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NORMALIZATIONS OF FUZZY BCC-IDEALS IN BCC-ALGEBRAS

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Abstract. We introduce the notion of normal fuzzy BCC-ideals, maximal fuzzy BCC-ideals and completely normal fuzzy BCC-ideals in BCC-algebras. We investigate some properties of normal (resp. maximal, completely normal) BCC-ideals. We show that every non-constant normal fuzzy BCC-ideal which is a maximal element of $(\mathcal{N}(X), \subseteq)$ takes only the values 0 and 1, and every maximal fuzzy BCC-ideal is completely normal.

1. Introduction

In 1966, Y. Imai and K. Iséki (cf. [7]) defined a class of algebras of type (2,0) called *BCK-algebras* which generalize the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, and on the other hand the notion of implication algebra (cf. [8]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem whether the class of BCK-algebras is a variety. That problem was solved by A. Wroński [11] who proved that BCK-algebras do not form a variety. In connection with this problem, Y. Komori (cf. [9]) introduced the notion of BCC-algebras, and W. A. Dudek (cf. [1], [2]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [5], W. A. Dudek and X. H. Zhang introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. W. A. Dudek and Y. B. Jun (cf. [3]) considered the fuzzification of BCC-ideals in BCC-algebras. They proved that every fuzzy BCC-ideal of a BCC-algebra is a fuzzy BCK-ideal, and showed that the converse is not true by providing a counterexample. They also proved that in a BCC-algebra every fuzzy BCK-ideal is a fuzzy BCC-subalgebra, and that in a BCK-algebra the notion of a fuzzy BCK-ideal and a fuzzy BCC-ideal coincide. The present authors (cf. [4]) described several

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properties of fuzzy BCC-ideals in BCC-algebras, and discussed an extension of fuzzy BCC-ideals. In this paper we establish the normalization of fuzzy BCC-ideals in BCC-algebras. We introduce the notion of normal fuzzy BCC-ideals, maximal fuzzy BCC-ideals and completely normal fuzzy BCC-ideals in BCC-algebras. We investigate some interesting properties of normal (resp. maximal, completely normal) BCC-ideals. We show that every non-constant normal fuzzy BCC-ideal which is a maximal element of $(\mathcal{N}(X), \subseteq)$ takes only the values 0 and 1, and every maximal fuzzy BCC-ideal is completely normal.

2. Preliminaries

In the present paper a binary multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written as $(xy \cdot zy) \cdot xz = 0$.

Definition 2.1. A non-empty set X with a constant 0 and a binary operation denoted by juxtaposition is called a *BCC-algebra* if for all $x, y, z \in X$ the following axioms hold:

(i) $(xy \cdot zy) \cdot xz = 0$,
(ii) $xx = 0$,
(iii) $0x = 0$,
(iv) $x0 = x$,
(v) $xy = 0$ and $yx = 0$ imply $x = y$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (cf. [2]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

(1) $$xy \cdot z = xz \cdot y.$$

On any BCC-algebra (similarly as in the case of BCK-algebras) one can define the natural order $\leq$ by putting

(2) $$x \leq y \Longleftrightarrow xy = 0.$$

It is not difficult to verify that this order is partial and 0 is its smallest element. Moreover, in any BCC-algebra X, the following are true:

(3) $$xy \leq x,$$

(4) $$xy \cdot zy \leq xz,$$

(5) $$x \leq y \text{ implies } xz \leq yz \text{ and } zy \leq zx.$$

Now we review some fuzzy logic concepts. A *fuzzy set* in a set X is a function $\mu : X \to [0,1]$. By X_μ we denote the set $\{x \in X : \mu(x) = \mu(0)\}$. For any fuzzy sets μ and ν in a set X, we define

$$\mu \subseteq \nu \Leftrightarrow \mu(x) \leq \nu(x) \quad \text{for all} \quad x \in X.$$

In the sequel, unless otherwise specified, X will denote a BCC-algebra.

Definition 2.2. A non-empty subset A of X is called a *BCC-ideal* of X if

(i) $0 \in A$,
(ii) $xy \cdot z \in A$ and $y \in A$ imply $xz \in A$, $\forall x, y, z \in X$.

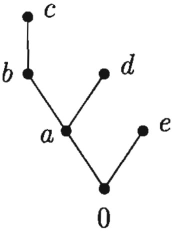
3. Normalization of fuzzy BCC-ideals

Definition 3.1. [3] A fuzzy set μ in X is called a *fuzzy BCC-ideal* of X if

(i) $\mu(0) \geq \mu(x)$, $\forall x \in X$,
(ii) $\mu(xz) \geq \min\{\mu(xy \cdot z), \mu(y)\}$, $\forall x, y, z \in X$.

