Slant Helices, Darboux Helices and Similar Curves in Dual Space $\mathbb{D}^3$

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ABSTRACT. In this paper, we give definitions and characterizations of slant helices, normalized Darboux helices and similar curves in dual space $\mathbb{D}^3$. First, we define dual slant helices and dual normalized Darboux helices and show that dual slant helices are also dual normalized Darboux helices. Then, we introduce the concept of dual similar curves and obtain that the family of dual slant helices forms a family of dual similar curves.

1. INTRODUCTION

Helix is one of the most interesting curves in science and nature. We can encounter helices in biology, fractal geometry, computer aided design, computer graphics, etc [1]. In differential geometry, a general helix is defined by the property that the tangent line of the curve makes a constant angle with a fixed straight line [14]. In 1802, Lancret stated that *"a curve is a general helix if and only if the ratio of curvature to torsion is constant"* [14]. There are many papers about helices and some of them include different types of helices and their properties. Barros proved the Lancret theorem for general helices in a space form by using killing vector fields along curves [2]. Izumiya and Takeuchi defined slant helices and conical geodesic curves [8]. Kula and Yaylı obtained that the indicatrices of a slant helix are spherical helices [9]. Moreover, they showed that a curve of constant precession is a slant helix. Kula et. al. investigated the relationships between slant helices and general helices in $\mathbb{R}^3$ [10]. They obtained differential equations which are characterizations of a slant helix. Lee, Choi and Jin studied dual slant helix and Mannheim partner curves in the dual space $\mathbb{D}^3$ [11]. Zıplar, Şenol and Yaylı introduced Darboux helices in $\mathbb{R}^3$ and gave the relations between Darboux helices and slant helices [18].

On the other hand, in local differential geometry, associated curves such as Bertrand curves, Mannheim curves and involute-evolute curves are very

fascinating research area. Recently, El-Sabbagh and Ali added a new one to these associated curves [6]. They defined a new family of curves and called a family of similar curves with variable transformation. Also, they introduced relationships between some special curves and similar curves.

This study consists of two original sections, Sections 3 and 4. In Section 3, the definitions and characterizations of slant helices and Darboux helices are introduced in dual space $\mathbb{D}^3$. It is shown that dual slant helices are also dual Darboux helices. In Section 4, dual similar curves are defined in $\mathbb{D}^3$ and it is obtained that the family of dual slant helices and of course in a special case dual Darboux helices forms a family of dual similar curves.

2. Preliminaries

A dual number, as introduced by W. Clifford, can be defined as an ordered pair of real numbers (a, a^*) where a is called the real part and a^* is called the dual part of the dual number. If both parts are nonzero, the dual number is said to be proper; if the real part is zero, it is called a pure dual number; and if the dual part is zero, it reduces to a real number. Dual numbers may be formally expressed as $\bar{a} = a + \varepsilon a^*$ where ε is the dual unit which is subjected to the following rules [15]:

$$\varepsilon \neq 0, \;\; 0\varepsilon = \varepsilon 0 = 0, \;\; 1\varepsilon = \varepsilon 1 = \varepsilon, \;\; \varepsilon^2 = 0.$$

We denote the set of dual numbers by $\mathbb{D}$, i.e.,

$$\mathbb{D} = \left\{ \, \bar{a} = a + \varepsilon a^* : \; a, a^* \in \mathbb{R}, \; \varepsilon^2 = 0 \, \right\}.$$

Two inner operations and equality on $\mathbb{D}$ are defined as follows [3, 7, 13]:
  (i) Addition : $(a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*)$
  (ii) Multiplication : $(a + \varepsilon a^*) \cdot (b + \varepsilon b^*) = ab + \varepsilon(ab^* + a^*b)$
  (iii) Equality : $a + \varepsilon a^* = b + \varepsilon b^*$ if and only if $a = b, \;\; a^* = b^*$.
  Since division by pure dual numbers is not defined, the set $\mathbb{D}$ of dual numbers with the above operations is a commutative ring, not a field.

A dual number $\bar{a} = a + \varepsilon a^*$ divided by a dual number $\bar{b} = b + \varepsilon b^*$, with $b \neq 0$, can be defined as

$$\frac{\bar{a}}{\bar{b}} = \frac{a}{b} + \varepsilon \frac{a^*b - ab^*}{b^2},$$

([3, 4]). We can define the function of a dual number $f(\bar{a})$ by expanding it formally in a Maclaurin series with ε as variable. Since $\varepsilon^n = 0$ for $n > 1$, we obtain

$$f(\bar{a}) = f(a + \varepsilon a^*) = f(a) + \varepsilon a^* f'(a),$$

where $f'(a)$ is derivative of $f(a)$ with respect to a [5].

In analogy with dual numbers, a dual vector referred to an arbitrarily chosen origin can be defined as an ordered pair of vectors (a, a^*), where

$a, a^* \in \mathbb{R}^3$ [16]. Also dual vectors can be expressed as $\tilde{a} = a + \varepsilon a^*$, where $a, a^* \in \mathbb{R}^3$ and $\varepsilon^2 = 0$. We denote the set of dual vectors by $\mathbb{D}^3$, i.e.,

$$\mathbb{D}^3 = \{ \tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) : \quad \bar{a}_i \in \mathbb{D}, \ i = 1, 2, 3 \}.$$

$\mathbb{D}^3$ is a module over the ring $\mathbb{D}$ and it is called dual space. For any dual vectors $\tilde{a} = a + \varepsilon a^*$ and $\tilde{b} = b + \varepsilon b^*$ in $\mathbb{D}^3$, the scalar product and the vector product are defined by

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \langle a, b \rangle + \varepsilon \left(\langle a, b^* \rangle + \langle a^*, b \rangle \right)$$

and

$$\tilde{a} \times \tilde{b} = a \times b + \varepsilon \left(a \times b^* + a^* \times b \right),$$

respectively [4, 7].

