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Existence and exponential stability of solutions for transmission system with varying delay in $\mathbb{R}$

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Abstract. In the present paper we are going to consider in a one dimension bounded domain a transmission system with a varying delay. Under suitable assumptions on the weights of the damping and the delay terms, we prove the well-possedness and the uniqueness of solution using the semigroup theory. Also we show the exponential stability by introducing an appropriate Lyaponov functional.

1. Introduction

It is well known that the PDEs with time delay have been much studied during the last years and their results is by now rather developed. See [1], [5, 6, 7, 14, 17, 18, 19]. In the classical theory of delayed wave equations, several main parts are joined in a fruitful way, it is very remarkable that the damped wave equation with varying delays occupies a similar position and arise in many applied problems.

We consider the transmission problem with a varying delay term,

$$(1)\qquad \begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1 u_t(x,t) \\ \qquad\qquad +\mu_2 u_t(x,t-\tau(t)) = 0, \quad in\ \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad (x,t) \in (L_1,L_2)\times(0,+\infty), \\ u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t),\ i=1,2, \\ au_x(L_i,t) = bv_x(L_i,t),\ i=1,2, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x\in\Omega, \\ u(x,t-\tau(t)) = f_0(x,t-\tau(t)),\ x\in\Omega,\ t\in[0,\bar{\tau}], \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ x\in\,]L_1,L_2[, \end{cases}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0,L_1[\cup]L_2,L_3[$, a,b,μ_1,μ_2 are positive constants.

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We assume, that there exist positive constants $\tau_0, \overline{\tau}$ such that

$$0 < \overline{\tau}_0 \leq \tau(t) \leq \overline{\tau}, \quad \forall t > 0. \tag{2}$$

Moreover, we assume that

$$\tau \in W^{2,\infty}([0,T]), \quad \forall T > 0, \tag{3}$$

$$\tau'(t) \leq d < 1, \quad \forall t > 0, \tag{4}$$

where d is a positive constant.

To motivate our work, let us mention the work [16], when the authors studied well-posedness and exponential stability of a problem with structural damping and boundary delay in both cases $\mu > 0$ and $\mu = 0$ in a bounded and smooth domain, where $k_2 = 0$. The analogous problem with boundary feedback has been introduced and studied by Xu, Yung, Li [19] in one-space dimension using a fine spectral analysis and in higher space dimension by the authors [14]. The case of time-varying delay has been already studied in [15] in one space dimension and in general dimension, with a possibly degenerate delay, in [16]. Both these papers deal with boundary feedback.

This paper improves the results in [4]; for $\tau(t) = \tau$, under suitable assumptions on the weight of the damping and the weight of the delay, he prove the existence and the uniqueness of the solution using the semigroup theory. Also he show the exponential stability of the solution by introducing a suitable Lyaponov functional..

Without delay, system (1) has been investigated in [3]; for $\Omega = [0, L_1]$, the authors showed the well-posedness and exponential stability of the total energy. Muñoz Rivera and Oquendo [13] studied the wave propagations over materials consisting of elastic and viscoelastic components; that is,

$$\begin{aligned} \rho_1 u_{tt} - \alpha_1 u_{xx} &= 0, \; x \in \,]0, L_0[\,, \; t > 0, \\ \rho_2 v_{tt} - \alpha_2 v_{xx} + \int_0^t g(t-s)\, v_{xx}(s)\, ds &= 0, \; x \in \,]L_0, L[\,, \; t > 0, \end{aligned} \tag{5}$$

with the boundary and initial conditions:

$$\begin{aligned} u(0,t) &= v(L,t), \; u(L_0,t) = v(L_0,t), \; t > 0, \\ \alpha_1 u_x(L_0,t) &= \alpha_2 v_x(L_0,t) - \int_0^t g(t-s)\, v_x(s)\, ds, \; t > 0, \end{aligned} \tag{6}$$

where ρ_1 and ρ_2 are densities of the materials and α_1, α_2 are elastic coefficients and g is positive exponential decaying function. They showed that the dissipation produced by the viscoelastic part is strong enough to produce an exponential decay of the solution, no matter how small is its size. Ma and Oquendo [9] considered transmission problem involving two Euler-Bernoulli equations modeling the vibrations of a composite beam. By using just one boundary damping term in the boundary, they showed the global existence and decay property of the solution. Marzocchi et al [10] investigated a 1-D semi-linear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero, no matter how small the damping subdomain is. A similar result has sheen shown by Messaoudi and Said-Houari [12], where a transmission problem in thermoelasticity of type III has been investigated. See also Marzocchi et al [11] for a multidimensional linear thermoelastic transmission problem. The effect of the delay in the stability of hyperbolic systems has been investigated by many people. See for instance [6, 7]. The aim of this article is to study effect of the varying delay in the stability of our system.

2. Well-posedness

Using the semigroup theory, we prove the existence and uniqueness of solution of system (1). As in [14], let us introduce the following new variable

$$z(x,\rho,t) = u_t(x, t-\tau(t)\rho). \tag{7}$$

Then, we obtain

$$\tau(t)\, z_t(x,\rho,t) + \big(1-\tau'(t)\rho\big)\, z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega\times(0,1)\times(0,+\infty). \tag{8}$$

Therefore, the first equation in problem (1) is become as

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \mu_1 u_t(x,t) \\ \qquad\qquad + \mu_2 z(x,1,t) = 0, \ (x,t)\in\Omega\times(0,+\infty), \\ \tau(t)\, z_t(x,\rho,t) + (1-\tau'(t)\rho)\, z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega\times(0,1)\times(0,+\infty), \end{cases}$$

which can be written as

$$\begin{cases} U' = \mathcal{A}(t)U, \\ U(0) = (u_0, v_0, u_1, v_1, f_0(\cdot, -\cdot\tau)), \end{cases}$$

where the operator $\mathcal{A}(t)$ is given by

$$\mathcal{A}(t)\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ a u_{xx} - \mu_1\varphi - \mu_2 z(.,1) \\ b v_{xx} \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \tag{9}$$

with the domain

$$D(\mathcal{A}(t)) = \left\{(u,v,\varphi,\psi,z)^T \in \mathcal{H}; z(.,0) = \varphi \text{ on } \Omega\right\}, \tag{10}$$

where

$$\mathcal{H} = X_* \times L^2(\Omega) \times L^2(L_1,L_2) \times L^2((\Omega)\times(0,1)), \tag{11}$$

where the space X_* is defined by

$$\begin{aligned} X_* = {} & \{(u,v)\in H^1(\Omega)\times H^1(L_1,L_2) : u(0) = u(L_3) = 0, \\ & u(L_i) = v(L_i),\ a u_x(L_i) = b v_x(L_i),\ i = 1,2\}. \end{aligned} \tag{12}$$

