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On the Logarithmic Integral and the Convolution

BRIAN FISHER AND FATMA AL-SIHERY

ABSTRACT. The logaritmic integral $\mathrm{li}(x)$ and its associated functions $\mathrm{li}_+(x)$ and $\mathrm{li}_-(x)$ are defined as locally summable functions on the real line. Some convolutions and neutrix convolutions of these functions and other functions are then found.

1. INTRODUCTION

The *logarithmic integral* $\mathrm{li}(x)$ (see Abramowitz and Stegun [1]) is defined by

$$\mathrm{li}(x) = \begin{cases} \displaystyle\int_0^x \frac{\mathrm{d}\,t}{\ln|t|}, & \text{for } |x| < 1, \\ \displaystyle\mathrm{PV}\int_0^x \frac{\mathrm{d}\,t}{\ln t}, & \text{for } x > 1, \\ \displaystyle\mathrm{PV}\int_0^x \frac{\mathrm{d}\,t}{\ln|t|}, & \text{for } x < -1 \end{cases}$$

$$= \begin{cases} \displaystyle\int_0^x \frac{\mathrm{d}\,t}{\ln|t|}, & \text{for } |x| < 1, \\ \displaystyle\lim_{\epsilon\to 0^+}\left[\int_0^{1-\epsilon} \frac{\mathrm{d}\,t}{\ln t} + \int_{1+\epsilon}^x \frac{\mathrm{d}\,t}{\ln t}\right], & \text{for } \ x > 1, \\ \displaystyle\lim_{\epsilon\to 0^+}\left[\int_0^{-1+\epsilon} \frac{\mathrm{d}\,t}{\ln|t|} + \int_{-1-\epsilon}^x \frac{\mathrm{d}\,t}{\ln|t|}\right], & \text{for } \ x < -1 \end{cases}$$

where PV denotes the Cauchy principal value of the integral. We will therefore write

$$\mathrm{li}(x) = \mathrm{PV}\int_0^x \frac{\mathrm{d}\,t}{\ln|t|}$$

for all values of x.

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The associated functions $\mathrm{li}_+(x)$ and $\mathrm{li}_-(x)$ are now defined by

$$\mathrm{li}_+(x) = H(x)\,\mathrm{li}(x), \quad \mathrm{li}_-(x) = H(-x)\,\mathrm{li}(x),$$

where $H(x)$ denotes Heaviside's function.

The classical definition of the convolution of two functions $f$and g is as follows:

Definition 1. Let f and g be functions. Then the *convolution* $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,\mathrm{d}\,t$$

for all points x for which the integral exist.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f \tag{1}$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{2}$$

Definition 1 can be extended to define the convolution $f * g$ of two distributions f and g in D' with the following definition, see Gel'fand and Shilov [10].

Definition 2. Let f and g be distributions in D'. Then the *convolution* $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle$$

for arbitrary φ in D, provided f and g satisfy either of the conditions

(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition then equations (1) and (2) are satisfied.

Before proving our main results, we need the following lemma which was proved in [5].

Lemma 1.

$$\mathrm{li}(x^r) = \mathrm{PV} \int_0^x t^{r-1} \frac{\mathrm{d}\,t}{\ln|t|}, \tag{3}$$

for $r = 1, 2, \ldots$.

The following theorem was also proved in [5].

Theorem 1. *The convolutions* $\mathrm{li}_+(x) * x_+^r$ *and* $\ln^{-1} x_+ * x_+^r$ *exist and*

$$\mathrm{li}_+(x) * x_+^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} x^i \,\mathrm{li}_+(x^{r-i+2}),$$

$$\ln^{-1} x_+ * x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} x^i \,\mathrm{li}_+(x^{r-i+1}),$$

for $r = 0, 1, 2, \ldots$.

We now prove the following generalization of Theorem 1.

