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Solvability of Boundary Value Problems for Second Order Impulsive Differential Equations on Whole Line with a Non-Carathédory Nonlinearity

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Abstract. We study a class of boundary value problems of the impulsive differential equations on whole lines with a non-Carathéodory nonlinearity. Sufficient conditions to guarantee the existence of solutions are established. A new Banach function space X and its relatively compact property of subset of X is proved. An example is given to illustrate the main results.

1. Introduction

The asymptotic theory of ordinary differential equations is an area in which there is great activity among a large number of investigators. In this theory, it is of great interest to investigate, in particular, the existence of solutions with prescribed asymptotic behavior, which are global in the sense that they are solutions on the whole line (or half line). The existence of global solutions with prescribed asymptotic behavior is usually formulated as the existence of solutions of boundary value problems on the whole line (or half line).

These problems arise from real world applications. For example, some chemical and biological systems can be modelled by an autocatalytic process (see, e.g. [23, 21]. In many of these process the system can support propagating wavefronts due to a combination of a reaction effect and a molecular diffusion. The pioneering model in this framework is due to Fisher, [20], who suggested the equation $u_t = u_{xx} + u(1-u)$ for studying the spatial spread of a favored gene in a population. The simplest generalization of that equation is the so called Fisher-Kolmogorov's equation $u_t = u_{xx} + f(u)$, where f is a given function with two zeroes, say $u = 0$ and $u = 1$, and positive on $(0, 1)$

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so that $u = 0$ and $u = 1$ are the only two stationary states of this equation. Equations like this one arises in many problems suggested, for instance, by the classical theory of population genetics or by certain flame propagation problems in chemical reactor theory (see, e.g. [6]). A travelling wavefront or travelling wave solution (t.w.s., in short) of this equation is a solution $u(t, x)$ having a constant profile, that is, such that $u(t, x) = \phi(x - ct)$, for some fixed $\phi(\xi)$ (called the wave shape) and a constant c (called the wave speed). Specially important for the applications are t.w.s. connecting the two stationary states, $u = 0$ and $u = 1$.

A simple calculation shows that if $u(t, x) = \phi(x - ct)$ is a t.w.s. of $u_t = u_{xx} + f(u)$, then the wave shape ϕ is a solution of the ODE $u'' + cu' + f(u) = 0$. When a t.w.s. connects the stationary states, its corresponding wave shape is a positive heteroclinic solution of $u'' + cu' + f(u) = 0$ that connects the equilibria 1 and 0, that is, a solution of $u'' + cu' + f(u) = 0$ defined on R and satisfying $u(t) \in (0, 1)$ for all $t \in \mathbb{R}$ and $\lim_{t\to-\infty} u(t) = 1$ and $\lim_{t\to+\infty} u(t) = 0$, see [1]. Thus it comes the following boundary value problem:

$$\left(e^{ct}u'\right)' + e^{ct}f(u) = 0,\ t \in \mathbb{R},\ \lim_{t\to-\infty} u(t) = 1,\ \lim_{t\to+\infty} u(t) = 0.$$

It is well known that the homogeneous and non-homogeneous Robin boundary value problems of the Lane-Emden equations are as follows, respectively,

$$-\Delta u(x) = u^p(x), x \in \Omega, \frac{\partial u}{\partial \overrightarrow{n}} + b(x)u = 0 \text{ on } \partial\Omega,$$
$$-\Delta u(x) = u^p(x) + \lambda f(x), x \in \Omega, \frac{\partial u}{\partial \overrightarrow{n}} + b(x)u = 0 \text{ on } \partial\Omega,$$

where Ω is a domain in the n-dimensional Euclidean space R^n, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Lane-Emden equations arise naturally from the study of various nonlinear phenomena, such as pattern formation, population evolution, chemical reaction and has attracted considerable attention in recent years [27, 29, 34].