Remark 2.1. Noting that the domain of $D(\mathcal{A})(t)$ is independent of the time t; i.e.,

$$D(\mathcal{A}\,(t)) = D(\mathcal{A}\,(0)),\ t > 0. \tag{13}$$

Let

$$U = (u, v, \varphi, \psi, z)^T, \quad \bar{U} = (\bar{u}, \bar{v}, \bar{\varphi}, \bar{\psi}, \bar{z})^T.$$

We define the standard inner product in $\mathcal{H}$ as follows:

$$\begin{aligned}\langle U, \bar{U}\rangle_{\mathcal{H}} &= \int_\Omega \{\varphi\bar{\varphi} + a u_x \bar{u}_x\} dx + \int_{L_1}^{L_2} \{\psi\bar{\psi} + b v_x \bar{v_x}\} dx \\ &\quad + \int_\Omega \int_0^1 z(x,\rho)\bar{z}(x,\rho)\, d\rho dx.\end{aligned}$$

Using semigroup arguments by the literature, we can obtain a well-posedness result (see [8]).

Theorem 2.1. *Assume that*

(i) $D(\mathcal{A}(0))$ is a dense subset of $\mathcal{H}$,
(ii) $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$ for all $t > 0$,
(iii) for all $t \in [0, T]$, $\mathcal{A}(t)$ generates a strongly continuous semigroup on $\mathcal{H}$ and the family $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ is stable with stability constants C and m independent of t (i.e. the semigroup $(S_t(s))_{s\geq 0}$ generated by $\mathcal{A}(t)$ satisfies $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$, for all $u \in \mathcal{H}$ and $s \geq 0$),
(iv) $\partial_t\mathcal{A}$ belongs to $L^\infty_*([0, T], B(D(\mathcal{A}(0)), \mathcal{H}))$, the space of equivalent classes of essentially bounded, strongly measurable functions from $[0, T]$ into the set $B(D(\mathcal{A}(0)), \mathcal{H})$ of bounded operators from $D(\mathcal{A}(0))$ into $\mathcal{H}$.

Then, problem (2) has a unique solution $U \in C([0, T], D(\mathcal{A}(0))) \cap C^1([0, T], \mathcal{H})$ for any initial datum in $D(\mathcal{A}(0))$.

Therefore, we will check the above assumptions for system (2).

Lemma 2.1. $D(\mathcal{A}(0))$ *is dense in* $\mathcal{H}$*,*

Proof. Let $(f, g, g_1, h_1, h_2)^T \in \mathcal{H}$ be orthogonal to all elements of $D(A(0))$, that is,

$$\begin{aligned}0 &= \langle (u, v, \varphi, \psi, z)^T, (f, g, g_1, h_1, h_2)^T\rangle_{\mathcal{H}} \\ &= \int_\Omega \{\varphi g_1 + a u_x f_x\} dx + \int_{L_1}^{L_2} \{\psi h_1 + b v_x g_x\} dx + \int_\Omega \int_0^1 z(x,\rho) h_2(x,\rho)\, d\rho dx,\end{aligned} \tag{14}$$

$\forall (u, v, \varphi, \psi, z)^T \in D(A(0))$.

Taking $u = v = \varphi = \psi = 0$ (then $u_x = v_x = 0$) and $z \in D(\Omega \times (0,1))$. As $(0,0,0,0,z)^T \in D(A(0))$, we obtain

$$\int_\Omega \int_0^1 z(x,\rho) h_2(x,\rho) d\rho dx = 0.$$

Since $D(\Omega \times (0,1))$ is dense in $L^2(\Omega \times (0,1))$, we deduce that $h_2 = 0$.

In the same way, by taking $u = v = \varphi = 0$ (then $u_x = v_x = 0$) and $\psi \in D(L_1, L_2)$. As $(0,0,0,\psi,0)^T \in D(A(0))$, we obtain

$$\int_{L_1}^{L_2} \psi h_1 \, dx = 0.$$

Since $D(L_1, L_2)$ is dense in $L^2(L_1, L_2)$, we deduce that $h_1 = 0$. Also for $u = v = 0$ (then $u_x = v_x = 0$) and $\varphi \in D(\Omega)$ we see that $g_1 = 0$. Therefore, for $(u,v) \in D(\Omega \times (L_1, L_2))$ (then $(u_x, v_x) \in D(\Omega \times (L_1, L_2))$) we obtain

$$\int_\Omega a u_x f_x \, dx + \int_{L_1}^{L_2} b v_x g_x dx = 0.$$

Since $D(\Omega \times (L_1, L_2))$ is dense in $L^2(\Omega \times (L_1, L_2))$, we deduce that $(f_x, g_x) = (0,0)$ then $(f,g) = (0,0)$ because $(f\,, g) \in X_*$. □

Assuming

$$\mu_2 \leq \sqrt{1-d}\,\mu_1. \tag{15}$$

In order to deduce a well-posedness result, we define on $\mathcal{H}$ the time dependent inner product

$$\begin{aligned} &\big\langle (u,v,\varphi,\psi,z)^T, (\bar{u},\bar{v},\bar{\varphi},\bar{\psi},\bar{z})^T \big\rangle_{\mathcal{H}} \\ &= \int_\Omega \{\varphi\bar{\varphi} + a u_x \bar{u}_x\} \, dx + \int_{L_1}^{L_2} \{\psi\bar{\psi} + b v_x \bar{v_x}\} dx \\ &+ \xi \tau(t) \int_\Omega \int_0^1 z(x,\rho) \bar{z}(x,\rho) d\rho dx, \end{aligned}$$

where ξ is the positive constant satisfying

$$\frac{\mu_2}{\sqrt{1-d}} \leq \xi \leq 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}. \tag{16}$$

Note that, from (15), such a constant ξ exists.

