

Hybrid mixed type iteration scheme for asymptotically nonexpansive mappings and total asymptotically nonexpansive non-self mappings

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ABSTRACT. In this paper, we proposed a new two-step iteration scheme of hybrid mixed type for two asymptotically nonexpansive self mappings and two total asymptotically nonexpansive non-self mappings and establish some strong convergence theorems for mentioned scheme and mappings in real Banach spaces. Our results extend and generalize several results from the current existing literature.

1. INTRODUCTION AND PRELIMINARIES

Let K be a nonempty subset of a real Banach space E . Let $T: K \rightarrow K$ be a nonlinear mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of four mappings S_1, S_2, T_1 and T_2 will be denoted by $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$.

Definition 1.1. A mapping $T: K \rightarrow K$ is said to be asymptotically nonexpansive [5] if there exists a positive sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$(1) \quad \|T^n(x) - T^n(y)\| \leq k_n \|x - y\|$$

for all $x, y \in K$ and $n \in \mathbf{N}$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on C , then T has a fixed point.

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Example 1.1. (See [5]) Let D be the closed unit ball in the Hilbert space $H = \ell_2$ and $T: D \rightarrow D$ a mapping defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_1^2, a_2x_2, a_3x_3, \dots)$$

where $\{a_i\}$ is a sequence of real numbers such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$.

Then

$$\|T(x) - T(y)\| \leq 2\|x - y\| \quad \text{for all } x, y \in D,$$

that is, T is Lipschitzian, but not nonexpansive.

Observe that

$$\|T^n(x) - T^n(y)\| \leq 2 \left(\prod_{i=2}^n a_i \right) \|x - y\| \quad \text{for all } x, y \in D, n \geq 2.$$

Here $k_n = 2 \prod_{i=2}^n a_i \rightarrow 1$ as $n \rightarrow \infty$. Therefore T is an asymptotically nonexpansive mapping.

Definition 1.2. A mapping $T: K \rightarrow K$ is said to be total asymptotically nonexpansive [1] if

$$(2) \quad \|T^n(x) - T^n(y)\| \leq \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n,$$

for all $x, y \in K$ and $n \in \mathbf{N}$, where $\{\mu_n\}$ and $\{\nu_n\}$ are nonnegative real sequences such that $\mu_n \rightarrow 0$ and $\nu_n \rightarrow 0$ as $n \rightarrow \infty$ and a strictly increasing continuous function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$.

From the above definition, we see that the class of total asymptotically nonexpansive mappings include the class of asymptotically nonexpansive mappings as a special case; see also [4] for more details.

Remark 1.1. From the above definition, it is clear that each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\nu_n = 0$, $\mu_n = k_n - 1$ for all $n \geq 1$, $\psi(t) = t$, $t \geq 0$.

Definition 1.3. A subset K of a Banach space E is said to be a retract of E if there exists a continuous mapping $P: E \rightarrow K$ (called a retraction) such that $P(x) = x$ for all $x \in K$. If, in addition P is nonexpansive, then P is said to be a nonexpansive retract of E .

If $P: E \rightarrow K$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

Definition 1.4. Let K be a nonempty closed and convex subset of a Banach space E . A non-self mapping $T: K \rightarrow E$ is said to be total asymptotically nonexpansive [14] if there exist sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $[0, \infty)$ with

$\mu_n \rightarrow 0$ and $\nu_n \rightarrow 0$ as $n \rightarrow \infty$ and a strictly increasing continuous function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$ such that

$$(3) \|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \leq \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n,$$

for all $x, y \in K$ and $n \in \mathbf{N}$.

Example 1.2. Let E be the real line with the usual norm $|\cdot|$, $C = [0, \infty)$ and P be the identity mapping. Assume that $T_1(x) = x$ and $T_2(x) = \sin x$ for all $x \in C$. Let ϕ be the strictly increasing continuous function such that $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\phi(0) = 0$. Let $\{\mu_n\}_{n \geq 1}$ and $\{\nu_n\}_{n \geq 1}$ be two nonnegative real sequences defined by $\mu_n = \frac{1}{n^2}$ and $\nu_n = \frac{1}{n^3}$ for all $n \geq 1$ with $\mu_n \rightarrow 0$ and $\nu_n \rightarrow 0$ as $n \rightarrow \infty$. Then T_1 and T_2 are total asymptotically nonexpansive mappings with common fixed point 0, that is, $F = F(T_1) \cap F(T_2) = \{0\}$.