One can see that the one-dimensional Robin boundary value problems of the Lane-Emden equations on whole line are as follows:

$$\begin{aligned} x''(t) + [x(t)]^p = 0, &\quad t \in \mathbb{R},\\ \lim_{t\to-\infty} x'(t) - \alpha \lim_{t\to-\infty} x(t) = 0,&\\ \lim_{t\to+\infty} x'(t) + \beta \lim_{t\to+\infty} x(t) = 0,&\\ x''(t) + [x(t)]^p + \lambda f(t) = 0, &\quad t \in \mathbb{R},\\ \lim_{t\to-\infty} x'(t) - \alpha \lim_{t\to-\infty} x(t) = 0,&\\ \lim_{t\to+\infty} x'(t) + \beta \lim_{t\to+\infty} x(t) = 0.& \end{aligned}$$

In recent years, the existence of solutions of boundary value problems of the differential equations governed by nonlinear differential operator $[\Phi(u')]'$

$= [|u'|^{p-2}u']'$ has been studied by many authors, see [7, 12, 13, 21, 22, 9, 11, 10, 14, 16, 17].

Impulsive differential equation is one of the main tools to study the dynamics of processes in which sudden changes occur. The theory of impulsive differential equation has recently received considerable attention, see [30, 31, 32]. However, the study on existence of positive solutions of nonlocal boundary value problems for impulsive differential equations on whole real line has not been sufficiently developed [2, 3, 5, 4, 8, 9, 26].

In all above mentioned papers, the boundary conditions are subjected to the two end points 0 and $+\infty$ (or $-\infty$ and $+\infty$) and the solutions obtained are defined on $[0, +\infty)$ (or R). An interesting question occurs: when one subjects the boundary conditions on two intermediate points ξ, η, how can we get solutions defined on R of a boundary value problem of differential equations on whole line? This is the first motivation of the present paper.

On the other hand, in known papers [10, 11, 12, 13, 14, 16, 17, 18, 33], concerning the differential equations $[\Phi(\rho(t)x'(t))]' + f(t, x(t), \rho(t)x'(t)) = 0$, it is supposed that $(t, u, v) \to f(t, u, \rho(t)v)$ is a Carathéodory function, i.e.,

(i) $t \to f(t, u, \rho(t)v)$ is integral on R for each $(u, v) \in \mathbb{R}^2$;

(ii) $(u, v) \to f(t, u, \rho(t)v)$ is continuous for almost all $t \in \mathbb{R}$;

(iii) for each $r > 0$ there exists a integral function $\phi_r : R \to \mathbb{R}$ such that $|f(t, u, \rho(t)v)| \le \phi_r(t)$ for almost all $t \in \mathbb{R}$ and $|u|, |v| \le r$.

To the best of our knowledge, there has been no paper concerning the solvability of boundary value problem of $[\Phi(\rho(t)x'(t))]' + f(t, x(t), \rho(t)x'(t)) = 0, t \in= R$ with $(t, u, v) \to p(t)f(t, u, v)$ being a non-Carathéodory function: $t \to f(t, u, \rho(t)v)$ is not integral on $= R$.

Motivated by mentioned papers, to fill this gap, we consider the following boundary value problem for the impulsive singular differential equation on the whole line

$$(1)\qquad \begin{cases} [\Phi(\rho(t)x'(t))]' + p(t)f(t, x(t), \rho(t)x'(t)) = 0, \text{ a.e. } t \in \mathbb{R}, \\ x(\xi) = \int_{-\infty}^{+\infty} m(s)\phi(s, x(s), \rho(s)x'(s))ds, \\ \rho(\eta)x'(\eta) = \int_{-\infty}^{+\infty} n(s)\psi(s, x(s), \rho(s)x'(s))ds, \\ \Delta x(t_i) = \lim\limits_{t \to t_i^+} x(t) - x(t_i) = I(t_i, x(t_i), \rho(t_i)x'(t_i)), i \in \mathbf{Z}, \\ \Delta\Phi(\rho(t_i)x'(t_i)) = \lim\limits_{t \to t_i^+} \Phi(\rho(t)x'(t)) - \Phi(\rho(t_i)x'(t_i)) \\ \qquad = J(t_i, x(t_i), \rho(t_i)x'(t_i)), i \in \mathbf{Z}, \end{cases}$$

where

(i) $\Phi(x) = |x|^{k-2}x$ with $k > 1$, the inverse of Φ is denoted by Φ^{-1} and $\Phi^{-1}(x) = |x|^{l-2}x$ with $\frac{1}{k} + \frac{1}{l} = 1$,