Example 3.2. [3] Let $X = \{0, a, b, c, d, e\}$ be a set with Cayley table and Hasse diagram as follows:

$\cdot$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	0	0	a
b	b	b	0	0	a	a
c	c	b	a	0	a	a
d	d	d	d	d	0	a
e	e	e	e	e	e	0



Then X is a BCC-algebra ([3]). Define a fuzzy set μ in X by $\mu(e) = 0.3$ and $\mu(x) = 0.5$ for all $x \neq e$. Then μ is a fuzzy BCC-ideal of X. □

Lemma 3.3. [4] *Every fuzzy BCC-ideal μ of X is order reversing.*

The following proposition is straightforward and omit the proof.

Proposition 3.4. *Let A be a non-empty subset of X and let μ_A be a fuzzy set in X defined by*

$$\mu_A(x) := \begin{cases} s & \text{if } x \in A, \\ t & \text{otherwise,} \end{cases}$$

for all $x \in X$ and all $s, t \in [0, 1]$ with $s > t$. Then μ_A is a fuzzy BCC-ideal of X if and only if A is a BCC-ideal of X. Moreover,

$$X_{\mu_A} := \{x \in X : \mu_A(x) = \mu_A(0)\} = A.$$

Definition 3.5. A fuzzy BCC-ideal μ of X is said to be *normal* if there exists $x \in X$ such that $\mu(x) = 1$.

Example 3.6. Let X be a BCC-algebra in Example 3.2. Then a fuzzy set μ in X defined by $\mu(e) = t < 1$ and $\mu(x) = 1$ for all $x \neq e$ is a normal fuzzy BCC-ideal of X.

We note that if μ is a normal fuzzy BCC-ideal of X, then clearly $\mu(0) = 1$, and hence μ is normal if and only if $\mu(0) = 1$.

Proposition 3.7. *Given a fuzzy BCC-ideal μ of X let μ^+ be a fuzzy set in X defined by $\mu^+(x) = \mu(x) + 1 - \mu(0)$ for all $x \in X$. Then μ^+ is a normal fuzzy BCC-ideal of X which contains μ.*

Proof. We have $\mu^+(0) = \mu(0) + 1 - \mu(0) = 1 \geq \mu^+(x)$ for all $x \in X$. Let $x, y, z \in X$. Then

$$\begin{aligned} \min\{\mu^+(xy \cdot z), \mu^+(y)\} &= \min\{\mu(xy \cdot z) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\} \\ &= \min\{\mu(xy \cdot z), \mu(y)\} + 1 - \mu(0) \\ &\leq \mu(xz) + 1 - \mu(0) \\ &= \mu^+(xz). \end{aligned}$$

This shows that μ^+ is a fuzzy BCC-ideal of X. Clearly $\mu \subseteq \mu^+$, completing the proof. □

Corollary 3.8. *Let μ and μ^+ be as in Proposition 3.7. If there is $x \in X$ such that $\mu^+(x) = 0$, then $\mu(x) = 0$.*

Proof. Since $\mu \subseteq \mu^+$, it is straightforward. □

Using Proposition 3.4, we know that for any BCC-ideal A of X the characteristic function χ_A of A is a normal fuzzy BCC-ideal of X. It is clear that μ is normal iff $\mu^+ = \mu$.

Proposition 3.9. *If μ is a fuzzy BCC-ideal of X, then $(\mu^+)^+ = \mu^+$. Moreover if μ is normal, then $(\mu^+)^+ = \mu$.*

Proof. Straightforward. □

Proposition 3.10. *If μ and ν are fuzzy BCC-ideals of X such that $\mu \subseteq \nu$ and $\mu(0) = \nu(0)$, then $X_\mu \subseteq X_\nu$.*

Proof. Let $x \in X_\mu$. Then $\nu(x) \geq \mu(x) = \mu(0) = \nu(0)$ and so $\nu(x) = \nu(0)$, i.e., $x \in X_\nu$, proving $X_\mu \subseteq X_\nu$. □

Corollary 3.11. *If μ and ν are normal fuzzy BCC-ideals of X such that $\mu \subseteq \nu$, then $X_\mu \subseteq X_\nu$.*

Proposition 3.12. *Let μ be a fuzzy BCC-ideal of X. If there exists a fuzzy BCC-ideal ν of X such that $\nu^+ \subseteq \mu$, then μ is normal.*

Proof. Assume that there exists a fuzzy BCC-ideal ν of X such that $\nu^+ \subseteq \mu$. Then $1 = \nu^+(0) \leq \mu(0)$, and so $\mu(0) = 1$ and we are done. □

Proposition 3.13. *Let μ be a fuzzy BCC-ideal of X and let $f\colon [0, \mu(0)] \to [0, 1]$ be an increasing function. Then a fuzzy set $\mu_f : X \to [0, 1]$ defined by $\mu_f(x) := f(\mu(x))$ for all $x \in X$ is a fuzzy BCC-ideal of X. In particular, if $f(\mu(0)) = 1$ then μ_f is normal; and if $f(t) \geq t$ for all $t \in [0, \mu(0)]$, then μ is contained in μ_f.*