Definition 1.5. A mapping $T: K \rightarrow K$ is said to be completely continuous if $\{Tx_n\}$ has a convergent subsequence in K whenever $\{x_n\}$ is bounded in K .

Definition 1.6. A mapping $T: K \rightarrow K$ is said to be semi-compact if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, then there exists a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $x_{n_r} \rightarrow x^* \in K$ strongly as $r \rightarrow \infty$.

In 2004, Chidume et al. [3] studied the following iteration scheme:

$$(4) \quad \begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}x_n + (1 - \alpha_n)x_n), \quad n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, and K is a nonempty closed convex subset of a real uniformly convex Banach space E , P is a nonexpansive retraction of E onto K , and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings in the intermediate sense in the framework of uniformly convex Banach spaces.

In 2006, Wang [12] generalized the iteration process (4) as follows:

$$(5) \quad \begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad n \geq 1, \end{aligned}$$

where $T_1, T_2: K \rightarrow E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1)$, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings.

In 2012, Guo et al. [6] generalized the iteration process (5) as follows:

$$x_1 = x \in K,$$

$$(6) \quad \begin{aligned} x_{n+1} &= P((1 - \alpha_n)S_1^n x_n + \alpha_n T_1 (PT_1)^{n-1} y_n), \\ y_n &= P((1 - \beta_n)S_2^n x_n + \beta_n T_2 (PT_2)^{n-1} x_n), \quad n \geq 1, \end{aligned}$$

where $S_1, S_2: K \rightarrow K$ are two asymptotically nonexpansive self mappings and $T_1, T_2: K \rightarrow E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1)$, and proved some strong and weak convergence theorems for mixed type asymptotically nonexpansive mappings.

Now, we define the hybrid mixed type iteration scheme.

Let E be a real Banach space, K be a nonempty closed convex subset of E and $P: E \rightarrow K$ be a nonexpansive retraction of E onto K . Let $S_1, S_2: K \rightarrow K$ be two asymptotically nonexpansive self mappings and $T_1, T_2: K \rightarrow E$ are two total asymptotically nonexpansive non-self mappings. Then the hybrid mixed type iteration scheme for the mentioned mappings is as follows:

$$(7) \quad \begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1 - \alpha_n)S_1^n x_n + \alpha_n T_1 (PT_1)^{n-1} y_n), \\ y_n &= P((1 - \beta_n)S_2^n x_n + \beta_n T_2 (PT_2)^{n-1} x_n), \quad n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$.

The purpose of this paper is to study newly proposed hybrid mixed type iteration scheme (7) and establish some strong convergence theorems in the setting of real Banach spaces.

Next, we need the following useful lemma to prove our main results.

Lemma 1.1. (See [11]) *Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + r_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^\infty \beta_n < \infty$ and $\sum_{n=1}^\infty r_n < \infty$, then

- (i) *$\lim_{n \rightarrow \infty} \alpha_n$ exists;*
- (ii) *In particular, if $\{\alpha_n\}_{n=1}^\infty$ has a subsequence which converges strongly to zero, then $\lim_{n \rightarrow \infty} \alpha_n = 0$.*

2. MAIN RESULTS

In this section, we prove some strong convergence theorems of iteration scheme (7) for two asymptotically nonexpansive mappings and two total asymptotically nonexpansive non-self mappings in real Banach spaces. First, we shall need the following lemma.

Lemma 2.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E . Let $S_1, S_2: K \rightarrow K$ be two asymptotically nonexpansive self mappings with sequences $\{h_n\}, \{k_n\} \in [1, \infty)$ and $T_1, T_2: K \rightarrow E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}, \{\mu_{n_1}\} \in [0, \infty)$ and $\{\nu_n\}, \{\nu_{n_1}\} \in [0, \infty)$. Let $\{x_n\}$ be the sequence defined by (7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. Suppose that $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$. If the following conditions hold:*

- (i) $\sum_{n=1}^{\infty} (h_n k_n - 1) < \infty$;
- (ii) $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \mu_{n_1} < \infty, \sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \nu_{n_1} < \infty$;
- (iii) *there exists a constant $M > 0$ such that $\psi(t) \leq M t, t \geq 0$.*

Then $\lim_{n \rightarrow \infty} \|x_n - q\|$ and $\lim_{n \rightarrow \infty} d(x_n, F)$ both exist for all $q \in F$.