Theorem 2. *The convolutions* $x^s \,\mathrm{li}_+(x) * x_+^r$ *and* $x^s \ln_+^{-1}(x) * x_+^r$ *exist and*

$$x^s \,\mathrm{li}_+(x) * x_+^r = \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i}}{(r+s-i+1)} [\mathrm{li}_+(x^{r+s+1}) - \mathrm{li}_+(x^{r+s-i+2})], \tag{4}$$

$$\begin{aligned} x^s \ln_+^{-1}(x) * x_+^r = {} & \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^{r-i+1} r}{(r+s-i)} [\ln_+(x^{r+s}) - \mathrm{li}_+(x^{r+s-i+1})] \\ & - \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i} s}{(r+s-i)} [\mathrm{li}_+(x^{r+s}) - \mathrm{li}_+(x^{r+s-i+1})], \end{aligned} \tag{5}$$

for $r = 0, 1, 2, \ldots$ *and* $s = 1, 2, \ldots$.

Proof. It is obvious that $x^s \,\mathrm{li}_+(x) * x_+^r = 0$ if $x < 0$. When $x > 0$, we have

$$\begin{aligned} x^s \,\mathrm{li}_+(x) * x_+^r &= \mathrm{PV} \int_0^x t^s (x-t)^r \int_0^t \frac{\mathrm{d}\,u}{\ln u} \,\mathrm{d}\,t \\ &= \mathrm{PV} \int_0^x \frac{1}{\ln u} \int_u^x t^s (x-t)^r \,\mathrm{d}\,t \,\mathrm{d}\,u \\ &= \mathrm{PV} \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i} x^i}{(r+s-i+1)} \int_0^x \frac{x^{r+s-i+1} - u^{r+s-i+1}}{\ln u} \,\mathrm{d}\,u \\ &= \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i}}{(r+s-i+1)} [\mathrm{li}_+(x^{r+s+1}) - \mathrm{li}_+(x^{r+s-i+2})], \end{aligned}$$

on using equation (3), proving equation (4).

Next, using equations (2) and (4), we have

$$\begin{aligned} & [x^s \ln^{-1} x_+ + s x^{s-1} \,\mathrm{li}_+(x)] * x_+^r = r x^s \,\mathrm{li}_+(x) * x_+^{r-1} \\ & = \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^{r-i+1} r}{(r+s-i)} [\ln_+(x^{r+s}) - \mathrm{li}_+(x^{r+s-i+1})] \\ & = x^s \ln^{-1} x_+ * x_+^r + \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i} s}{(r+s-i)} [\mathrm{li}_+(x^{r+s}) - \mathrm{li}_+(x^{r+s-i+1})] \end{aligned}$$

and equation (5) follows. □

Theorem 3. *The convolutions* $x^s \operatorname{li}_-(x) * x_-^r$ *and* $x^s \ln^{-1} x_- * x_-^r$ *exist and*

$$x^s \operatorname{li}_-(x) * x_-^r = \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i+1}}{(r+s-i+1)} [\operatorname{li}_-(x^{r+s+1}) - \operatorname{li}_-(x^{r+s-i+2})], \tag{6}$$

$$\begin{aligned} x^s \ln_-^{-1}(x) * x_-^r = & \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^{r-i} r}{(r+s-i)} [\ln_-(x^{r+s}) - \operatorname{li}_-(x^{r+s-i+1})] \\ & + \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i} s}{(r+s-i)} [\operatorname{li}_-(x^{r+s}) - \operatorname{li}_-(x^{r+s-i+1})], \end{aligned} \tag{7}$$

for $r = 0, 1, 2, \ldots$ *and* $s = 1, 2, \ldots$.

Proof. It is obvious that $x^s \operatorname{li}_-(x) * x_-^r = 0$ if $x > 0$.

When $x < 0$, we have

$$\begin{aligned} x^s \operatorname{li}_-(x) * x_-^r &= PV \int_x^0 t^s (x-t)^r \int_t^0 \frac{\mathrm{d}\,u}{\ln|u|} \,\mathrm{d}\,t \\ &= \mathrm{PV} \int_x^0 \frac{1}{\ln|u|} \int_x^u t^s (x-t)^r \,\mathrm{d}\,t \,\mathrm{d}\,u \\ &= PV \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i+1} x^i}{(r+s-i+1)} \int_x^0 \frac{x^{r+s-i+1} - u^{r+s-i+1}}{\ln|u|} \,\mathrm{d}\,u \\ &= \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i+1}}{(r+s-i+1)} [\operatorname{li}_-(x^{r+s+1}) - \operatorname{li}_-(x^{r+s-i+2})], \end{aligned}$$

on using equation (3), proving equation (6).