The norm of a dual vector $\tilde{a}$ is given by

$$\|\tilde{a}\| = \|a\| + \varepsilon \frac{\langle a, a^* \rangle}{\|a\|}.$$

(See [7]). A dual vector $\tilde{a}$ with norm $1 + \varepsilon 0$ is called dual unit vector. The set of dual unit vectors is denoted by

$$\tilde{S}^2 = \left\{ \tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) \in \mathbb{D}^3 : \ \langle \tilde{a}, \tilde{a} \rangle = (1, 0) \right\}$$

and called dual unit sphere [3, 7].

A dual angle, subtended by two oriented lines in space as introduced by Study in 1903, is defined as $\bar{\theta} = \theta + \varepsilon \theta^*$, where θ is the projected angle and θ^* is the shortest distance between the two lines [13].

$\tilde{\alpha}(t) = \alpha(t) + \varepsilon \alpha^*(t)$ is a curve in dual space $\mathbb{D}^3$ and is called dual space curve or curve in dual space, where $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ and $\alpha^*(t) = (\alpha_1^*(t), \alpha_2^*(t), \alpha_3^*(t))$ are real valued curves in the real space $\mathbb{R}^3$. If the functions $\alpha_i(t)$ and $\alpha_i^*(t)$, $(1 \leq i \leq 3)$ are differentiable then the dual space curve

$$\begin{aligned} \tilde{\alpha} : I &\subset \mathbb{R} \to \mathbb{D}^3 \\ t \to \tilde{\alpha}(t) &= (\alpha_1(t) + \varepsilon \alpha_1^*(t), \ \alpha_2(t) + \varepsilon \alpha_2^*(t), \ \alpha_3(t) + \varepsilon \alpha_3^*(t)) \\ &= \alpha(t) + \varepsilon \alpha^*(t) \end{aligned}$$

is differentiable in $\mathbb{D}^3$. The real part $\alpha(t)$ of the dual space curve $\tilde{\alpha} = \tilde{\alpha}(t)$ is called indicatrix. The dual arc-length of the dual space curve $\tilde{\alpha}(t)$ from t_1 to t is defined by

$$(1) \quad \bar{s} = \int_{t_1}^{t} \|\tilde{\alpha}'(t)\| \, \mathrm{d}\, t = \int_{t_1}^{t} \|\vec{\alpha}'(t)\| \, \mathrm{d}\, t + \varepsilon \int_{t_1}^{t} \left\langle T, \alpha^{*\,'}(t) \right\rangle \mathrm{d}\, t = s + \varepsilon s^*,$$

where T is unit tangent vector of the indicatrix $\alpha(t)$ [12, 17].

Let $\tilde{\alpha}(\bar{s})$ be a dual space curve with dual arc length parameter $\bar{s}$. Dual unit tangent vector of $\tilde{\alpha}$ is defined by

$$(2) \qquad\qquad\qquad \tilde{T} = \frac{\mathrm{d}\, \tilde{\alpha}}{\mathrm{d}\, \bar{s}}.$$

Differentiating $\tilde{T}$ with respect to dual arc length parameter $\bar{s}$ we have

$$(3) \qquad \tilde{T}' = \frac{\mathrm{d}\,\tilde{T}}{\mathrm{d}\,\bar{s}} = \frac{\mathrm{d}^2\,\tilde{\alpha}}{\mathrm{d}\,\bar{s}^2} = \bar{\kappa}\tilde{N},$$

where $\bar{\kappa} = \bar{\kappa}(\bar{s})$ is called dual curvature. We restrict that $\bar{\kappa}(\bar{s})$ is never pure dual number. The dual unit vector $\tilde{N} = (1/\bar{\kappa})\tilde{T}'$ is called dual unit principal normal vector of $\tilde{\alpha}$. The dual unit vector $\tilde{B}$ defined by $\tilde{B} = \tilde{T} \times \tilde{N}$ is called dual unit binormal vector of $\tilde{\alpha}$. The dual frame $\left\{ \tilde{T}(\bar{s}),\ \tilde{N}(\bar{s}),\ \tilde{B}(\bar{s}) \right\}$ is called moving dual Frenet frame along the dual space curve $\tilde{\alpha}(\bar{s})$ in $\mathbb{D}^3$. For the curve $\tilde{\alpha}$, the dual Frenet derivative formulae can be given in matrix form as

$$(4) \qquad \frac{\mathrm{d}}{\mathrm{d}\,\bar{s}} \begin{bmatrix} \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{bmatrix} = \begin{bmatrix} 0 & \bar{\kappa} & 0 \\ -\bar{\kappa} & 0 & \bar{\tau} \\ 0 & -\bar{\tau} & 0 \end{bmatrix} \begin{bmatrix} \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{bmatrix},$$

where $\bar{\tau} = \bar{\tau}(s)$ is called the dual torsion of $\tilde{\alpha}$ [17]. Then the dual Darboux vector of $\tilde{\alpha}(\bar{s})$ is defined by $\tilde{W} = \bar{\tau}\tilde{T} + \bar{\kappa}\tilde{B}$ which gives the derivative formulae (4) as follows

$$\frac{\mathrm{d}\,\tilde{T}}{\mathrm{d}\,\bar{s}} = \tilde{W} \times \tilde{T}, \quad \frac{\mathrm{d}\,\tilde{N}}{\mathrm{d}\,\bar{s}} = \tilde{W} \times \tilde{N}, \quad \frac{\mathrm{d}\,\tilde{B}}{\mathrm{d}\,\bar{s}} = \tilde{W} \times \tilde{B}.$$

The unit dual Darboux vector of $\tilde{\alpha}(\bar{s})$ is defined by $\tilde{W}_0 = \dfrac{\bar{\tau}\tilde{T} + \bar{\kappa}\tilde{B}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}}$.