Proof. Let $q \in F$ and let $\rho_n = \max\{\mu_n, \mu_{n_1}\}$, $\theta_n = \max\{\nu_n, \nu_{n_1}\}$ with $\sum_{n=1}^{\infty} \rho_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$. From (7), we have

$$\begin{aligned}
 \|y_n - q\| &= \|P((1 - \beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} x_n) - P(q)\| \\
 &\leq \|(1 - \beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} x_n - q\| \\
 &= \|(1 - \beta_n)(S_2^n x_n - q) + \beta_n(T_2(PT_2)^{n-1} x_n - q)\| \\
 &\leq (1 - \beta_n)\|S_2^n x_n - q\| + \beta_n\|T_2(PT_2)^{n-1} x_n - q\| \\
 &\leq (1 - \beta_n)k_n\|x_n - q\| + \beta_n[\|x_n - q\| + \mu_{n_1}\psi(\|x_n - q\|) + \nu_{n_1}] \\
 &\leq (1 - \beta_n)k_n\|x_n - q\| + \beta_n[\|x_n - q\| + \rho_n M\|x_n - q\| + \theta_n] \\
 &\leq (1 - \beta_n)k_n\|x_n - q\| + \beta_n k_n\|x_n - q\| + \rho_n M\|x_n - q\| + \theta_n \\
 (8) \quad &= (k_n + \rho_n M)\|x_n - q\| + \theta_n.
 \end{aligned}$$

Again using (7), we have

$$\begin{aligned}
 \|x_{n+1} - q\| &= \|P((1 - \alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n) - P(q)\| \\
 &\leq \|(1 - \alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n - q\| \\
 &= \|(1 - \alpha_n)(S_1^n x_n - q) + \alpha_n(T_1(PT_1)^{n-1} y_n - q)\| \\
 &\leq (1 - \alpha_n)\|S_1^n x_n - q\| + \alpha_n\|T_1(PT_1)^{n-1} y_n - q\| \\
 &\leq (1 - \alpha_n)h_n\|x_n - q\| + \alpha_n[\|y_n - q\| + \mu_n\psi(\|y_n - q\|) + \nu_n] \\
 &\leq (1 - \alpha_n)h_n\|x_n - q\| + \alpha_n[\|y_n - q\| + \rho_n M\|y_n - q\| + \theta_n] \\
 (9) \quad &\leq (1 - \alpha_n)h_n\|x_n - q\| + \alpha_n(1 + \rho_n M)\|y_n - q\| + \theta_n.
 \end{aligned}$$

Using equation (8) in (9), we obtain

$$\begin{aligned}
 \|x_{n+1} - q\| &\leq (1 - \alpha_n)h_n\|x_n - q\| + \alpha_n(1 + \rho_n M) \times \\
 &\quad [(k_n + \rho_n M)\|x_n - q\| + \theta_n] + \theta_n \\
 &\leq (1 - \alpha_n)h_n\|x_n - q\| + \alpha_n(1 + \rho_n M) \times \\
 &\quad (k_n + \rho_n M)\|x_n - q\| + (1 + \rho_n M)\theta_n + \theta_n \\
 &= [(1 - \alpha_n)h_n + \alpha_n(1 + \rho_n M)(k_n + \rho_n M)]\|x_n - q\| \\
 &\quad + (2 + \rho_n M)\theta_n
 \end{aligned}$$

$$\begin{aligned}
&\leq [(1 - \alpha_n)h_n k_n + \alpha_n h_n k_n + \rho_n M(1 + k_n + \rho_n M)] \|x_n - q\| \\
&\quad + (2 + \rho_n M)\theta_n \\
&= [1 + (h_n k_n - 1) + \rho_n M(1 + k_n + \rho_n M)] \|x_n - q\| \\
&\quad + (2 + \rho_n M)\theta_n \\
(10) \quad &\leq (1 + t_n) \|x_n - q\| + R\theta_n
\end{aligned}$$

where $t_n = (h_n k_n - 1) + \rho_n M(1 + k_n + \rho_n M)$ and for some $R > 0$. Since $\sum_{n=1}^{\infty} (h_n k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \rho_n < \infty$, it follows that $\sum_{n=1}^{\infty} t_n < \infty$. Now, since $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$, it follows from Lemma 1.1 that $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists.