Next, using equations (2) and (6), we have

$$\begin{aligned} &[-x^s \ln^{-1} x_- + s x^{s-1} \operatorname{li}_-(x)] * x_-^r = -r x^s \operatorname{li}_-(x) * x_-^{r-1} \\ &= \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^{r-i} r}{(r+s-i)} [\ln_+(x^{r+s}) - \operatorname{li}_+(x^{r+s-i+1})] \\ &= x^s \ln^{-1} x_- * x_-^r + \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i+1} s}{(r+s-i)} [\operatorname{li}_-(x^{r+s}) - \operatorname{li}_-(x^{r+s-i+1})] \end{aligned}$$

and equation (7) follows. □

The above definition of the convolution is rather restrictive and so a neutrix convolution was defined in [3]. In order to define the neutrix convolution, we first of all let τ be a function in D, see [11], satisfying the the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

The following definition of the non-commutative neutrix convolution was given in [3].

Definition 3. Let f and g be distributions in D' and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the *non-commutative neutrix convolution* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}_{n \in N}$, provided the limit h exists in the sense that

$$\underset{n\to\infty}{\text{N}-\lim}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in D, where N is the neutrix, see van der Corput [2], having domain N' the positive reals and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \ \lambda > 0, \ r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

It is easily seen that any results proved with the original definition of the convolution hold with the new definition of the neutrix convolution. The following results proved in [3] hold, first showing that the neutrix convolution is a generalization of the convolution.

Theorem 4. *Let f and g be distributions in D', satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

Theorem 5. *Let f and g be distributions in D' and suppose that the neutrix convolution $f \circledast g$ exists. Then the neutrix convolution $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'.$$

If $\underset{n\to\infty}{\text{N}-\lim}\langle (f\tau_n') * g, \varphi\rangle$ *exists and equals* $\langle h, \varphi\rangle$ *for all* φ *in* D, *then* $f' \circledast g$ *exists and*

$$(f \circledast g)' = f' \circledast g + h.$$

In the following, we need to extend our set of negligible functions to include finite linear sums of the functions $n^s \operatorname{li}(n^r)$ and $n^s \ln^{-r} n$, $(n > 1)$ for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$.

Before proving our next results, we need the following lemmas, which were proved in [5].

Lemma 2.

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} \tau_n(t)\,\mathrm{li}(t)t^r\,dt = 0 \tag{8}$$

for $r = 1, 2, \ldots$.

Lemma 3.

$$\mathrm{N}\!-\!\lim_{n\to\infty} \mathrm{li}[(x+n)^r] = 0, \tag{9}$$

$$\mathrm{N}\!-\!\lim_{n\to\infty} n^r\,\mathrm{li}[(x+n)] = 0 \tag{10}$$

for $r = 1, 2, \ldots$.

The next theorem was also proved in [5].

Theorem 6. *The neutrix convolutions* $\mathrm{li}_+(x) \circledast x^r$ *and* $\ln^{-1} x_+ \circledast x^r$ *exist and*

$$\mathrm{li}_+(x) \circledast x^r = 0,$$

$$\ln^{-1} x_+ \circledast x^r = 0$$

for $r = 0, 1, 2, \ldots$ *and* $s = 1, 2, \ldots$.

We now prove some further results involving the neutrix convolution. First of all we have the following generalization of Theorem 6.

Theorem 7. *The neutrix convolutions* $x^s\,\mathrm{li}_+(x) \circledast x^r$ *and* $x^s \ln^{-1} x_+ \circledast x^r$ *exist and*

$$x^s\,\mathrm{li}_+(x) \circledast x^r = 0, \tag{11}$$

$$x^s \ln^{-1} x_+ \circledast x^r = 0 \tag{12}$$

for $r = 0, 1, 2, \ldots$ *and* $s = 1, 2, \ldots$.