Now we give the following theorem which we will use in the following sections.

Theorem 2.1. *Let $\tilde{\alpha}(\bar{s})$ be a dual curve parametrized by dual arclength $\bar{s}$. Suppose $\tilde{\alpha} = \tilde{\alpha}(\bar{\theta})$ is another parametric representation of this dual curve by the parameter $\bar{\theta} = \int \bar{\kappa}(\bar{s})\,\mathrm{d}\,\bar{s}$. Then the dual unit tangent vector $\tilde{T}$ satisfies the following differential equation:*

$$(5) \qquad \left(\frac{1}{\bar{f}(\bar{\theta})} \tilde{T}''(\bar{\theta}) \right)' + \left(\frac{1 + \bar{f}^2(\bar{\theta})}{\bar{f}(\bar{\theta})} \right) \tilde{T}'(\bar{\theta}) - \frac{\bar{f}'(\bar{\theta})}{\bar{f}^2(\bar{\theta})} \tilde{T}(\bar{\theta}) = 0$$

where $\bar{f}(\bar{\theta}) = \dfrac{\bar{\tau}(\bar{\theta})}{\bar{\kappa}(\bar{\theta})}$ and prime shows the derivative with respect to $\bar{\theta}$.

Proof. Let $\tilde{\alpha}(\bar{s})$ be a unit speed dual curve. We can write this curve in another parametric representation $\tilde{\alpha} = \tilde{\alpha}(\bar{\theta})$, where $\bar{\theta} = \int \bar{\kappa}(\bar{s})\,\mathrm{d}\,\bar{s}$, and we have new dual Frenet equations as follows:

$$(6) \qquad \begin{bmatrix} \tilde{T}'(\bar{\theta}) \\ \tilde{N}'(\bar{\theta}) \\ \tilde{B}'(\bar{\theta}) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \bar{f}(\bar{\theta}) \\ 0 & -\bar{f}(\bar{\theta}) & 0 \end{bmatrix} \begin{bmatrix} \tilde{T}(\bar{\theta}) \\ \tilde{N}(\bar{\theta}) \\ \tilde{B}(\bar{\theta}) \end{bmatrix},$$

where $\bar{f}(\bar{\theta}) = \frac{\bar{\tau}(\bar{\theta})}{\bar{\kappa}(\bar{\theta})}$. If we use the first and second equation of the new Frenet formulae we have

(7)
$$\frac{\mathrm{d}\,\tilde{N}}{\mathrm{d}\,\bar{\theta}} = \frac{\mathrm{d}^2\,\tilde{T}}{\mathrm{d}\,\bar{\theta}^2} = -\tilde{T} + \bar{f}\,\tilde{B}$$

and we get the dual unit binormal vector as

(8)
$$\tilde{B}(\bar{\theta}) = \frac{1}{\bar{f}(\bar{\theta})}\left[\tilde{T}''(\bar{\theta}) + \tilde{T}(\bar{\theta})\right].$$

Differentiating the equation (8) and using the third equation of equation (6), we obtain the dual vector differential equation as desired. □

3. Slant Helices and Darboux Helices in Dual Space

In this section, we will give the definitions and characterizations of dual slant helix and dual Darboux helix.

Definition 3.1. ([11]) Let $\tilde{\alpha} : \mathbb{D} \to \mathbb{D}^3$ be a dual space curve with dual Frenet frame $\left\{\tilde{T}(\bar{s}),\ \tilde{N}(\bar{s}),\ \tilde{B}(\bar{s})\right\}$. Then $\tilde{\alpha}$ is called a *dual slant helix* if the dual unit principal normal vectors $\tilde{N}$ make a constant dual angle $\bar{\phi}$ with a fixed dual unit vector $\tilde{d}$ in dual space.

Definition 3.2. Let $\tilde{\alpha} : \mathbb{D} \to \mathbb{D}^3$ be a dual space curve with dual Frenet frame $\left\{\tilde{T}(\bar{s}),\ \tilde{N}(\bar{s}),\ \tilde{B}(\bar{s})\right\}$. Then $\tilde{\alpha}$ is called a *dual Darboux helix* if the dual Darboux vectors $\tilde{W}$ make a constant dual angle $\bar{\phi}$ with a fixed dual unit vector $\tilde{d}$ in dual space. Moreover, $\tilde{\alpha}$ is called a *dual normalized Darboux helix* if the dual unit Darboux vectors $\tilde{W}_0$ make a constant dual angle with a fixed dual unit vector $\tilde{d}$ in dual space.

After these definitions, we can give the following characterizations:

Theorem 3.1. *Let $\tilde{\alpha} : \mathbb{D} \to \mathbb{D}^3$ be a dual curve parameterized by its dual arc length. The dual curve $\tilde{\alpha}$ is a dual slant helix with constant dual angle $\bar{\phi}$ such that $\left\langle \tilde{N},\ \tilde{d} \right\rangle = \cos\bar{\phi} = \bar{n}$ is constant if and only if*

(9)
$$\frac{\bar{\tau}(\bar{s})}{\bar{\kappa}(\bar{s})} = \pm\frac{\bar{m}\int \bar{\kappa}(\bar{s})\,\mathrm{d}\,\bar{s}}{\sqrt{1 - \bar{m}^2\left(\int \bar{\kappa}(\bar{s})\,\mathrm{d}\,\bar{s}\right)^2}},$$

holds along the curve, where $\bar{m} = \dfrac{\bar{n}}{\sqrt{1 - \bar{n}^2}}$ is a dual constant.