Now, taking the infimum over all $q \in F$ in (10), we have

$$(11) \quad d(x_{n+1}, F) \leq [1 + t_n]d(x_n, F) + R\theta_n$$

for all $n \in \mathbf{N}$. it follows from $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \theta_n < \infty$ and Lemma 1.1 that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. This completes the proof. \square

Theorem 2.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E . Let $S_1, S_2: K \rightarrow K$ be two asymptotically nonexpansive self mappings with sequences $\{h_n\}, \{k_n\} \in [1, \infty)$ and $T_1, T_2: K \rightarrow E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}, \{\mu_{n_1}\} \in [0, \infty)$ and $\{\nu_n\}, \{\nu_{n_1}\} \in [0, \infty)$. Let $\{x_n\}$ be the sequence defined by (7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. Suppose that $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$ is closed. If the following conditions hold:*

- (i) $\sum_{n=1}^{\infty} (h_n k_n - 1) < \infty$;
- (ii) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \mu_{n_1} < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \nu_{n_1} < \infty$;
- (iii) there exists a constant $M > 0$ such that $\psi(t) \leq M t$, $t \geq 0$.

Then $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2 if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Proof. The necessity is obvious. Indeed, if $x_n \rightarrow q \in F$ as $n \rightarrow \infty$, then

$$d(x_n, F) = \inf_{q \in F} d(x_n, q) \leq \|x_n - q\| \rightarrow 0 \quad (n \rightarrow \infty).$$

Thus $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

Conversely, suppose that $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. By Lemma 2.1, we have that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. Further, by assumption $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, from (11) and Lemma 1.1(ii), we conclude that $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence in E . Indeed, from (10), we have

$$\|x_{n+1} - q\| \leq [1 + t_n] \|x_n - q\| + R\theta_n$$

for each $n \in \mathbf{N}$, with $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \theta_n < \infty$ and $q \in F$. For any $m, n, m > n \in \mathbf{N}$, we have

$$\|x_m - q\| \leq [1 + t_{m-1}] \|x_{m-1} - q\| + R\theta_{m-1}$$

$$\begin{aligned}
&\leq e^{t^{m-1}} \|x_{m-1} - q\| + R\theta_{m-1} \\
&\quad \vdots \\
&\leq \left(e^{\sum_{i=n}^{m-1} t_i} \right) \|x_n - q\| + R \left(e^{\sum_{i=n+1}^{m-1} t_i} \right) \sum_{i=n}^{m-1} \theta_i \\
&\leq R' \|x_n - q\| + RR' \sum_{i=n}^{m-1} \theta_i
\end{aligned}$$

where $R' = e^{\sum_{i=n}^{\infty} t_i}$.

Thus for any $q \in F$, we have

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - q\| + \|x_m - q\| \\
&\leq \|x_n - q\| + R' \|x_n - q\| + RR' \sum_{i=n}^{m-1} \theta_i \\
&\leq (R' + 1) \|x_n - q\| + RR' \sum_{i=n}^{\infty} \theta_i
\end{aligned}$$

Taking the infimum over all $q \in F$, we obtain

$$\|x_n - x_m\| \leq (R' + 1)d(x_n, F) + RR' \sum_{i=n}^{\infty} \theta_i.$$

Thus it follows from $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ and $\theta_n \rightarrow 0$ as $n \rightarrow \infty$ that $\{x_n\}$ is a Cauchy sequence in K . Since K is closed subset of E , the sequence $\{x_n\}$ converges strongly to some $p^* \in K$. Next, we show that $p^* \in F$. Now, $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ gives that $d(p^*, F) = 0$. Since F is closed, $p^* \in F$. Thus p^* is a common fixed point of S_1, S_2, T_1 and T_2 . This completes the proof. \square