Lemma 2.2. *Let $\Phi = (u,v,\varphi,\psi,z)^T$, then*

$$\|\Phi\|_t \leq \|\Phi\|_s e^{\frac{d}{2\tau_0}|t-s|}, \; \forall t, s \in [0,T], \tag{17}$$

where d is a positive constant.

Proof. For all $s, t \in [0, T]$, we have

$$\begin{aligned}
&\|\Phi\|_t^2 - \|\Phi\|_s^2 e^{\left(\frac{d}{2\overline{\tau}_0}\right)|t-s|} \\
&= \left(1 - e^{\left(\frac{d}{2\overline{\tau}_0}\right)|t-s|}\right)\left(\int_\Omega (\{\varphi^2 + au_x^2\})dx + \int_{L_1}^{L_2}\{\psi^2 + bv_x^2\}dx\right) \\
&+ \xi\left(\tau(t) - \tau(s)e^{\left(\frac{d}{2\overline{\tau}_0}\right)|t-s|}\right)\int_\Omega\int_0^1 z^2(x,\rho)d\rho dx.
\end{aligned}$$

We notice that

$$e^{\left(\frac{d}{2\overline{\tau}_0}\right)|t-s|} \geq 1.$$

Moreover

$$\tau(t) - \tau(s)e^{\left(\frac{d}{2\overline{\tau}_0}\right)|t-s|} \leq 0,$$

for some $d > 0$.

Indeed,

$$\tau(t) = \tau(s) + \tau'(a)(t-s),$$

where $a, b \in (s, t)$, and thus,

$$\frac{\tau(t)}{\tau(s)} = 1 + \frac{|\tau'(a)|}{\tau(s)}|t-s|,$$

By (4), τ' is bounded on $[0, T]$ and therefore, recalling also (2),

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{d}{\overline{\tau}_0}|t-s| \leq e^{\frac{d}{2\overline{\tau}_0}|t-s|},$$

thus

$$\frac{\tau(t)}{\tau(s)} \leq e^{\frac{d}{2\overline{\tau}_0}|t-s|}.$$

This complete the proof. □

Lemma 2.3. *Under condition (16) the operator*

$$\mathcal{A}_1(t) = \mathcal{A}(t) - \kappa(t)I, \tag{18}$$

is dissipative, and

$$\frac{d}{dt}\mathcal{A}_1(t) \in L_*^\infty([0,T], B(D(A(0)), \mathcal{H})),$$

where

$$\kappa(t) = \frac{\sqrt{\tau'^2(t) + 1}}{2\tau(t)}. \tag{19}$$

Proof. Taking $U = (u, v, \varphi, \psi, z)^T \in D(\mathcal{A}(t))$. Then, for a fixed t,

$$\left\langle \mathcal{A}(t)\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \end{pmatrix}\right\rangle_t$$

$$
= \left\langle \begin{pmatrix} \varphi \\ \psi \\ au_{xx} - \mu_1 \varphi - \mu_2 z(.,1) \\ bv_{xx} \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \end{pmatrix} \right\rangle_t
$$

$$
= \int_\Omega \{(au_{xx} - \mu_1\varphi - \mu_2 z(.,1))\,\varphi + a\varphi_x u_x\} dx + \int_{L_1}^{L_2} \{bv_{xx}\psi + b\psi_x v_x\} dx
$$

$$
- \xi\tau(t) \int_\Omega \int_0^1 \frac{1 - \tau'(t)\rho}{\tau(t)} z_\rho z(x,\rho) d\rho dx\,.
$$

$$
= a\int_\Omega \varphi u_{xx} dx - \mu_1 \int_\Omega \varphi^2 dx - \mu_2 \int_\Omega \varphi z(.,1) dx + a\int_\Omega \varphi_x u_x dx
$$

$$
+ b\int_{L_1}^{L_2} v_{xx}\psi dx + b\int_{L_1}^{L_2} \psi_x v_x\, dx - \xi \int_\Omega \int_0^1 \left(1 - \tau'(t)\rho\right) z_\rho z(x,\rho) d\rho dx.
$$

Integrating by parts, we obtain

$$
\int_\Omega \int_0^1 \left(1 - \tau'(t)\rho\right) z_\rho z(x,\rho) d\rho dx
$$

$$
= \frac{\tau'(t)}{2} \int_\Omega \int_0^1 z^2(x,\rho) d\rho dx + \frac{1}{2}\int_\Omega \left\{ z^2(x,1)\left(1 - \tau'(t)\right) - z^2(x,0)\right\} dx.
$$

We get

$$
\begin{aligned}
\langle \mathcal{A}(t) U, U\rangle_t = {} & a\int_\Omega \varphi u_{xx} dx - \mu_1 \int_\Omega \varphi^2 dx - \mu_2 \int_\Omega \varphi z(.,1) dx + a\int_\Omega \varphi_x u_x dx \\
& + b\int_{L_1}^{L_2} v_{xx}\psi dx + b\int_{L_1}^{L_2} \psi_x v_x\, dx + \frac{\xi}{2}\int_\Omega z^2(x,0) dx \\
& - \frac{\xi\tau'(t)}{2} \int_\Omega\int_0^1 z^2(x,\rho) d\rho dx - \frac{\xi\left(1 - \tau'(t)\right)}{2}\int_\Omega z^2(x,1) dx,
\end{aligned}
$$

by fact that $z(x,0) = \varphi(x)$

$$
\begin{aligned}
\langle \mathcal{A}(t) U, U\rangle_t & \\
= {} & a\left[u_x\varphi\right]_{\partial\Omega} + b\left[v_x\psi\right]_{L_1}^{L_2} - \mu_1 \int_\Omega \varphi^2 dx - \mu_2 \int_\Omega \varphi z(.,1) dx \\
& + \frac{\xi}{2}\int_\Omega \varphi^2 dx - \frac{\xi\left(1 - \tau'(t)\right)}{2}\int_\Omega z^2(x,1) dx + \frac{\xi\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho) d\rho,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\langle \mathcal{A}(t) U, U\rangle_t = {} & a\left[u_x\varphi\right]_{\partial\Omega} + b\left[v_x\psi\right]_{L_1}^{L_2} - \left(\mu_1 - \frac{\xi}{2}\right)\int_\Omega \varphi^2 dx - \mu_2 \int_\Omega \varphi z(.,1) dx \\
& - \frac{\xi\left(1 - \tau'(t)\right)}{2}\int_\Omega z^2(x,1) dx + \frac{\xi\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho) d\rho,
\end{aligned}
\tag{21}
$$

