014), pp. 479–484.

[8] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, 259 (1983).

[9] M. K. Aouf, R. M. El-Ashwah and Ahmed M. Abd-Eltawab, *New Subclasses of Biunivalent Functions Involving Dziok-Srivastava Operator*, International Scholarly Research Notices, Article ID 387178, (2013), 5 pp.

[10] B. A. Frasin and M. K. Aouf, *New subclasses of bi-univalent functions*, Applied Mathematics Letters, Vol. **24**, (2011), pp. 1569–1573.

[11] S. G. Hamidi and J. M. Jahangiri,*Faber polynomial coefficient estimates for analytic bi-close-to-convex functions*, C. R. Acad. Sci. Paris, Ser.I, Vol. **352**, No. **1** (2014), pp. 17–20.

[12] J. M. Jahangiri and G. S. Hamidi, *Coefficient estimates for certain classes of bi-univalent functions*, International Journal of Mathematics and Mathematical Sciences, Article ID 190560, (2013), 4 p.

[13] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., Vol. **18**, (1967), pp. 63–68.

[14] N. Magesh and J. Yamini, *Coefficient bounds for a certain subclass of bi-univalent functions*, International Mathematical Forum, Vol. **8**, No. **27** (2013), pp. 1337–1344.

[15] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Archive for Rational Mechanics and Analysis, Vol. **32**, (1969), pp. 110–112.

[16] K. Padmanabhan and R. Parvatham, *Properties of a class of functions with bounded boundary rotation*, Ann. Polon. Math., Vol. **31**, (1975), pp. 311-323.

[17] S. Porwal and M. Darus, *On a new subclass of bi-univalent functions*, J. Egypt. Math. Soc., Vol. **21**, No. **3** (2013), pp. 190-193.

[18] H. M. Srivastava, A. K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Applied Mathematics Letters, Vol. **23**, No. **10** (2010), pp. 1188-1192.

[19] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat, Vol. **27**, No. **5** (2013), pp. 831-842.

[20] Q. H. Xu, Y. C. Gui and H. M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Applied Mathematics Letters, Vol. **25**, (2012), pp. 990-994.

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