Proof. Note that $\mu(x) \leq \mu(0)$ for all $x \in X$. Since f is increasing, it follows that

$$\mu_f(0) = f(\mu(0)) \geq f(\mu(x)) = \mu_f(x)$$

for all $x \in X$. For any $x, y, z \in X$, we have

$$\begin{aligned} \min\{\mu_f(xy \cdot z), \mu_f(y)\} &= \min\{f(\mu(xy \cdot z)), f(\mu(y))\} \\ &= f(\min\{\mu(xy \cdot z), \mu(y)\}) \\ &\leq f(\mu(xz)) = \mu_f(xz). \end{aligned}$$

Hence μ_f is a fuzzy BCC-ideal of X. If $f(\mu(0)) = 1$, then clearly μ_f is normal. Assume that $f(t) \geq t$ for all $t \in [0, \mu(0)]$. Then $\mu_f(x) = f(\mu(x)) \geq \mu(x)$ for all $x \in X$, which proves that μ is contained in μ_f. □

Denote by $\mathcal{N}(X)$ the set of all normal fuzzy BCC-ideals of X. Note that $\mathcal{N}(X)$ is a poset under the set inclusion.

Theorem 3.14. *Let $\mu \in \mathcal{N}(X)$ be a non-constant such that it is a maximal element of $(\mathcal{N}(X), \subseteq)$. Then μ takes only the values 0 and 1.*

Proof. Note that $\mu(0) = 1$ since μ is normal. Let $x \in X$ be such that $\mu(x) \neq 1$. We claim that $\mu(x) = 0$. If not, then there exists $a \in X$ such that $0 < \mu(a) < 1$. Let ν be a fuzzy set in X defined by $\nu(x) := \frac{1}{2}(\mu(x) + \mu(a))$ for all $x \in X$. Then clearly ν is well-defined, and we have that for all $x \in X$,

$$\nu(0) = \frac{1}{2}(\mu(0) + \mu(a)) = \frac{1}{2}(1 + \mu(a)) \geq \frac{1}{2}(\mu(x) + \mu(a)) = \nu(x).$$

For any $x, y, z \in X$ we obtain

$$\begin{aligned} \nu(xz) &= \tfrac{1}{2}(\mu(xz) + \mu(a)) \\ &\geq \tfrac{1}{2}(\min\{\mu(xy \cdot z), \mu(y)\} + \mu(a)) \\ &= \min\{\tfrac{1}{2}(\mu(xy \cdot z) + \mu(a)), \tfrac{1}{2}(\mu(y) + \mu(a))\} \\ &= \min\{\nu(xy \cdot z), \nu(y)\}. \end{aligned}$$

Hence ν is a fuzzy BCC-ideal of X. It follows from Proposition 3.7 that $\nu^+ \in \mathcal{N}(X)$ where ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(0)$ for all $x \in X$.

Clearly $\nu^+(x) \geq \mu(x)$ for all $x \in X$. Note that

$$\begin{aligned} \nu^+(a) &= \nu(a) + 1 - \nu(0) \\ &= \tfrac{1}{2}(\mu(a) + \mu(a)) + 1 - \tfrac{1}{2}(\mu(0) + \mu(a)) \\ &= \tfrac{1}{2}(\mu(a) + 1) > \mu(a) \end{aligned}$$

and $\nu^+(a) < 1 = \nu^+(0)$. Hence ν^+ is non-constant, and μ is not a maximal element of $\mathcal{N}(X)$. This is a contradiction. □

We construct a new fuzzy BCC-ideal from old. Let $t > 0$ be a real number. If $\alpha \in [0,1]$, α^t shall mean the positive root in case $t < 1$. We define $\mu^t : L \to [0,1]$ by $\mu^t(x) := (\mu(x))^t$ for all $x \in X$.

Proposition 3.15. *If μ is a fuzzy BCC-ideal of X, then so is μ^t and $X_{\mu^t} = X_\mu$.*

Proof. For any $x, y, z \in X$, we have $\mu^t(0) = (\mu(0))^t \geq (\mu(x))^t = \mu^t(x)$ and

$$\begin{aligned} \mu^t(xz) &= (\mu(xz))^t \\ &\geq (\min\{\mu(xy \cdot z), \mu(y)\})^t \\ &= \min\{(\mu(xy \cdot z))^t, (\mu(y))^t\} \\ &= \min\{\mu^t(xy \cdot z), \mu^t(y)\}. \end{aligned}$$

Hence μ^t is a fuzzy BCC-ideal of X. Now

$$\begin{aligned} X_{\mu^t} &= \{x \in L : \mu^t(x) = \mu^t(0)\} \\ &= \{x \in X : (\mu(x))^t = (\mu(0))^t\} \\ &= \{x \in X : \mu(x) = \mu(0)\} \\ &= X_\mu. \end{aligned}$$

This completes the proof. □

Corollary 3.16. *If $\mu \in \mathcal{N}(X)$, then so is μ^t.*

Proof. Straightforward. □

Definition 3.17. Let μ be a fuzzy BCC-ideal of X. Then μ is said to be *maximal* if

(i) μ is non-constant.
(ii) μ^+ is a maximal element of the poset $(\mathcal{N}(X), \subseteq)$.