Using Young's inequality, the third condition of (1) and the equality $\varphi(L_2) = \psi(L_2)$,we obtain

$$\langle \mathcal{A}(t)U, U\rangle_t \leq -\left(\mu_1 - \frac{\xi}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)\int_\Omega \varphi^2 dx$$

(22)
$$-\left(\frac{\xi(1-d)}{2} - \frac{\mu_2\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1)dx + k(t)\langle U, U\rangle,$$

where

(23)
$$\kappa(t) = \frac{\left(\tau'(t)^2 + 1\right)^{\frac{1}{2}}}{2\tau(t)}.$$

Consequently, using (16), we deduce that

$$\langle \mathcal{A}(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \leq 0.$$

Which means that the operator

$$\mathcal{A}(t)_1(t) = \mathcal{A}(t) - k(t)I,$$

is dissipative.

Moreover,

$$\kappa'(t) = \frac{\tau''(t)\tau'(t)}{2\tau(t)(\tau'^2+1)^{\frac{1}{2}}} - \frac{\tau'(t)(\tau'^2(t)+1)^{\frac{1}{2}}}{2\tau(t)^2},$$

is bounded on $[0, T]$ for all $T > 0$ (by (2) and (3) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = (0, 0, 0, 0, \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1)}{\tau(t)^2}z_\rho)^T,$$

with

$$\frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1)}{\tau(t)^2},$$

is bounded on $[0, T]$. Thus

(24)
$$\frac{d}{dt}\mathcal{A}_1(t) \in L_*^\infty([0, T], B(D(\mathcal{A}(0)), \mathcal{H})),$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0, T]$ into $B(D(\mathcal{A}(0)), \mathcal{H})$. □

Lemma 2.4. *For fixed $t > 0$ and $\lambda > 0$, the operator $\lambda I - \mathcal{A}(t)$ is surjective.*

Proof. Let $(f, g, g_1, h_1, h_2)^T \in \mathcal{H}$, we seek $U = (u, v, \varphi, \psi, z)^T \in D(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t))\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \\ g_1 \\ h_1 \\ h_2 \end{pmatrix},$$

that is verifying

$$\begin{aligned} \lambda u-\varphi &= f, \\ \lambda v-\psi &= g, \\ \lambda\varphi - au_{xx}+\mu_1\varphi+\mu_2 z(.,1) &= g_1, \\ \lambda\psi - bv_{xx} &= h_1, \\ \lambda z - \frac{\tau'(t)\rho-1}{\tau(t)} z_\rho &= h_2. \end{aligned} \tag{25}$$

Suppose that we have found (u,v) with the appropriate regularity. Then

$$\varphi = \lambda u - f, \tag{26}$$

$$\psi = \lambda v - g. \tag{27}$$

It is clear that $\varphi \in H^1(\Omega)$ and $\psi \in H^1(L_1,L_2)$, furthermore, by (25), we can find z as $z(x,0)=\varphi(x)$, $x\in\Omega$, using the approach as in Nicaise and Pignotti [14], we obtain, by using the equation in (25)

$$z(x,\rho)=\varphi(x)e^{-\lambda\rho\tau(t)}+\tau(t)\,e^{-\lambda\rho\tau(t)}\int_0^\rho h_2(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma,$$

if $\tau'(t)=0$, and

$$\begin{aligned} z(x,\rho) &= \varphi(x)e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)} \\ &\quad + e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)}\int_0^\rho \frac{h_2(x,\sigma)\tau(t)}{1-\tau'(t)\sigma}e^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\sigma)}d\sigma, \end{aligned}$$

otherwise.

By using (25), (26) and (27), the functions u,v satisfy the following equations

$$\lambda^2 u - au_{xx}+\mu_1 z(.,0)+\mu_2 z(.,1) = g_1+\lambda f, \tag{28}$$

$$\lambda^2 v - bv_{xx} = h_1+\lambda g. \tag{29}$$

Since

$$\begin{aligned} z(x,1) &= \lambda u e^{-\lambda\tau(t)} - f e^{-\lambda\tau(t)} + \tau(t)\,e^{-\lambda\tau(t)}\int_0^1 h_2(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma \\ &= \lambda u e^{-\lambda\tau(t)} + z_0(x), \end{aligned}$$

with

$$z_0(x) = -fe^{-\lambda\tau(t)}+\tau(t)\,e^{-\lambda\tau(t)}\int_0^1 h_2(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma, \text{ for } x\in\Omega,$$

if $\tau'(t)=0$, and

$$\begin{aligned} z(x,1) &= \lambda u e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))} - f e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))} \\ &\quad + e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))}\int_0^1 \frac{h_2(x,\sigma)\tau(t)}{1-\tau'(t)\sigma}e^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\sigma)}d\sigma, \end{aligned}$$

$$= \lambda u e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + z_0(x), \text{ for } x \in \Omega,$$

with

$$z_0(x) = -f e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} \int_0^1 \frac{h_2(x,\sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\sigma)} d\sigma,$$

otherwise.