Theorem 2.2. *Let E be a real Banach space, K be a nonempty closed convex subset of E . Let $S_1, S_2: K \rightarrow K$ be two asymptotically nonexpansive self mappings with sequences $\{h_n\}, \{k_n\} \in [1, \infty)$ and $T_1, T_2: K \rightarrow E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}, \{\mu_{n_1}\} \in [0, \infty)$ and $\{\nu_n\}, \{\nu_{n_1}\} \in [0, \infty)$. Let $\{x_n\}$ be the sequence defined by (7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. Suppose that $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Assume that one of S_1, S_2, T_1 and T_2 is completely continuous and $\lim_{n \rightarrow \infty} \|x_n - S_i x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$. If the following conditions hold:*

- (i) $\sum_{n=1}^{\infty} (h_n k_n - 1) < \infty$;
- (ii) $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \mu_{n_1} < \infty, \sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \nu_{n_1} < \infty$;
- (iii) there exists a constant $M > 0$ such that $\psi(t) \leq M t, t \geq 0$.

Then $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2 .

Proof. Without loss of generality we can assume that S_1 is completely continuous. Since $\{x_n\}$ is bounded by Lemma 2.1, there exists a subsequence $\{S_1x_{n_k}\}$ of $\{S_1x_n\}$ such that $\{S_1x_{n_k}\}$ converges strongly to some $q^* \in K$. Moreover, by hypothesis of the theorem we know that

$$\lim_{k \rightarrow \infty} \|x_{n_k} - S_1x_{n_k}\| = \lim_{k \rightarrow \infty} \|x_{n_k} - S_2x_{n_k}\| = 0$$

and

$$\lim_{k \rightarrow \infty} \|x_{n_k} - T_1x_{n_k}\| = \lim_{k \rightarrow \infty} \|x_{n_k} - T_2x_{n_k}\| = 0$$

which implies that

$$\|x_{n_k} - q^*\| \leq \|x_{n_k} - S_1x_{n_k}\| + \|S_1x_{n_k} - q^*\| \rightarrow 0$$

as $k \rightarrow \infty$ and so $x_{n_k} \rightarrow q^* \in K$. Thus, by the continuity of S_1, S_2, T_1 and T_2 , we have

$$\|q^* - S_iq^*\| = \lim_{k \rightarrow \infty} \|x_{n_k} - S_ix_{n_k}\| = 0$$

and

$$\|q^* - T_iq^*\| = \lim_{k \rightarrow \infty} \|x_{n_k} - T_ix_{n_k}\| = 0$$

for $i = 1, 2$. Thus it follows that $q^* \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Again, since $\lim_{n \rightarrow \infty} \|x_n - q^*\|$ exists by Lemma 2.1, we have $\lim_{n \rightarrow \infty} \|x_n - q^*\| = 0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2 . This completes the proof. \square

Theorem 2.3. Let E be a real Banach space, K be a nonempty closed convex subset of E . Let $S_1, S_2: K \rightarrow K$ be two asymptotically nonexpansive self mappings with sequences $\{h_n\}, \{k_n\} \in [1, \infty)$ and $T_1, T_2: K \rightarrow E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}, \{\mu_{n_1}\} \in [0, \infty)$ and $\{\nu_n\}, \{\nu_{n_1}\} \in [0, \infty)$. Let $\{x_n\}$ be the sequence defined by (7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. Suppose that $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Assume that one of S_1, S_2, T_1 and T_2 is semi-compact and $\lim_{n \rightarrow \infty} \|x_n - S_ix_n\| = \lim_{n \rightarrow \infty} \|x_n - T_ix_n\| = 0$ for $i = 1, 2$. If the following conditions hold:

- (i) $\sum_{n=1}^{\infty} (h_n k_n - 1) < \infty$;
- (ii) $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \mu_{n_1} < \infty, \sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \nu_{n_1} < \infty$;
- (iii) there exists a constant $M > 0$ such that $\psi(t) \leq Mt, t \geq 0$.

Then $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2 .