Proof. Let $\tilde{d}$ be a fixed dual unit vector which makes a constant dual angle $\bar{\phi} = \pm \arccos(\bar{n})$ with the dual unit principal normal vector $\tilde{N}$, i.e.,

$$\tag{10} \left\langle \tilde{N}, \, \tilde{d} \right\rangle = \cos \bar{\phi} = \bar{n},$$

where $\bar{n} \in \mathbb{D}$ is a constant. If we differentiate equation (10) with respect to $\bar{\theta} = \int \bar{\kappa}(\bar{s}) \, \mathrm{d}\,\bar{s}$ and use the new dual Frenet equation (6), we get

$$\tag{11} \left\langle -\tilde{T}(\bar{\theta}) + \bar{f}(\bar{\theta})\tilde{B}(\bar{\theta}), \, \tilde{d} \right\rangle = 0.$$

Therefore,

$$\tag{12} \left\langle \tilde{T}, \, \tilde{d} \right\rangle = \bar{f} \left\langle \tilde{B}, \, \tilde{d} \right\rangle.$$

We can put $\left\langle \tilde{B}, \, \tilde{d} \right\rangle = \bar{b}$ and write

$$\tag{13} \tilde{d} = \bar{f} \, \bar{b} \, \tilde{T} + \bar{n} \, \tilde{N} + \bar{b} \, \tilde{B}.$$

Since the dual vector $\tilde{d}$ is a dual unit vector, we get $\bar{b} = \pm\sqrt{\frac{1-\bar{n}^2}{1+\bar{f}^2}}$. Hence, dual unit vector $\tilde{d}$ can be written as

$$\tag{14} \tilde{d} = \pm\bar{f}\sqrt{\frac{1-\bar{n}^2}{1+\bar{f}^2}} \, \tilde{T} + \bar{n} \, \tilde{N} \pm \sqrt{\frac{1-\bar{n}^2}{1+\bar{f}^2}} \, \tilde{B}.$$

If we differentiate equation (11) again and use equation (6), we have

$$\tag{15} \left\langle \bar{f}' \, \tilde{B} - (1+\bar{f}^2)\tilde{N}, \, \tilde{d} \right\rangle = 0.$$

Substituting equation (14) into equation (15), we obtain following differential equation

$$\tag{16} \frac{\bar{f}'}{\left(1+\bar{f}^2\right)^{3/2}} = \pm\bar{m},$$

where $\bar{m} = \frac{\bar{n}}{\sqrt{1-\bar{n}^2}}$. Integrating above equation, we get

$$\tag{17} \frac{\bar{f}}{\sqrt{1+\bar{f}^2}} = \pm\bar{m}(\bar{\theta} + \bar{c}_1).$$

where $\bar{c}_1$ is a dual integration constant. We can use a parameter change $\bar{\theta} \to \bar{\theta} - \bar{c}_1$ to eliminate the integration constant. Then, $\bar{f}$ can be found from equation (17) as

$$\tag{18} \bar{f}(\bar{\theta}) = \pm\frac{\bar{m} \, \bar{\theta}}{\sqrt{1 - \bar{m}^2\bar{\theta}^2}}.$$

Since $\bar{\tau}(\bar{s}) = \bar{\kappa}(\bar{s})\bar{f}(\bar{s})$ and $\bar{\theta} = \int \bar{\kappa}(\bar{s}) \, \mathrm{d}\,\bar{s}$, from (18) we obtain desired result.

Conversely, suppose that

$$\frac{\bar{\tau}(\bar{s})}{\bar{\kappa}(\bar{s})} = \pm \frac{\bar{m} \int \bar{\kappa}(\bar{s}) \, \mathrm{d}\,\bar{s}}{\sqrt{1 - \bar{m}^2 \left(\int \bar{\kappa}(\bar{s}) \, \mathrm{d}\,\bar{s} \right)^2}},$$

holds. The function $\bar{f}$ can be written as $\bar{f}(\bar{\theta}) = \pm \dfrac{\bar{m} \, \bar{\theta}}{\sqrt{1 - \bar{m}^2 \bar{\theta}^2}}$, where $\bar{\theta} = \int \bar{\kappa}(\bar{s}) \, \mathrm{d}\,\bar{s}$. Let us consider dual vector

$$(19) \qquad \tilde{d} = \bar{n} \left(\bar{\theta} \tilde{T} + \tilde{N} \pm \frac{1}{\bar{m}} \sqrt{1 - \bar{m}^2 \bar{\theta}^2} \tilde{B} \right).$$

We must show that the dual vector $\tilde{d}$ is a dual constant vector. Differentiating equation (19) and using dual Frenet formulae (6), we get

$$(20) \qquad \tilde{d}' = 0$$

The equation (20) shows that dual vector $\tilde{d}$ is a dual constant vector and $\left\langle \tilde{N}, \, \tilde{d} \right\rangle = \bar{n}$ is constant. This finishes the proof of Theorem 3.1. $\qquad \square$

Theorem 3.2. *Let $\tilde{\alpha} : \mathbb{D} \to \mathbb{D}^3$ be a dual curve parameterized by its dual arc length. Then $\tilde{\alpha}$ is a dual slant helix if and only if*

$$(21) \qquad \frac{\bar{\kappa}^2}{(\bar{\tau}^2 + \bar{\kappa}^2)^{3/2}} \left(\frac{\bar{\tau}}{\bar{\kappa}} \right)',$$

is a dual constant.