(ii) $p : \mathbb{R} \to [0, \infty)$ with $p \in L^1_{loc}(\mathbb{R})$ and $\int_{-\infty}^{0} p(t)dt = \int_0^{+\infty} p(s)ds = +\infty$

(iii) $\rho : \mathbb{R} \to [0, \infty)$ with $\frac{\Phi^{-1}(\tau(\cdot))}{\rho(\cdot)} \in L^1_{loc}(\mathbb{R})$ and $\int_{-\infty}^{0} \frac{\Phi^{-1}(\tau(u))}{\rho(u)}du = \int_0^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)}du = +\infty$ with $\tau(t) = 1 + \left|\int_\eta^t p(s)ds\right|$,

(iv) $f, \phi, \psi : \mathbb{R}^3 \to \mathbb{R}$ are weak Carathéodory functions (see Definition 2.1 in Section 2), $m, n \in L^1(\mathbb{R})$,

(v) $\mathbf{Z} = \{0, \pm 1, \pm 2, \cdots\}$, $\{t_i : i \in \mathbf{Z}\}$ is a increasing sequence with $\lim_{i \to -\infty} t_i = -\infty$ and $\lim_{i \to +\infty} t_i = +\infty$,

(vi) $I, J : \{t_i : i \in \mathbf{Z}\} \times \mathbb{R}^2 \to \mathbb{R}$ are discrete Carathéodory functions (see Definition 2.2 in Section 2).

The homogeneous form of boundary conditions in (1) is as follows: $x(\xi) = 0, \rho(\eta)x'(\eta) = 0$, which comes from a generalization of the boundary conditions $a \lim_{t \to -\infty} x(t) - bx(\xi) = c \lim_{t \to +\infty}(t) - d\rho(\eta)x'(\eta) = 0$. This kind of boundary conditions arise in the study of heat flow problems involving a bar of infinite length with two controllers at $t = -\infty$ and $t = +\infty$ adding or removing heat according to the temperatures and the change of temperature detected by two sensors at $t = \xi$ and $t = \eta$ respectively. But these two controllers do not work well. Problem (1) is also a generalization of the initial value problem, but the solutions gotten are defined on $\mathbb{R}$. One knows that $x''(t) = [1 + x'(t)]^2, t \in \mathbb{R}, x'(0) = 0, x(0) = 1$ (corresponding to (1), $\Phi(x) = x, \rho(t) = 1, p(t) = 1$ and $f(t, x, x') = 1 + x'^2$, $I = J = 0$ and (i)-(vi) are satisfied) has solution $x(t) = -\ln|\cos t|$, it blowup at time $t = \pm\frac{\pi}{2}$. From this point, it is of interest and meaningful to study the existence of solutions of (1).

The purpose is to establish sufficient conditions for the existence of solutions of BVP(1). Since (ii) and f is a weak Carathédory function, then $(t, u, v) \to p(t)f(t, u, \rho(t)v)$ is a non-Carathéodory function on $\mathbb{R}^3$. The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the existence result of solutions of BVP(1) is proved in Section 3. Finally in Section 4, an example is given to illustrate the main results.

2. Preliminary Results

In this section, we present some background definitions. The preliminary results are given too. Denote

$$\tau(t) = 1 + \left|\int_\eta^t p(s)ds\right|, \quad \sigma(t) = 1 + \left|\int_\xi^t \frac{\Phi^{-1}(\tau(u))}{\rho(u)}du\right|.$$

It is easy to show that σ, τ are continuous on R and $\lim_{t \to \pm\infty} \sigma(t) = \lim_{t \to \pm\infty} \tau(t) = +\infty$.