The system (25)-(26) can be reformulated as

$$\text{(30)} \quad \begin{cases} \int_\Omega \left(\lambda^2 u - a u_{xx} + \mu_1 \lambda u + \lambda \mu_2 u e^{-\lambda \tau(t)}\right) \omega dx \\ \qquad = \int_\Omega \left(\mu_1 f + g_1 + \lambda f - \mu_2 z_0(x)\right) \omega dx, \\ \int_{L_1}^{L_2} \left(\lambda^2 v - b v_{xx}\right) \tilde{\omega} dx = \int_{L_1}^{L_2} \left(h_1 + \lambda g\right) \tilde{\omega} dx, \end{cases}$$

for any $(\omega, \tilde{\omega}) \in X_*$, if $\tau'(t) = 0$, and

$$\text{(31)} \quad \begin{cases} \int_\Omega \left(\lambda^2 u - a u_{xx} + \mu_1 \lambda u + \lambda \mu_2 u e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))}\right) \omega dx \\ \qquad = \int_\Omega \left(\mu_1 f + g_1 + \lambda f - \mu_2 z_0(x)\right) \omega dx, \\ \int_{L_1}^{L_2} \left(\lambda^2 v - b v_{xx}\right) \tilde{\omega} dx = \int_{L_1}^{L_2} \left(h_1 + \lambda g\right) \tilde{\omega} dx, \end{cases}$$

otherwise.

Integrating by parts, we obtain

$$\text{(32)} \quad \begin{cases} \int_\Omega \left(\lambda^2 + \mu_1 \lambda + \lambda \mu_2 e^{-\lambda \tau(t)}\right) u \omega dx + a \int_\Omega u_x \omega_x dx - a \left[u_x \omega\right]_{\partial\Omega} \\ \qquad = \int_\Omega \left(\mu_1 f + g_1 + \lambda f - \mu_2 z_0(x)\right) \omega dx, \\ \int_{L_1}^{L_2} \lambda^2 v \tilde{\omega} dx + b \int_{L_1}^{L_2} v_x \tilde{\omega}_x dx - b \left[v_x \tilde{\omega}\right]_{L_1}^{L_2} = \int_{L_1}^{L_2} \left(h_1 + \lambda g\right) \tilde{\omega} dx, \end{cases}$$

if $\tau'(t) = 0$, and

$$\text{(33)} \quad \begin{cases} \int_\Omega \left(\lambda^2 + \mu_1 \lambda + \lambda \mu_2 e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))}\right) u \omega dx + a \int_\Omega u_x \omega_x dx - a \left[u_x \omega\right]_{\partial\Omega} \\ \qquad = \int_\Omega \left(\mu_1 f + g_1 + \lambda f - \mu_2 z_0(x)\right) \omega dx, \\ \int_{L_1}^{L_2} \lambda^2 v \tilde{\omega} dx + b \int_{L_1}^{L_2} v_x \tilde{\omega}_x dx - b \left[v_x \tilde{\omega}\right]_{L_1}^{L_2} = \int_{L_1}^{L_2} \left(h_1 + \lambda g\right) \tilde{\omega} dx, \end{cases}$$

otherwise. The problem (32) and (33) is equivalent to the problem,

$$\text{(34)} \quad \Phi((u,v),(\omega,\tilde{\omega})) = l(\omega,\tilde{\omega}),$$

where the bilinear form $\Phi : (X_* \times X_*) \to \mathbb{R}$ and the linear form $l : X_* \to \mathbb{R}$ are defined by

$$\Phi((u,v),(\omega,\tilde{\omega})) = \int_\Omega \left(\lambda^2 + \mu_1 \lambda + \lambda \mu_2 e^{-\lambda \tau(t)}\right) u \omega dx + a \int_\Omega u_x \omega_x dx - a \left[u_x \omega\right]_{\partial\Omega} + \int_{L_1}^{L_2} \lambda^2 v \tilde{\omega} dx + b \int_{L_1}^{L_2} v_x \tilde{\omega}_x dx - b \left[v_x \tilde{\omega}\right]_{L_1}^{L_2},$$

$$l(\omega,\tilde{\omega})=\int_{\Omega}\left(\mu_1 f+g_1+\lambda f-\mu_2 y_0(x)\right)\omega dx+\int_{L_1}^{L_2}\left(h_1+\lambda g\right)\tilde{\omega}dx,$$

if $\tau'(t)=0$, and

$$\begin{aligned}&\Phi((u,v),(\omega,\tilde{\omega}))\\&=\int_{\Omega}\left(\lambda^2+\mu_1\lambda+\lambda\mu_2 e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))}\right)u\omega dx+a\int_{\Omega}u_x\omega_x dx-a\left[u_x\omega\right]_{\partial\Omega}\\&+\int_{L_1}^{L_2}\lambda^2 v\tilde{\omega}dx+b\int_{L_1}^{L_2}v_x\tilde{\omega}_x dx-b\left[v_x\tilde{\omega}\right]_{L_1}^{L_2},\end{aligned}$$

$$l(\omega,\tilde{\omega})=\int_{\Omega}\left(\mu_1 f+g_1+\lambda f-\mu_2 z_0(x)\right)\omega dx+\int_{L_1}^{L_2}\left(h_1+\lambda g\right)\tilde{\omega}dx,$$

otherwise.

Using the properties of the space X_*, it is clear that Φ is continuous and coercive, and l is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\omega,\tilde{\omega})\in X_*$, problem (34) admits a unique solution $(u,v)\in X_*$. It follows from (32) and (33) that $(u,v)\in\{\left(H^2(\Omega)\times H^2(L_1,L_2)\right)\cap X_*\}$. In conclusion, we have found $U=(u,v,\varphi,\psi,z)^T\in D(\mathcal{A}(t))$, which verifies (25), and thus $(\lambda I-\mathcal{A}(t))$ is surjective for some $\lambda>0$ and $t>0$. Again as $\kappa(t)>0$, this proves that

$$\lambda I-\mathcal{A}_1(t)=(\lambda+\kappa(t))I-\mathcal{A}(t),\quad\text{is surjective,}\tag{35}$$

for any $\lambda>0$ and $t>0$. ☐

Theorem 2.2. *The operator A generates a C_0-semigroup on H. For any $U_0\in\mathcal{H}$, the problem (9) possesses a unique weak solution $U\in C([0,+\infty),\mathcal{H})$. Moreover, if $U_0\in D(\mathcal{A}(0))$, then U is a strong solution , i. e*