Proof. Since $\tilde{\alpha}$ is a dual slant helix, we have $\left\langle \tilde{N}, \, \tilde{d} \right\rangle = \cos \bar{\phi} = \text{constant}$. So, there exist smooth functions $\bar{a}_1 = \bar{a}_1(\bar{s})$ and $\bar{a}_3 = \bar{a}_3(\bar{s})$ of dual arc length $\bar{s}$ such that

$$(22) \qquad \tilde{d} = \bar{a}_1 \tilde{T} + \cos \bar{\phi} \tilde{N} + \bar{a}_3 \tilde{B}.$$

Since $\tilde{d}$ is a constant dual vector, we get $\tilde{d}' = 0$. Then if we take the derivative of (22) and use the dual Frenet formulae, we obtain

$$(23) \qquad (\bar{a}_1' - \bar{\kappa} \cos \bar{\phi}) \tilde{T} + (\bar{a}_1 \bar{\kappa} - \bar{a}_3 \bar{\tau}) \tilde{N} + (\bar{a}_3' + \bar{\tau} \cos \bar{\phi}) \tilde{B} = 0$$

where the prime shows the derivative with respect to $\bar{s}$. Since the dual Frenet frame $\left\{ \tilde{T}, \tilde{N}, \tilde{B} \right\}$ is linearly independent, we have

$$(24) \qquad \begin{cases} \bar{a}_1' - \bar{\kappa} \cos \bar{\phi} = 0, \\ \bar{a}_1 \bar{\kappa} - \bar{a}_3 \bar{\tau} = 0, \\ \bar{a}_3' + \bar{\tau} \cos \bar{\phi} = 0. \end{cases}$$

From the second equation of the system (24) we obtain

$$(25) \qquad \bar{a}_1 = \left(\frac{\bar{\tau}}{\bar{\kappa}} \right) \bar{a}_3.$$

Since $\tilde{d}$ is a constant dual vector, $\left\|\tilde{d}\right\|$ is constant. Then (25) gives that

$$(26) \qquad \left(\frac{\bar{\tau}}{\bar{\kappa}}\right)^2 \bar{a}_3^2 + \bar{a}_3^2 + \cos^2 \bar{\phi} = \text{constant}$$

and from (26) we can write

$$(27) \qquad \left(\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)^2 + 1\right) \bar{a}_3^2 = \bar{n}^2$$

where $\bar{n}^2$ is a non-zero dual constant. From (27) we have

$$(28) \qquad \bar{a}_3 = \pm \frac{\bar{n}}{\sqrt{\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)^2 + 1}}.$$

By taking derivative of (28) and using the third equation of system (24) we get

$$(29) \qquad \frac{\bar{\kappa}^2}{(\bar{\tau}^2 + \bar{\kappa}^2)^{3/2}} \left(\frac{\bar{\tau}}{\bar{\kappa}}\right)' = \text{constant},$$

which is desired function.

Conversely, let us assume that the function (29) is satisfied. We define the dual vector

$$(30) \qquad \tilde{d} = \frac{\bar{\tau}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \tilde{T} + \cos \bar{\phi} \tilde{N} + \frac{\bar{\kappa}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \tilde{B},$$

where $\bar{\phi}$ is a dual constant angle between dual vectors $\tilde{d}$ and $\tilde{N}$. If we take derivative of equation (30) and use dual Frenet formulae (4) and (29), we can easily see that $\tilde{d}' = 0$, which gives that $\tilde{d} = \text{constant}$. On the other hand, $\left\langle \tilde{N}, \tilde{d} \right\rangle = \cos \bar{\phi}$ is constant and which means that the curve $\tilde{\alpha}$ is a dual slant helix. $\qquad \square$

Theorem 3.3. *Every dual slant helix is also a dual normalized Darboux helix with same dual axis.*

Proof. Let $\tilde{\alpha}$ be a slant helix in $\mathbb{D}^3$. Then from (30) the axis of dual slant helix is

$$(31) \qquad \tilde{d} = \frac{\bar{\tau}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \tilde{T} + \cos \bar{\phi} \tilde{N} + \frac{\bar{\kappa}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \tilde{B},$$

where $\bar{\phi}$ is constant dual angle between the vectors $\tilde{d}$ and $\tilde{N}$. Considering unit dual Darboux vector $\tilde{W}_0$, equality (31) can be written as follows

$$(32) \qquad \tilde{d} = \tilde{W}_0 + \cos \bar{\phi} \tilde{N},$$

which shows that $\tilde{d}$ lies on the dual plane spanned by $\tilde{W}_0$ and $\tilde{N}$. Since $\bar{\phi}$ is a dual constant, if the dual angle between the dual vectors $\tilde{d}$ and $\tilde{W}_0$ is $\bar{\lambda}$, from (32) and equality $\left\langle \tilde{d},\ \tilde{W}_0 \right\rangle = \left\| \tilde{d} \right\| \cos\bar{\lambda}$, we have

$$(33) \qquad \cos\bar{\lambda} = \frac{1}{\sqrt{1 + \cos^2\bar{\phi}}},$$

is constant. Then we have $\left\langle \tilde{d},\ \tilde{W}_0 \right\rangle$ is constant, i.e, $\tilde{\alpha}$ is also a normalized Darboux helix in $\mathbb{D}^3$ with same dual axis. $\qquad\square$

Theorem 3.4. *Let $\tilde{\alpha}$ be a dual slant helix with dual Frenet frame $\left\{ \tilde{T}, \tilde{N}, \tilde{B} \right\}$ and non-zero dual curvatures $\bar{\kappa}$ and $\bar{\tau}$. The curvatures $\bar{\kappa}$ and $\bar{\tau}$ satisfy the following non-linear equation system,*

$$\left(\frac{\bar{\tau}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \right)' - \bar{\mu}\bar{\kappa} = 0, \quad \left(\frac{\bar{\kappa}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \right)' + \bar{\mu}\bar{\tau} = 0.$$

Proof. Since $\tilde{\alpha}$ is a dual slant helix, the fixed dual unit vector is given by,

$$(34) \qquad \tilde{d} = \frac{\bar{\tau}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \tilde{T} + \cos\bar{\phi}\tilde{N} + \frac{\bar{\kappa}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \tilde{B},$$

where $\bar{\kappa}$ and $\bar{\tau}$ are dual curvatures of $\tilde{\alpha}$. Taking the derivative in each part of the equation (34), we get

$$(35) \qquad \tilde{d}' = \left(\frac{\bar{\tau}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \right)' \tilde{T} + \left(\frac{\bar{\kappa}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \right)' \tilde{B} + \cos\bar{\phi}(-\bar{\kappa}\tilde{T} + \bar{\tau}\tilde{B})$$

