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DIFFERENCE EQUATIONS AND NEW EQUIVALENTS OF THE KUREPA HYPOTHESIS

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Abstract. By making use of the difference equations we deduce, for the first time, two new equivalent statements of the Guy's unsolved problem B44, also known as the Kurepa hypothesis for the left factorial.

1. Introduction

In what follows the sum denoted by $!n$ and defined as

$$!n = 0! + 1! + 2! + \cdots + (n-2)! + (n-1)!, \quad n \in N := \{1, 2, 3, \dots\},$$

where $n!$ is, as usual, the factorial

$$0! = 1, n! = n(n-1)!, \quad n \in N,$$

is referred to as the Kurepa left factorial.

The Kurepa left factorial $!n$ appears in the following assertion

$$GCD(!n, n!) = 2, n > 1, \qquad [2,\text{p.}147] \tag{1.1}$$

$GCD(!n, n!)$ is the greatest common divisor of numbers $!n$ and $n!$, which we call the Kurepa hypothesis for left factorial. The Kurepa hypothesis is listed as the problem B44 of Guy's [1].

Kurepa showed that the following assertion

$$!p \not\equiv 0, \quad (\text{mod } p), \quad p > 2 \text{ is a prime number}, \quad [2, \text{ p. } 149, \text{ Th. } 2.4] \tag{1.2}$$

is equivalent to (1.1). Also, for every $p > 2$, the Kurepa hypothesis is equivalent to

$$!p \not\equiv \sum_{k=1}^{p-2} \frac{(-k)^{k+1}}{k!}, \quad (\text{mod } p). \quad [4] \tag{1.3}$$

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2. Statement of the results

Let a sequence (x_n) be a particular solution of the difference equation

$$h_{n+2} + d_n h_{n+1} + g_n h_n = 0, \tag{2.1}$$

where (d_n) and (g_n) are given sequences. In this case, it is well-known that, a general solution of the equation (2.1) is as follows

$$h_n = Cx_n + Dx_n \sum_{k=1}^{n-1} \prod_{l=1}^{k-1} \frac{g_l x_l}{x_{l+2}}, \quad [3, \text{ p. } 158, \text{ Entry. } 4.10] \tag{2.2}$$

C and D being some constants.

Theorem 1. *Let p be a prime and define a sequence (A_k) by*

$$A_\circ = 0, A_1 = 1, A_k = (k-1)(A_{k-1} + A_{k-2}), \quad (k = 2, 3, \dots p-2).$$

Then

$$!p \equiv A_{p-2}, \pmod p.$$

Remark. Several first terms of (A_k) are: $A_\circ = 0$, $A_1 = 1$, $A_2 = 1$, $A_3 = 4$, $A_4 = 15$, $A_5 = 76, \dots$

Theorem 2. *Define a two-dimensional sequence $(A_k(n))$ by*

$$A_\circ(n) = 0, \quad A_1(n) = \frac{(-1)^{n+1}}{(n-2)!},$$
$$A_k(n) = -(n-k-1)A_{k-1}(n) + (n-k)A_{k-2}(n), \qquad (k = 2, 3, \dots, p-2).$$

Then for every prime p we have

$$!p \equiv A_{p-2}(p) \pmod p.$$

3. Proof of the results

Proof of Theorem 1. First, note that $B_k = k!$ is a particular solution of the following equation

$$A_{k+2} - (k+1)A_{k+1} - (k+1)A_k = 0. \tag{3.1}$$

Then, it follows by (2.2) where

$$d_k = g_k = -(k+1),$$

that a general solution of the equation (3.1) is given by

$$(3.2) \qquad A_k = CB_k + DB_k \sum_{r=1}^{k-1} \prod_{l=1}^{r-1} \frac{-(l+1)B_l}{B_{l+2}},$$

$$A_k = B_k \left(C + D \sum_{r=1}^{k-1} \prod_{l=1}^{r-1} \frac{-(l+1)!}{(l+2)!} \right),$$

$$(3.3) \qquad A_k = B_k \left(C + 2D \sum_{r=2}^{k} \frac{(-1)^r}{r!} \right).$$

Since $A_3 = 4$ and $B_3 = 6$ and $A_4 = 15$ and $B_4 = 24$ we have

$$(3.4) \qquad C = 1, \quad D = -\frac{1}{2}.$$

On using the equations (3.3) and (3.4) we obtain

$$(3.5) \qquad \frac{A_k}{B_k} = 1 - \sum_{r=2}^{k} \frac{(-1)^r}{r!}.$$

Set $k = p - 2$ in (3.5). Then

$$(3.6) \qquad A_{p-2} = (p-2)! \sum_{r=1}^{p-2} \frac{(-1)^{r+1}}{r!}.$$

Finally, the equations (1.3) and (3.6) give

$$A_{p-2} \equiv (p-2)!(!p) \pmod{p},$$

and our result follows by the Wilson theorem.

Proof of Theorem 2. Since, $B_k = 1$ is a particular solution of the equation

$$(3.7) \qquad A_{k+2}(n) + (n-k-3)A_{k+1}(n) - (n-k-2)A_k(n),$$

then by (2.2) where

$$d_k = n - k - 3, \quad g_k = -(n-k-2),$$

we have its general solution in the form

$$(3.8) \qquad A_k(n) = C + D \sum_{r=1}^{k-1} \prod_{l=1}^{r-1} -(n-l-2).$$

The constants C and D are obtained without difficulty as follows

$$(3.9) \qquad C = \frac{(-1)^{n+1}}{(n-2)!} \quad \text{and} \quad D = \frac{(-1)^n}{(n-3)!},$$

by using $A_3(n)$ and $B_3(n)$ and $A_4(n)$ and $B_4(n)$. Further, the equations (3.8) and (3.9) lead us to

$$A_k(n) = \frac{(-1)^{n+1}}{(n-2)!} + \frac{(-1)^n}{(n-3)!}\sum_{r=1}^{k-1}\prod_{l=1}^{r-1} -(n-l-2),$$

$$A_k(n) = \frac{(-1)^{n+1}}{(n-2)!} + \frac{(-1)^n}{2(n-3)!}[!n-(n-1)!-2], \tag{3.10}$$

Set $k = n-2$ in (3.10). Then

$$A_{n-2}(n) \equiv \frac{(-1)^{n+1}}{(n-2)!} + \frac{(-1)^{n+1}}{(n-2)!}[!n-1] \pmod{n},$$

and our result follows by the Wilson theorem.

4. References

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