$$U\in C([0,+\infty),D(\mathcal{A}(0)))\cap C^1([0,+\infty),\mathcal{H}).$$

Proof. Results (17), (18) and (35) imply that the family $\mathcal{A}_1=\{\mathcal{A}_1(t):t\in[0,T]\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of t. Therefore, all assumptions of Theorem 2.1 are satisfied by (13), Lemma2.1–Lemma2.4, and thus, the problem

$$\begin{aligned}\tilde{U}'&=\mathcal{A}_1(t)\tilde{U},\\\tilde{U}(0)&=U_0,\end{aligned}$$

has a unique solution $\tilde{U}\in C([0,+\infty),D(\mathcal{A}(0)))\cap C^1([0,+\infty),\mathcal{H})$ for $U_0\in D(\mathcal{A}(0))$. The requested solution of (2) is then given by

$$U(t)=e^{B(t)}\tilde{U}(t),$$

with $B(t)=\int_0^t\kappa(s)ds$ because

$$\begin{aligned}U'(t)&=\kappa(t)e^{B(t)}\tilde{U}(t)+e^{B(t)}\tilde{U}'(t),\\U'(t)&=\kappa(t)e^{B(t)}\tilde{U}(t)+e^{B(t)}\mathcal{A}_1(t)\tilde{U}(t),\end{aligned}$$

$$U'(t) \;=\; \mathcal{A}(t)e^{B(t)}\tilde{U}(t) = \mathcal{A}(t)U(t).$$

This concludes the proof. □

3. Stability result

In this section we study the asymptotic behavior of the system (1). For any regular solution of (1), we give the total energy as

$$E(t) = \frac{1}{2}\int_{\Omega} u_t^2(x,t)dx + \frac{a}{2}\int_{\Omega} u_x^2(x,t)dx + \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)dx$$
$$+\frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)dx + \frac{\xi}{2}\int_{\Omega}\int_{t-\tau(t)}^{t} u_t^2(x,s)dsdx, \tag{36}$$

where ξ is the positive constant defined by (16). Our next main result reads as.

Theorem 3.1. *Let (u,v) be the solution of (1). Assume that $\mu_2 > \mu_1$ and*

$$\frac{a}{b} < \frac{L_3 + L_1 - L_2}{2(L_2 - L_1)}. \tag{37}$$

Then there exist two positive constants W and w, such that

$$E(t) \le We^{-wt}, \quad \forall t \ge 0. \tag{38}$$

To prove Theorem 3.1, we use the following lemmas. First, we will need an explicit formula of energy derivative. The following energy functional law holds.

Lemma 3.1. *Let (u,v,z) be the solution of (1). Assume that $\mu_1 \ge \mu_2$. Then we have the inequality*

$$\frac{dE(t)}{dt} \;\le\; \left(-\mu_1 + \frac{\mu_2\sqrt{1-d}}{2} + \frac{\xi}{2}\right)\int_{\Omega} u_t^2(x,t)dx$$
$$-\left(\frac{\xi\,(1-d)}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)\int_{\Omega} u_t^2\left(x,t-\tau\left(t\right)\right)dx. \tag{39}$$

Proof. From (36) we have

$$\frac{dE_1(t)}{dt} = a\int_{\Omega} u_t(x,t)u_{xx}\left(x,t\right)dx - \mu_1\int_{\Omega} u_t^2(x,t)dx$$
$$-\mu_2\int_{\Omega} u_t(x,t)u_t\left(x,t-\tau\left(t\right)\right)dx + a\int_{\Omega} u_{xt}(x,t)u_x(x,t)dx.$$

where

$$E_1(t) = \frac{1}{2}\int_{\Omega} u_t^2(x,t)dx + \frac{a}{2}\int_{\Omega} u_x^2(x,t)dx.$$

Using system (2), and integrating by parts, we obtain

$$\frac{dE_1(t)}{dt} = a\int_{\Omega} u_t(x,t)u_{xx}\left(x,t\right)dx - \mu_1\int_{\Omega} u_t^2(x,t)dx$$

$$- \mu_2 \int_\Omega u_t(x,t) u_t\left(x, t-\tau(t)\right) dx$$
$$+ a[u_x u_t]_{\partial\Omega} - a \int_\Omega u_{xx}(x,t)\, u_t(x,t)\, dx.$$

applying Young's inequality

$$\frac{dE_1(t)}{dt} \leq -\left(\mu_1 - \frac{\mu_2\sqrt{1-d}}{2}\right)\int_\Omega u_t^2(x,t)dx + \frac{\mu_2}{2\sqrt{1-d}}\int_\Omega u_t^2\left(x, t-\tau(t)\right)dx + a[u_x u_t]_{\partial\Omega}. \tag{40}$$

On the other hand,

$$\frac{dE_2(t)}{dt} = b[v_x v_t]_{L_1}^{L_2}. \tag{41}$$

where

$$E_2(t) = \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)dx.$$

Using the fact that

$$\frac{d}{dt}\int_\Omega \int_{t-\tau(t)}^{t} u_t^2(x,s)dsdx = \int_\Omega u_t^2(x,t)dx - \left(1-\tau'(t)\right)\int_\Omega u_t^2(x,t-\tau(t))dx, \tag{42}$$

collecting (40), (41), (42), using boundary conditions and applying Young's inequality, we show that (39) holds. The proof is complete. □

Following [2], we define the functional

$$I(t) = \int_\Omega \int_{t-\tau(t)}^{t} e^{s-t} u_t^2(x,s)dsdx,$$

and state the following lemma.

Lemma 3.2. *Let (u,v) be the solution of (2). Then*

$$\frac{dI(t)}{dt} \leq \int_\Omega u_t^2(x,t)dx - (1-d)\, e^{-\bar{\tau}} \int_\Omega u_t^2(x,t-\tau(t))dx - e^{-\bar{\tau}} \int_\Omega \int_{t-\tau(t)}^{t} u_t^2(x,s)dsdx. \tag{43}$$

Now, we define the functional $\mathfrak{D}(t)$ as follows

$$\mathfrak{D}(t) = \int_\Omega u u_t dx + \frac{\mu_1}{2}\int_\Omega u^2 dx + \int_{L_1}^{L_2} v v_t dx. \tag{44}$$

Then, we have the following estimate.