Since the system $\left\{ \tilde{T},\ \tilde{B} \right\}$ is linearly independent, we have

$$(36) \qquad \left(\frac{\bar{\tau}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \right)' - \bar{\mu}\bar{\kappa} = 0, \quad \left(\frac{\bar{\kappa}}{\sqrt{\bar{\tau}^2 + \bar{\kappa}^2}} \right)' + \bar{\mu}\bar{\tau} = 0.$$

where $\cos\bar{\phi} = \bar{\mu}$. $\qquad\square$

4. Similar Curves in Dual Space

In this section, we will give the definition of dual similar curves with variable transformation. Then we will present some theorems concerning the relations between dual Frenet elements of dual similar curves.

Definition 4.1. Let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two curves in $\mathbb{D}^3$ parameterized by dual arclengths $\bar{s}_\alpha$ and $\bar{s}_\beta$ with dual curvatures $\bar{\kappa}_\alpha$ and $\bar{\kappa}_\beta$, dual torsions $\bar{\tau}_\alpha$ and $\bar{\tau}_\beta$ and dual Frenet frames $\left\{ \tilde{T}_\alpha,\ \tilde{N}_\alpha,\ \tilde{B}_\alpha \right\}$ and $\left\{ \tilde{T}_\beta,\ \tilde{N}_\beta,\ \tilde{B}_\beta \right\}$,

respectively. $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ are called dual similar curves with variable transformation $\bar{\lambda}_\beta^\alpha$ if there exists a variable transformation

$$(37) \qquad \bar{s}_\alpha = \int \bar{\lambda}_\beta^\alpha(\bar{s}_\beta) \ \mathrm{d}\,\bar{s}_\beta$$

of the dual arclengths such that dual unit tangent vectors are the same for two dual curves, i.e.,

$$(38) \qquad \tilde{T}_\beta(\bar{s}_\beta) = \tilde{T}_\alpha(\bar{s}_\alpha),$$

for all corresponding values of parameters under the transformation $\bar{\lambda}_\beta^\alpha$ which is arbitrary dual function of dual arclength $\bar{s}_\beta$. It is provided that $\bar{\lambda}_\beta^\alpha \bar{\lambda}_\alpha^\beta = 1$. All dual curves satisfying equation (38) are called a family of dual similar curves with variable transformations.

Theorem 4.1. *Let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two regular curves in $\mathbb{D}^3$. $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ are similar curves in $\mathbb{D}^3$ with variable transformation if and only if the dual unit principal normal vectors are the same, i.e.,*

$$(39) \qquad \tilde{N}_\beta(\bar{s}_\beta) = \tilde{N}_\alpha(\bar{s}_\alpha)$$

under the particular variable transformation

$$(40) \qquad \bar{\lambda}_\alpha^\beta = \frac{\mathrm{d}\,\bar{s}_\beta}{\mathrm{d}\,\bar{s}_\alpha} = \frac{\bar{\kappa}_\alpha}{\bar{\kappa}_\beta}.$$

Proof. Let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two regular similar curves with variable transformations in $\mathbb{D}^3$. If we differentiate the equation (38) with respect to $\bar{s}_\beta$ we have

$$(41) \qquad \bar{\kappa}_\beta(\bar{s}_\beta)\tilde{N}_\beta(\bar{s}_\beta) = \bar{\kappa}_\alpha(\bar{s}_\alpha)\tilde{N}_\alpha(\bar{s}_\alpha)\frac{\mathrm{d}\,\bar{s}_\alpha}{\mathrm{d}\,\bar{s}_\beta}.$$

From the equation (41), we obtain the two equations (39) and (40).

Conversely, let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two regular curves in $\mathbb{D}^3$ satisfying two equations (39) and (40). Multiplying equation (39) by $\kappa_\beta(\bar{s}_\beta)$ and integrating the result with respect to $\bar{s}_\beta$ we get

$$(42) \qquad \int \bar{\kappa}_\beta(\bar{s}_\beta)\tilde{N}_\beta(\bar{s}_\beta) \ \mathrm{d}\,\bar{s}_\beta = \int \bar{\kappa}_\beta(\bar{s}_\beta)\tilde{N}_\beta(\bar{s}_\beta)\frac{d\bar{s}_\beta}{\mathrm{d}\,\bar{s}_\alpha} \ \mathrm{d}\,\bar{s}_\alpha.$$

Using equations (39) and (40), the equation (42) becomes

$$(43) \qquad \int \bar{\kappa}_\beta(\bar{s}_\beta)\tilde{N}_\beta(\bar{s}_\beta) \ \mathrm{d}\,\bar{s}_\beta = \int \bar{\kappa}_\alpha(\bar{s}_\alpha)\tilde{N}_\alpha(\bar{s}_\alpha) \ \mathrm{d}\,\bar{s}_\alpha.$$

If we use the first equation of Frenet formulae (4) and integrate the result, we obtain the equation (38) which completes proof. $\qquad\square$

Theorem 4.2. *Let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two regular curves in $\mathbb{D}^3$. Then $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ are similar curves in $\mathbb{D}^3$ with variable transformation if and only if the dual unit binormal vectors are the same, i.e.,*

$$(44) \qquad \tilde{B}_\beta(\bar{s}_\beta) = \tilde{B}_\alpha(\bar{s}_\alpha)$$

under the particular transformation

$$(45) \qquad \bar{\lambda}_\alpha^\beta = \frac{\mathrm{d}\,\bar{s}_\beta}{\mathrm{d}\,\bar{s}_\alpha} = \frac{\bar{\tau}_\alpha}{\bar{\tau}_\beta}.$$