Proof. Since by hypothesis $\lim_{n \rightarrow \infty} \|x_n - S_ix_n\| = \lim_{n \rightarrow \infty} \|x_n - T_ix_n\| = 0$ for $i = 1, 2$ and one of S_1, S_2, T_1 and T_2 is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $p^* \in K$. Moreover, by the continuity of S_1, S_2, T_1 and T_2 , we have $\|p^* - S_ip^*\| = \lim_{j \rightarrow \infty} \|x_{n_j} - S_ix_{n_j}\| = 0$ and $\|p^* - T_ip^*\| = \lim_{j \rightarrow \infty} \|x_{n_j} - T_ix_{n_j}\| = 0$

for $i = 1, 2$. Thus it follows that $p^* \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Since $\lim_{n \rightarrow \infty} \|x_n - p^*\|$ exists by Lemma 2.1, we have $\lim_{n \rightarrow \infty} \|x_n - p^*\| = 0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2 . This completes the proof. \square

Theorem 2.4. *Let E be a real Banach space, K be a nonempty closed convex subset of E . Let $S_1, S_2: K \rightarrow K$ be two asymptotically nonexpansive self mappings with sequences $\{h_n\}, \{k_n\} \in [1, \infty)$ and $T_1, T_2: K \rightarrow E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}, \{\mu_{n_1}\} \in [0, \infty)$ and $\{\nu_n\}, \{\nu_{n_1}\} \in [0, \infty)$. Let $\{x_n\}$ be the sequence defined by (7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. Suppose that $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$ and the following conditions hold:*

- (i) $\sum_{n=1}^{\infty} (h_n k_n - 1) < \infty$;
- (ii) $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \mu_{n_1} < \infty, \sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \nu_{n_1} < \infty$;
- (iii) there exists a constant $M > 0$ such that $\psi(t) \leq M t, t \geq 0$.

If S_1, S_2, T_1 and T_2 satisfy the following conditions:

(C₁) $\lim_{n \rightarrow \infty} \|x_n - S_i x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$;

(C₂) there exists a continuous function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$f(d(x, F)) \leq a_1 \|x - S_1 x\| + a_2 \|x - S_2 x\| + a_3 \|x - T_1 x\| + a_4 \|x - T_2 x\|$$

for all $x \in K$, and a_1, a_2, a_3, a_4 are nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 = 1$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2 .

Proof. It follows from the hypothesis that

$$\begin{aligned} \lim_{n \rightarrow \infty} f(d(x_n, F)) &\leq a_1 \|x_n - S_1 x_n\| + a_2 \|x_n - S_2 x_n\| \\ &\quad + a_3 \|x_n - T_1 x_n\| + a_4 \|x_n - T_2 x_n\| \\ &= 0. \end{aligned}$$

That is,

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0.$$

Since $f: [0, \infty) \rightarrow [0, \infty)$ is a continuous function and $f(0) = 0$, therefore we have

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0.$$

Therefore, Theorem 2.1 implies that $\{x_n\}$ must converges strongly to a common fixed point of S_1, S_2, T_1 and T_2 . This completes the proof. \square

Example 2.1. Let $E = \mathbb{R}$ be the real line with the usual norm $\|\cdot\| = |\cdot|$, $K = [-1, 1]$ and P be the identity mapping. For each $x \in K$, define two

mappings $T, S: K \rightarrow K$ by

$$T(x) = \begin{cases} -2 \sin \frac{x}{2}, & \text{if } x \in [0, 1], \\ 2 \sin \frac{x}{2}, & \text{if } x \in [-1, 0). \end{cases}$$

and

$$S(x) = \begin{cases} \frac{x}{2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and uniformly L -Lipschitzian mapping with $L = \sup_{n \geq 1} \{k_n\}$ and hence it is a total asymptotically nonexpansive mapping by remark 1.1 and S is an asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$. Also $F(T) = \{0\}$ is the unique fixed point of T and $F(S) = \{0\}$ is the unique fixed point of S , that is, $F = F(S) \cap F(T) = \{0\}$ is the unique hybrid common fixed point of S and T .

3. CONCLUDING REMARKS

In this paper, we study hybrid mixed type iteration scheme for two asymptotically nonexpansive self mappings and two total asymptotically nonexpansive non-self mappings and establish some strong convergence theorems using completely continuous and semi-compactness conditions in the framework of real Banach spaces. Our results extend and generalize the corresponding results of [2, 3, 6, 7, 8, 9, 10, 11, 12, 13].

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