Definition 2.1. $F : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is called a weak Carathédory function, that is

(i) $(u, v) \to F\left(t, \sigma(t)u, \Phi^{-1}(\tau(t))v\right)$ is continuous on $\mathbb{R}^2$ for a.e. $t \in \mathbb{R}$,

(ii) for each $r > 0$, there exists nonnegative function $M_r \geq 0$ such that $|u|, |v| \leq r$ implies

$$\left|F\left(t, \sigma(t)u, \Phi^{-1}(\tau(t))v\right)\right| \leq M_r, a.e.t \in \mathbb{R}.$$

Definition 2.2. $H : \{t_i : i \in Z\} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a discrete Carathédory function, that is

(i) $(x, y) \to H\left(t_s, \sigma(t_s)x, \Phi^{-1}(\tau(t_s))y\right)$ is continuous on $\mathbb{R}^2$ for all $s \in \mathbf{Z}$,

(ii) for each $r > 0$, there exists nonnegative constants $M_{ir} \geq 0 (i \in Z)$ such that $|x|, |y| \leq r$ implies

$$\left|H\left(t_s, \sigma(t_s)x, \Phi^{-1}(\tau(t_s))y\right)\right| \leq M_{sr}, s \in \mathbf{Z}, \sum_{s=-\infty}^{+\infty} M_{sr} < +\infty.$$

Definition 2.3. Let X be a real Banach space. An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Choose

$$X = \left\{ x : \mathbb{R} \to \mathbb{R} : \begin{array}{c} x|_{(t_s, t_{s+1}]},\ \rho x'|_{(t_s, t_{s+1}]} \text{ is continuous}, s \in \mathbf{Z}, \\ \text{and there exist the limits} \\ \lim_{t\to t_s^+} x(t),\ \lim_{t\to t_s^+} \rho(t)x'(t), s \in \mathbf{Z}, \\ \lim_{t\to-\infty} \frac{x(t)}{\sigma(t)},\ \lim_{t\to+\infty} \frac{x(t)}{\sigma(t)} \\ \lim_{t\to-\infty} \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))},\ \lim_{t\to+\infty} \frac{\rho(t)x'(t)}{\Phi^{1}(\tau(t))} \end{array} \right\}.$$

For $x \in X$, define

$$||x|| = \max\left\{\sup_{t\in\mathbb{R}} \frac{|x(t)|}{\sigma(t)}, \sup_{t\in\mathbb{R}} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\tau(t))}\right\}.$$

Lemma 2.1. *X is a Banach space with $||\cdot||$ defined.*

Proof. It is easy to see that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X. Then $||x_u - x_v|| \to 0,\ u, v \to +\infty$. It follows that $x_u \in X$ and

$$\sup_{t\in\mathbb{R}} \frac{|x_u(t)-x_v(t)|}{\sigma(t)} \to 0, u, v \to +\infty,$$

$$\sup_{t\in\mathbb{R}} \frac{\rho(t)|x_u'(t)-x_v'(t)|}{\Phi^{-1}(\tau(t))} \to 0, u, v \to +\infty.$$

So

$$\sup_{t\in(t_s,t_{s+1}]} \frac{|x_u(t)-x_v(t)|}{\sigma(t)} \to 0, u,v\to+\infty, \lim_{t\to t_s^+}\frac{x_u(t)}{\sigma(t)} \text{ exists}, s\in N,$$

$$\sup_{t\in(t_s,t_{s+1}]} \frac{\rho(t)|x_u'(t)-x_v'(t)|}{\Phi^{-1}(\tau(t))} \to 0, u,v\to+\infty, \lim_{t\to t_s^+}\frac{\rho(t)x_u'(t)}{\Phi^{-1}(\tau(t))} \text{ exists}, s\in \mathbb{Z}.$$

Then there exists functions $x_{s,0}, y_{s,0}\in C^0[t_s,t_{s+1}]$ such that $\lim_{u\to+\infty}\frac{x_u(t)}{\sigma(t)} = x_{s,0}(t)$ and $\lim_{u\to+\infty}\frac{\rho(t)x_u'(t)}{\Phi^{-1}(\tau(t))} = y_{s,0}(t)$ uniformly on $[t_s,t_{s+1}]$.