Lemma 3.3. *The functional $\mathfrak{D}(t)$ satisfies*

$$\frac{d\mathfrak{D}(t)}{dt} \leq -\left(a-\epsilon_0 c_0^2\right)\int_\Omega u_x^2\left(x,t\right)dx - b\int_{L_1}^{L_2} v_x^2\left(x,t\right)dx$$

$$+\int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2\,dx + C\left(\epsilon_0\right)\int_\Omega u_t^2\left(x,t-\tau\left(t\right)\right)dx. \tag{45}$$

Proof. Taking the derivative of $\mathfrak{D}(t)$ with respect to t and using (1), we find that

$$\frac{d\mathfrak{D}(t)}{dt} = \int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx - a\int_\Omega u_x^2\left(x,t\right)dx - b\int_{L_1}^{L_2} v_x^2\left(x,t\right)dx$$

$$-\mu_2\int_\Omega u u_t\left(x,t-\tau\left(t\right)\right)dx + a\left[uu_x\right]_{\partial\Omega} + b\left[vv_x\right]_{L_1}^{L_2}. \tag{46}$$

Using the boundary conditions , we have

$$a\left[uu_x\right]_{\partial\Omega} + b\left[vv_x\right]_{L_1}^{L_2} = 0. \tag{47}$$

On the other hand, we have by Poincaré's and Young's inequalities,

$$\mu_2\int_\Omega uu_t\left(x,t-\tau\left(t\right)\right)dx \leq \epsilon_0\int_\Omega u^2 dx + C(\epsilon_0)\int_\Omega u_t^2\left(x,t-\tau\left(t\right)\right)dx$$

$$\leq \epsilon_0 c_0^2\int_\Omega u_x^2 dx + C(\epsilon_0)\int_\Omega u_t^2\left(x,t-\tau\left(t\right)\right)dx,$$

where c_0 is the Poincaé's constant. Consequently, plugging the above estimates into (46), we find (45). □

Now, inspired by [10], we introduce the functional

$$q(x) = \begin{cases} x-\frac{L_1}{2}, & x\in[0,L_1],\\ x-\frac{L_2+L_3}{2}, & x\in[L_2,L_3],\\ \frac{L_2-L_3-L_1}{2(L_2-L_1)}(x-L_1)+\frac{L_1}{2}, & x\in[L_1,L_2]. \end{cases} \tag{48}$$

Next, in order to construct the Lyapounov function, we define the functionals

$$\mathcal{L}_1(t) = -\int_\Omega q(x)u_x u_t dx, \quad \mathcal{L}_2(t) = -\int_{L_1}^{L_2} q(x)v_x v_t dx.$$

Then, we have the following estimates.

Lemma 3.4. *For any $\epsilon_2 > 0$, we have the estimates*

(49)

$$\frac{d\mathcal{L}_1(t)}{dt} \leq C\left(\epsilon_2\right)\int_\Omega u_t^2 dx + \left(\frac{a}{2}+\epsilon_2\right)\int_\Omega u_x^2 dx$$

$$+C\left(\epsilon_2\right)\int_\Omega u_t^2\left(x,t-\tau\left(t\right)\right)dx - \frac{a}{4}[\left(L_3-L_2\right)u_x^2\left(L_3,t\right)+L_1 u_x^2\left(L_2,t\right)],$$

and

$$\frac{d\mathcal{L}_2(t)}{dt} \leq \frac{L_2-L_3-L_1}{4\left(L_2-L_1\right)}\left(\int_{L_1}^{L_2} v_t^2 dx+\int_{L_1}^{L_2} b v_x^2 dx\right) + \frac{b}{4}\left(\left(L_3-L_2\right) v_x^2\left(L_2, t\right)+L_1 v_x^2\left(L_1, t\right)\right). \tag{50}$$

Proof. Taking the derivative of $\mathcal{L}_1(t)$ with respect to t and using (1), we obtain

$$\frac{d\mathcal{L}_1(t)}{dt} = -\int_\Omega q(x) u_{xt} u_t dx - a \int_\Omega q(x) u_x u_{xx}(x,t)\, dx + \mu_1 \int_\Omega q(x) u_x u_t(x,t)\, dx + \mu_2 \int_\Omega q(x) u_x u_t(x, t-\tau(t))\, dx. \tag{51}$$

Integrating by parts,

$$\int_\Omega q(x) u_{tx} u_t dx = -\frac{1}{2}\int_\Omega q'(x) u_t^2 dx + \frac{1}{2}[q(x) u_t^2]_{\partial\Omega}. \tag{52}$$

On the other hand,

$$\int_\Omega a q(x) u_{xx} u_x dx = -\frac{1}{2}\int_\Omega a q'(x) u_x^2 dx + \frac{1}{2}[a q(x) u_x^2]_{\partial\Omega}. \tag{53}$$

Substituting (52) and (53) in (51), we find that

$$\frac{d\mathcal{L}_1(t)}{dt} = \frac{1}{2}\int_\Omega q'(x) u_t^2 dx - \frac{1}{2}[q(x) u_t^2]_{\partial\Omega} + \frac{1}{2}\int_\Omega a q'(x) u_x^2 dx - \frac{1}{2}[a q(x) u_x^2]_{\partial\Omega} + \mu_1 \int_\Omega q(x) u_x u_t(x,t)\, dx + \mu_2 \int_\Omega q(x) u_x u_t(x, t-\tau(t))\, dx. \tag{54}$$

Using Young's inequality and (48), equation (54) becomes

$$\frac{d\mathcal{L}_1(t)}{dt} \leq C(\epsilon_2)\int_\Omega u_t^2 dx + \left(\frac{a}{2}+\epsilon_2\right)\int_\Omega u_x^2 dx + C(\epsilon_2)\int_\Omega u_t^2(x, t-\tau(t))\, dx - \frac{a}{2}[q(x) u_x^2]_{\partial\Omega} - \frac{1}{2}[q(x) u_t^2]_{\partial\Omega}. \tag{55}$$

Since $q(L_1) > 0$ and $q(L_2) < 0$, by using the boundary condition, we have

$$\frac{1}{2}[q(x) u_t^2]_{\partial\Omega} \geq 0. \tag{56}$$

Also, we have

$$-\frac{a}{2}[q(x) u_x^2]_{\partial\Omega} = -\frac{a\left(L_3-L_2\right)}{4}[u_x^2(L_3,t) + u_x^2(L_2,t)]. \tag{57}$$

Taking into account (56) and (57), then (55) gives (49).