Proof. We put $[x^s\,\mathrm{li}_+(x)]_n = x^s\,\mathrm{li}_+(x)\tau_n(x)$. Then the convolution $[x^s\,\mathrm{li}_+(x)]_n * x^r$ exists and

$$[x^s\,\mathrm{li}_+(x)]_n * x^r = \int_0^n t^s\,\mathrm{li}(t)(x-t)^r\,\mathrm{d}\,t + \int_n^{n+n^{-n}} \tau_n(t)t^s\,\mathrm{li}(t)(x-t)^r\,\mathrm{d}\,t, \tag{13}$$

where

$$\begin{aligned}
\int_0^n t^s\,\mathrm{li}(t)(x-t)^r\,\mathrm{d}\,t &= \mathrm{PV}\int_0^n t^s(x-t)^r \int_0^t \frac{\mathrm{d}\,u}{\ln u}\,\mathrm{d}\,t \\
&= \mathrm{PV}\int_0^n \frac{1}{\ln u}\int_u^n t^s(x-t)^r\,\mathrm{d}\,t\,\mathrm{d}\,u \\
&= \mathrm{PV}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{r-i}x^i}{(r+s-i+1)}\int_0^n \frac{n^{r+s-i+1}-u^{r+s-i+1}}{\ln u}\,\mathrm{d}\,u \\
&= \sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{r-i}x^i}{(r+s-i+1)}[n^{r+s-i+1}\,\mathrm{li}_+(n) - \mathrm{li}_+(n^{r+s-i+2})].
\end{aligned}$$

It follows that

$$\underset{n\to\infty}{\text{N}-\lim}\int_0^n t^s \operatorname{li}(t)(x-t)^r \,\mathrm{d}\,t = 0. \tag{14}$$

Using Lemma 2, we have

$$\int_n^{n+n^{-n}} \tau_n(t)t^s \operatorname{li}(t)(x-t)^r \,\mathrm{d}\,t = 0 \tag{15}$$

and equation (11) now follows from equations (13), (14) and (15).

Differentiating equation (11) and using Theorem 5, we get

$$\begin{aligned}
[sx^{s-1}\operatorname{li}_+(x) + x^s \ln^{-1} x_+] \circledast x^r + \underset{n\to\infty}{\text{N}-\lim}[x^s \operatorname{li}_+(x)\tau_n'(x)] * x^r &= \\
= x^s \ln^{-1} x_+ \circledast x^r + \underset{n\to\infty}{\text{N}-\lim}[x^s \operatorname{li}_+(x)\tau_n'(x)] * x^r & \\
= 0, &
\end{aligned} \tag{16}$$

where, on integration by parts, we have

$$\begin{aligned}
[x^s \operatorname{li}_+(x)\tau_n'(x)] * x^r &= \int_n^{n+n^{-n}} \tau_n'(t)\operatorname{li}(t)t^s(x-t)^r \,\mathrm{d}\,t \\
&= -\operatorname{li}(n)n^s(x-n)^r - \int_n^{n+n^{-n}} \ln^{-1}(t)t^s(x-t)^r\tau_n(t)\,\mathrm{d}\,t \\
&\quad + \int_n^{n+n^{-n}} [r\operatorname{li}(t)t^s(x-t)^{r-1} - s\operatorname{li}(t)t^{s-1}(x-t)^r]\tau_n(t)\,\mathrm{d}\,t.
\end{aligned} \tag{17}$$

It is clear that

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} \ln^{-1}(t)t^s(x-t)^r\tau_n(t)\,\mathrm{d}\,t = 0 \tag{18}$$

and now equation (12) follows from Lemma 2 and equations (16), (17) and (18). □

Theorem 8. *The neutrix convolutions $x^s \operatorname{li}_-(x) \circledast x^r$ and $x^s \ln^{-1} x_- \circledast x^r$ exist and*

$$x^s \operatorname{li}_-(x) \circledast x^r = 0, \tag{19}$$

$$x^s \ln^{-1} x_- \circledast x^r = 0 \tag{20}$$

for $r = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots$.

Proof. The proofs of equations (19) and (20) are similar to the proofs of Theorems 3 and 7. □

Corollary 8.1. *The neutrix convolutions $x^s \operatorname{li}(x) \circledast x^r$ and $x^s \ln^{-1}|x| \circledast x^r$ exist and*

$$x^s \operatorname{li}(x) \circledast x^r = 0, \tag{21}$$

$$x^s \ln^{-1}|x| \circledast x^r = 0 \tag{22}$$

for $r = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots$.