Proof. Let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two regular similar curves with variable transformation in $\mathbb{D}^3$. Then from Definition 4.1 and Theorem 4.1., there exists a variable transformation of dual arc lengths such that the dual unit tangent vectors and the dual unit principle normal vectors are the same. From equations (38) and (39) we have

$$(46) \qquad \tilde{B}_\beta(\bar{s}_\beta) = \tilde{T}_\beta(\bar{s}_\beta) \times \tilde{N}_\beta(\bar{s}_\beta) = \tilde{T}_\alpha(\bar{s}_\alpha) \times \tilde{N}_\alpha(\bar{s}_\alpha) = \tilde{B}_\alpha(\bar{s}_\alpha).$$

Conversely, let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two regular curves in $\mathbb{D}^3$ with the same dual unit binormal vector under the particular transformation $\bar{\lambda}_\alpha^\beta = \frac{\mathrm{d}\,\bar{s}_\beta}{\mathrm{d}\,\bar{s}_\alpha} = \frac{\bar{\tau}_\alpha}{\bar{\tau}_\beta}$ of the dual arclengths. Differentiating the equation (44) with respect to $\bar{s}_\beta$ we get

$$(47) \qquad -\bar{\tau}_\beta(\bar{s}_\beta)\tilde{N}_\beta(\bar{s}_\beta) = -\bar{\tau}_\alpha(\bar{s}_\alpha)\tilde{N}_\alpha(\bar{s}_\alpha)\frac{d\bar{s}_\alpha}{d\bar{s}_\beta}.$$

The equation (47) leads to the following two equations

$$(48) \qquad \tilde{N}_\beta(\bar{s}_\beta) = \tilde{N}_\alpha(\bar{s}_\alpha)$$

and

$$(49) \qquad \frac{\mathrm{d}\,\bar{s}_\alpha}{\mathrm{d}\,\bar{s}_\beta} = \frac{\bar{\tau}_\beta(\bar{s}_\beta)}{\bar{\tau}_\alpha(\bar{s}_\alpha)}.$$

From the hypothesis and equation (48) we have

$$(50) \qquad \tilde{T}_\beta(\bar{s}_\beta) = \tilde{N}_\beta(\bar{s}_\beta) \times \tilde{B}_\beta(\bar{s}_\beta) = \tilde{N}_\alpha(\bar{s}_\alpha) \times \tilde{B}_\alpha(\bar{s}_\alpha) = \tilde{T}_\alpha(\bar{s}_\alpha),$$

which completes the proof. $\qquad\square$

Theorem 4.3. *Let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two regular curves in $\mathbb{D}^3$. Then $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ are dual similar curves with variable transformation if and only if the ratios of dual torsion to dual curvature are the same for all dual curves, i.e.,*

$$(51) \qquad \frac{\bar{\tau}_\beta(\bar{s}_\beta)}{\bar{\kappa}_\beta(\bar{s}_\beta)} = \frac{\bar{\tau}_\alpha(\bar{s}_\alpha)}{\bar{\kappa}_\alpha(\bar{s}_\alpha)},$$

under the particular variable transformation $\bar{\lambda}_\alpha^\beta = \dfrac{\mathrm{d}\,\bar{s}_\beta}{\mathrm{d}\,\bar{s}_\alpha} = \dfrac{\bar{\kappa}_\alpha}{\bar{\kappa}_\beta}$ keeping equal dual total curvatures, i.e.,

$$\tag{52} \bar{\theta}_\beta(\bar{s}_\beta) = \int \bar{\kappa}_\beta \ \mathrm{d}\,\bar{s}_\beta = \int \bar{\kappa}_\alpha \ \mathrm{d}\,\bar{s}_\alpha = \bar{\theta}_\alpha(\bar{s}_\alpha),$$

of the arclengths.

Proof. Let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two dual similar curves with variable transformations. Then Definition 4.1 and Theorem 4.2 gives that there exists a variable transformation of arclengths such that the dual unit tangent and the dual unit binormal vectors are the same. If we differentiate the equations (38) and (44) with respect to $\bar{s}_\beta$, we have

$$\tag{53} \bar{\kappa}_\beta(\bar{s}_\beta)\tilde{N}_\beta(\bar{s}_\beta) = \bar{\kappa}_\alpha(\bar{s}_\alpha)\tilde{N}_\alpha(\bar{s}_\alpha)\frac{\mathrm{d}\,\bar{s}_\alpha}{\mathrm{d}\,\bar{s}_\beta},$$

$$\tag{54} -\bar{\tau}_\beta(\bar{s}_\beta)\tilde{N}_\beta(\bar{s}_\beta) = -\bar{\tau}_\alpha(\bar{s}_\alpha)\tilde{N}_\alpha(\bar{s}_\alpha)\frac{\mathrm{d}\,\bar{s}_\alpha}{\mathrm{d}\,\bar{s}_\beta}.$$

From equations (53) and (54), we obtain the following two equations

$$\tag{55} \bar{\kappa}_\beta(\bar{s}_\beta) = \bar{\kappa}_\alpha(\bar{s}_\alpha)\frac{\mathrm{d}\,\bar{s}_\alpha}{\mathrm{d}\,\bar{s}_\beta},$$

and

$$\tag{56} \bar{\tau}_\beta(\bar{s}_\beta) = \bar{\tau}_\alpha(\bar{s}_\alpha)\frac{\mathrm{d}\,\bar{s}_\alpha}{\mathrm{d}\,\bar{s}_\beta}.$$

From equations (55) and (56), we obtain equation (51) under the variable transformation (52).