By the same method, taking the derivative of $\mathcal{L}_2(t)$ with respect to t, we obtain

(58)

$$\begin{aligned}
\frac{d\mathcal{L}_2(t)}{dt} &= -\int_{L_1}^{L_2} q(x)v_{xt}v_t dx - \int_{L_1}^{L_2} q(x)v_x v_{tt} dx \\
&= \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2 dx + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2 dx - \frac{1}{2}[q(x)v_t^2]_{L_1}^{L_2} - \frac{b}{2}[q(x)u_x^2]_{L_1}^{L_2} \\
&\leq \frac{L_2 - L_3 - L_{1,}}{4\left(L_2 - L_{1,}\right)}\left(\int_{L_1}^{L_2} v_t^2 dx + \int_{L_1}^{L_2} bv_x^2 dx\right) \\
&+ \frac{b}{4}\left(\left(L_3 - L_2\right)v_x^2\left(L_2, t\right) + L_1 v_x^2\left(L_1, t\right)\right).
\end{aligned}$$

which is exactly (50). □

We define the Lyapunov functional

(59) $$\mathfrak{L}(t) = NE(t) + I(t) + \gamma_2\mathfrak{D}(t) + \gamma_3\mathcal{L}_1(t) + \gamma_4\mathcal{L}_2(t),$$

where N, γ_2, γ_3 and γ_4 are positive constants.

Proof of the Theorem 3.1. Now, it is clear from the boundary conditions, that

(60) $$a^2u_x^2(L_i, t) = b^2v_x^2(L_i, t), \quad i = 1, 2.$$

Taking the derivative of (59) with respect to t and making use of (39)-(49) and taking into account (60), we obtain

(61)

$$\begin{aligned}
\frac{d\mathfrak{L}(t)}{dt} \leq\ & \left\{N\left(-\mu_1 + \frac{\mu_2\sqrt{1-d}}{2} + \frac{\xi}{2}\right) + 1 + \gamma_2 + \gamma_3 C\left(\epsilon_2\right)\right\}\int_\Omega u_t^2(x,t)dx \\
& + \left\{N\left(\frac{\mu_2}{2\sqrt{1-d}} - \frac{\xi\left(1-d\right)}{2}\right) - \left(1-d\right)e^{-\bar{\tau}}\right. \\
& \left. + \gamma_2 C\left(\epsilon_0\right) + \gamma_3 C\left(\epsilon_2\right)\right\}\int_\Omega u_t^2\left(x, t-\tau\left(t\right)\right)dx \\
& + \left(\gamma_2\left(-a + \epsilon_0 c_0^2\right) + \gamma_3\epsilon_2 + \frac{\gamma_3 a}{2}\right)\int_\Omega u_x^2\left(x,t\right)dx \\
& + \left(\gamma_4\frac{L_2 - L_3 - L_{1,}}{4\left(L_2 - L_{1,}\right)} - \gamma_2 b\right)\int_{L_1}^{L_2} v_x^2\left(x,t\right)dx \\
& + \left(\gamma_2 + \gamma_4\frac{L_2 - L_3 - L_1}{4\left(L_2 - L_1\right)}\right)\int_{L_1}^{L_2} v_t^2 dx \\
& - e^{-\bar{\tau}}\int_\Omega\int_{t-\tau(t)}^t u_t^2(x,s)dsdx - \left(\gamma_3 - \frac{a}{b}\gamma_4\right)\frac{a\left(L_3 - L_2\right)}{4}u_x^2\left(L_2, t\right) \\
& - \left(\gamma_3 - \frac{a}{b}\gamma_4\right)\frac{aL_1}{4}u_x^2\left(L_1, t\right).
\end{aligned}$$

At this point, we choose our constants in (61), carefully, such that all the coefficients in (61) will be negative. Indeed, under the assumption (37), we can always find γ_2, γ_3 and γ_4 such that

$$\frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\gamma_4 + \gamma_2 < 0, \quad \gamma_3 > \frac{a}{b}\gamma_4, \quad \gamma_2 > \frac{\gamma_3}{2}. \tag{62}$$

Once the above constants are fixed, we may choose ϵ_2 and ϵ_0 small enough such that

$$\gamma_2\epsilon_0 c_0^2 + \gamma_3\epsilon_2 < a\left(\gamma_2 - \frac{\gamma_3}{2}\right).$$

Finally, keeping in mind (2) and choosing N large enough such that the first and the second coefficients in (61) are negatives.

Consequently, from the above, we deduce that there exist a positive constant η_1, such that (61) becomes

$$\begin{aligned}\frac{d\mathfrak{L}(t)}{dt} \leq &-\eta_1 \int_\Omega \left(u_t^2(x,t) + u_x^2(x,t) + u_t^2(x, t-\tau(t))\right)dx \\ &- \eta_1 \int_\Omega \left(v_t^2(x,t) + v_x^2(x,t)\right)dx - \eta_1 \int_\Omega \int_{t-\tau(t)}^t u_t^2(x,s)ds dx.\end{aligned} \tag{63}$$

Consequently, recalling (36), we deduce that there exist also $\eta_2 > 0$, such that

$$\frac{d\mathfrak{L}(t)}{dt} \leq -\eta_2 E(t), \quad \forall t \geq 0. \tag{64}$$

On the other hand, it is not hard to see that from (59) and for N large enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq \mathfrak{L}(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \tag{65}$$

Combining (64) and (65), we deduce that there exists $\Lambda > 0$ for which the estimate

$$\frac{d\mathfrak{L}(t)}{dt} \leq -\Lambda\mathfrak{L}(t), \quad \forall t \geq 0, \tag{66}$$

holds. Integrating (64) over $(0,t)$ once again, then (38) holds. Then, the proof is complete. □

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