Proof. Equation (21) follows on adding equation (11) and (19) and equation (22) follows on adding equations (12) and (20). □

Theorem 9. *The neutrix convolutions* $x^r \circledast x^s \operatorname{li}_+(x)$ *and* $x^r \circledast x^s \ln^{-1} x_+$ *exist and*

$$x^r \circledast x^s \operatorname{li}_+(x) = 0, \tag{23}$$

$$x^r \circledast x^s \ln^{-1} x_+ = 0 \tag{24}$$

for $r = 0, 1, 2, \ldots$.

Proof. This time we put $(x^r)_n = x^r \tau_n(x)$ for $r = 0, 1, 2, \ldots$. Then the convolution $(x^r)_n * x^s \operatorname{li}_+(x)$ exists and

$$\begin{aligned} (x^r)_n * x^s \operatorname{li}_+(x) &= \int_0^{x+n} t^s \operatorname{li}(t)(x-t)^r \, \mathrm{d}\, t \\ &\quad + \int_{x+n}^{x+n+n^{-n}} \tau_n(x-t) t^s \operatorname{li}(t)(x-t)^r \, \mathrm{d}\, t, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \int_0^{x+n} t^s \operatorname{li}(t)(x-t)^r \, \mathrm{d}\, t &= PV \int_0^{x+n} t^s (x-t)^r \int_0^t \ln^{-1} u \, \mathrm{d}\, u \, \mathrm{d}\, t \\ &= PV \int_0^{x+n} \ln^{-1} u \int_u^{x+n} t^s (x-t)^r \, \mathrm{d}\, t \, \mathrm{d}\, u \\ = PV \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i} x^i (x+n)^{r+s-i+1}}{r+s-i+1} & \int_0^{x+n} \frac{\mathrm{d}\, u}{\ln u} \\ - PV \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} x^i & \int_0^{x+n} \frac{u^{r+s-i+1}}{(r+s-i+1) \ln u} \, \mathrm{d}\, u \\ = \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i} x^i (x+n)^{r+s-i+1}}{r+s-i+1} & \operatorname{li}(x+n) \\ - \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i} x^i}{r+s-i+1} & \operatorname{li}[(x+n)^{r+s-i+2}], \end{aligned}$$

on using Lemma 1.

Hence, on using Lemma 3, we have

$$\mathrm{N-}\lim_{n \to \infty} \int_0^{x+n} t^s \operatorname{li}(t)(x-t)^r \, \mathrm{d}\, t = 0. \tag{26}$$

Further, using lemma 2, it is easily seen that

$$\lim_{n \to \infty} \int_{x+n}^{x+n+n^{-n}} \tau_n(x-t) \operatorname{li}(t)(x-t)^r \, \mathrm{d}\, t = 0 \tag{27}$$

and equation (23) follows from equations (25), (26) and (27).

Differentiating equation (23), using Theorem 5, gives

$$x^r \circledast x^s \ln^{-1} x_+ + sx^r \circledast x^{s-1} \operatorname{li}_+(x) = 0$$

and equation (24) follows on using equation (23). □

Theorem 10. *The neutrix convolutions $x^r \circledast \operatorname{li}_-(x)$ and $x^r \circledast \ln^{-1} x_-$ exist and*

$$x^r \circledast \operatorname{li}_-(x) = 0, \tag{28}$$

$$x^r \circledast \ln^{-1} x_- = 0 \tag{29}$$

for $r = 0, 1, 2, \ldots$.

Proof. The proofs of equations (28) and (29) are similar to the proofs of Theorems 3 and 9. □

Corollary 10.1. *The neutrix convolutions $x^r \circledast \operatorname{li}(x)$ and $x^r \circledast \ln^{-1} |x|$ exist and*

$$x^r \circledast \operatorname{li}(x) = 0, \tag{30}$$

$$x^r \circledast \ln^{-1} |x| = 0 \tag{31}$$

for $r = 0, 1, 2, \ldots$.

Proof. Equation (30) follows on adding equation (23) and (28) and equation (31) follows on adding equations (24) and (29). □

For further results involving the neutrix convolution, see [4], [7], [8], [6] and [9].

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Brian Fisher
Department of Mathematics
University of Leicester
Leicester, LE1 7RH
England
E-mail address: fbr@le.ac.uk

Fatma Al-Sihery
Department of Mathematics
King Abdulaziz University
Jeddah
Saudi Arabia
E-mail address: falserehi@kau.edu.sa