Conversely, let $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ be two dual curves such that the equation (51) is satisfied under the variable transformation (52) of the dual arc lengths. From the Theorem 2.1, we know that the tangent vectors $\tilde{T}_\alpha(\bar{s}_\alpha)$ and $\tilde{T}_\beta(\bar{s}_\beta)$ of two curves satisfy vector differential equations as follows:

$$\tag{57} \left(\frac{1}{\bar{f}_\alpha(\bar{\theta}_\alpha)}\tilde{T}''_\alpha(\bar{\theta}_\alpha)\right)' + \left(\frac{1+\bar{f}_\alpha^2(\bar{\theta}_\alpha)}{\bar{f}_\alpha(\bar{\theta}_\alpha)}\right)\tilde{T}'_\alpha(\bar{\theta}_\alpha) - \frac{\bar{f}'_\alpha(\bar{\theta}_\alpha)}{\bar{f}_\alpha^2(\bar{\theta}_\alpha)}\tilde{T}_\alpha(\bar{\theta}_\alpha) = 0,$$

$$\tag{58} \left(\frac{1}{\bar{f}_\beta(\bar{\theta}_\beta)}\tilde{T}''_\beta(\bar{\theta}_\beta)\right)' + \left(\frac{1+\bar{f}_\beta^2(\bar{\theta}_\beta)}{\bar{f}_\beta(\bar{\theta}_\beta)}\right)\tilde{T}'_\beta(\bar{\theta}_\beta) - \frac{\bar{f}'_\beta(\bar{\theta}_\beta)}{\bar{f}_\beta^2(\bar{\theta}_\beta)}\tilde{T}_\beta(\bar{\theta}_\beta) = 0,$$

respectively, where

$$\bar{f}_\alpha(\bar{\theta}_\alpha) = \frac{\bar{\tau}_\alpha(\bar{\theta}_\alpha)}{\bar{\kappa}_\alpha(\bar{\theta}_\alpha)}, \bar{f}_\beta(\bar{\theta}_\beta) = \frac{\bar{\tau}_\beta(\bar{\theta}_\beta)}{\bar{\kappa}_\beta(\bar{\theta}_\beta)}, \bar{\theta}_\alpha = \int \bar{\kappa}_\alpha(\bar{s}_\alpha) \ \mathrm{d}\,\bar{s}_\alpha, \bar{\theta}_\beta = \int \bar{\kappa}_\beta(\bar{s}_\beta) \ \mathrm{d}\,\bar{s}_\beta.$$

From the equation (51) we have $\bar{f}_\beta(\bar{\theta}_\beta) = \bar{f}_\alpha(\bar{\theta}_\alpha)$ under the variable transformation $\bar{\theta}_\beta = \bar{\theta}_\alpha$. Under the equation (51) and the transformation (52), two equations (57) and (58) are the same. So the solutions are the same, i.e., the dual unit tangent vectors are the same. This means that the dual curves $\tilde{\alpha}(\bar{s}_\alpha)$ and $\tilde{\beta}(\bar{s}_\beta)$ are dual similar curves with variable transformation. This completes the proof. $\qquad\square$

After these characterizations we can give the following special cases:

Case 1. If the dual curve $\tilde{\alpha}$ is a general dual helix, i.e., $\frac{\bar{\tau}_\alpha}{\bar{\kappa}_\alpha} = \cot\bar{\phi} = \bar{m}$ is a constant and $\bar{\phi}$ is dual constant angle between dual unit tangent vector and a fixed dual unit vector, then from Theorem 4.3, any dual similar curve $\tilde{\beta}$ of this dual helix has the property $\frac{\bar{\tau}_\beta}{\bar{\kappa}_\beta} = \bar{m}$. Thus we have the following corollary:

Corollary 4.1. *The family of general dual helices with fixed dual angle $\bar{\phi}$ between a fixed dual unit vector and dual unit tangent vector forms a family of dual similar curves with variable transformation.*

Case 2. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two dual slant helices such that the transformation (52) is satisfied. If we use the relation (9) and (52), it is easy to prove that:

$$\frac{\bar{\tau}_\beta}{\bar{\kappa}_\beta} = \frac{\bar{m}\,\bar{\theta}_\beta}{\sqrt{1 - \bar{m}^2\,\bar{\theta}_\beta^2}} = \frac{\bar{m}\,\bar{\theta}_\alpha}{\sqrt{1 - \bar{m}^2\,\bar{\theta}_\alpha^2}} = \frac{\bar{\tau}_\alpha}{\bar{\kappa}_\alpha},$$

where $\bar{m} = \cot\bar{\phi}$ is a constant and $\bar{\phi}$ is the angle between the dual unit principal normal vector of $\tilde{\alpha}$ and a fixed dual unit vector. Thus we have the following corollary:

Corollary 4.2. *The family of dual slant helices forms a family of dual similar curves with variable transformation.*

Case 3. Let $\tilde{\alpha}$ be a dual normalized Darboux helix with $\bar{\kappa}^2 + \bar{\tau}^2 =$ constant. Then from Theorem 3.4 we have that $\tilde{\alpha}$ is a dual slant helix. Thus we can give the following corollary.

Corollary 4.3. *The family of dual normalized Darboux helices with $\bar{\kappa}^2 + \bar{\tau}^2 =$ constant forms a family of dual similar curves with variable transformation.*

5. Conclusion

The characterizations of special curves are important and fascinating problem of differential geometry. These curves are characterized by relationships between the curvatures and torsions of curves and well-known examples of such curves are helices and slant helices which have been studied in different spaces such as Euclidean space and Minkowski space. But there are no many studies on these curves in dual space which is a more general space than the others. In this space, a dual curve consists of two real curves. So, the characterizations of dual special curves include the characterizations of real space curves. This paper gives some new characterizations of dual slant helix. Moreover, the dual normalized Darboux helix and dual similar curves are introduced and the relationships between these special dual curves are obtained.

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