Theorem 3.18. *If μ is a maximal fuzzy BCC-ideal of X, then*

(i) *μ is normal.*
(ii) *μ takes only the values 0 and 1.*
(iii) *$\mu_{X_\mu} = \mu$.*
(iv) *X_μ is a maximal BCC-ideal of X.*

Proof. Let μ be a maximal fuzzy BCC-ideal of X. Then μ^+ is a non-constant maximal element of the poset $(\mathcal{N}(X), \subseteq)$. It follows from Theorem 3.14 that μ^+ takes only the values 0 and 1. Note that $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(0)$, and $\mu^+(x) = 0$ if and only if $\mu(x) = \mu(0) - 1$. By Corollary 3.8, we have $\mu(x) = 0$, that is, $\mu(0) = 1$. Hence μ is normal, and clearly $\mu^+ = \mu$. This proves (i) and (ii).

(iii) Clearly $\mu_{X_\mu} \subseteq \mu$ and μ_{X_μ} takes only the values 0 and 1. Let $x \in X$. If $\mu(x) = 0$, then obviously $\mu \subseteq \mu_{X_\mu}$. If $\mu(x) = 1$, then $x \in X_\mu$, and so $\mu_{X_\mu}(x) = 1$. This shows that $\mu \subseteq \mu_{X_\mu}$.

(iv) X_μ is a proper BCC-ideal of X because μ is non-constant. Let A be a BCC-ideal of X such that $X_\mu \subseteq A$. Noticing that, for any BCC-ideals A and B of X, $A \subseteq B$ if and only if $\mu_A \subseteq \mu_B$, then we obtain $\mu = \mu_{X_\mu} \subseteq \mu_A$. Since μ and μ_A are normal and since $\mu = \mu^+$ is a maximal element of $\mathcal{N}(X)$, we have that either $\mu = \mu_A$ or $\mu_A = \mathbf{1}$ where $\mathbf{1} : X \to [0, 1]$ is a fuzzy set defined by $\mathbf{1}(x) := 1$ for all $x \in X$. The later case implies that $A = X$. If $\mu = \mu_A$, then $X_\mu = X_{\mu_A} = A$ by Proposition 3.4. This proves that X_μ is a maximal BCC-ideal of X, ending the proof. □

Definition 3.19. A normal fuzzy BCC-ideal μ of X is said to be *completely normal* if there exists $x \in X$ such that $\mu(x) = 0$. Denote by $\mathcal{C}(X)$ the set of all completely normal fuzzy BCC-ideals of X.

We note that $\mathcal{C}(X) \subseteq \mathcal{N}(X)$ and the restriction of the partial ordering $\subseteq$ of $\mathcal{N}(X)$ gives a partial ordering of $\mathcal{C}(X)$.

Proposition 3.20. *Any non-constant maximal element of* $(\mathcal{N}(X), \subseteq)$ *is also a maximal element of* $(\mathcal{C}(X), \subseteq)$.

Proof. Let μ be a non-constant maximal element of $(\mathcal{N}(X), \subseteq)$. By Theorem 3.14, μ takes only the values 0 and 1, and so $\mu(0) = 1$ and $\mu(x) = 0$ for some $x \in X$. Hence $\mu \in \mathcal{C}(X)$. Assume that there exists $\nu \in \mathcal{C}(X)$ such that $\mu \subseteq \nu$. It follows that $\mu \subseteq \nu$ in $\mathcal{N}(X)$. Since μ is maximal in $(\mathcal{N}(X), \subseteq)$ and since ν is non-constant, therefore $\mu = \nu$. Thus μ is maximal element of $(\mathcal{C}(X), \subseteq)$, ending the proof. □

Theorem 3.21. *Every maximal fuzzy BCC-ideal of X is completely normal.*

Proof. Let μ be a maximal fuzzy BCC-ideal of X. Then by Theorem 3.18, μ is normal and $\mu = \mu^+$ takes only the values 0 and 1. Since μ is non-constant, it follows that $\mu(0) = 1$ and $\mu(x) = 0$ for some $x \in X$. Hence μ is completely normal, ending the proof. □

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