Define $x_0(t)=x_{s,0}(t), y_0(t)=y_{s,0}(t)$ for all $t\in(t_s,t_{s+1}](s\in N)$. Then $x_0,y_0:\mathbb{R}\to\mathbb{R}$ is well defined on $\mathbb{R}$ and

$$\lim_{u\to+\infty}\frac{x_u(t)}{\sigma(t)} = x_0(t), \ \lim_{u\to+\infty}\frac{\rho(t)x_u'(t)}{\Phi^{-1}(\tau(t))} = y_0(t), t\in\mathbb{R}.$$

It follows that

$$\sup_{t\in(t_s,t_{s+1}]}\left|\frac{x_u(t)}{\sigma(t)} - x_0(t)\right| \to 0, u\to+\infty,$$

$$\sup_{t\in(t_s,t_{s+1}]}\left|\frac{\rho(t)x_u(t)}{\Phi^{-1}(\tau(t))} - y_0(t)\right|, u\to+\infty.$$

Step 1. Prove that $\sigma(\cdot)x_0(\cdot), \Phi^{-1}(\tau(\cdot))y_0(\cdot)\in C^0(t_s,t_{s+1}]$ and the limits $\lim_{t\to t_s^+}\sigma(t)x_0(t)$ and $\lim_{t\to t_s^+}\Phi^{-1}(\tau(t))y_0(t)$ exist.

We have for $\bar{t}_0\in(t_s,t_{s+1}]$ that

$$\begin{aligned}|\sigma(t)x_0(t)-\sigma(\bar{t}_0)x_0(\bar{t}_0)| &\le |\sigma(t)x_0(t)-x_N(t)| + \left|x_N(t)-x_N(\bar{t}_0)\right| \\ &\quad + \left|x_N(\bar{t}_0)-\sigma(\bar{t}_0)x_0(\bar{t}_0)\right|\end{aligned}$$

$$\le 2\sup_{t\in(t_s,t_{s+1}]}\sigma(t)\left|\frac{x_N(t)}{\sigma(t)} - x_0(t)\right| + |x_N(t)-x_N(t_0)|$$

$$\le 2\max_{t\in[t_s,t_{s+1}]}\sigma(t)\sup_{t\in(t_s,t_{s+1}]}\left|\frac{x_N(t)}{\sigma(t)} - x_0(t)\right| + |x_N(t)-x_N(t_0)|.$$

Since $\sup_{t\in(t_s,t_{s+1}]}\left|\frac{x_u(t)}{\sigma(t)} - x_0(t)\right| \to 0, u\to+\infty$ and $x_u(t)$ is continuous on $(t_s,t_{s+1}]$, then for any $\epsilon>0$ we can choose N and $\delta>0$ such that $\sup_{t\in(t_s,t_{s+1}]}\left|\frac{x_N(t)}{\sigma(t)} - x_0(t)\right| < \epsilon$ and $\left|x_N(t)-x_N(\bar{t}_0)\right| < \epsilon$ for all $|t-\bar{t}_0|<\delta$. Thus $|\sigma(t)x_0(t)-\sigma(\bar{t}_0)x_0(\bar{t}_0)| < 3\epsilon$ foe all $|t-\bar{t}_0|<\delta$. So $\sigma(\cdot)x_0(\cdot)\in C^0(t_s,t_{s+1}]$. Similarly we can prove that $\Phi^{-1}(\tau(\cdot))y_0(\cdot)\in C^0(t_s,t_{s+1}]$. We see easily that the limits $\lim_{t\to t_s^+}\sigma(t)x_0(t)$ and $\lim_{t\to t_s^+}\Phi^{-1}(\tau(t))y_0(t)$ exist.

Step 2. Prove that the limits $\lim_{t\to\pm\infty}x_0(t)$, $\lim_{t\to\pm\infty}y_0(t)$ exist.

Suppose that $\lim\limits_{t\to-\infty} x_u(t) = A_u^-$. By $\sup\limits_{t\in R} \frac{|x_u(t)-x_v(t)|}{\sigma(t)} \to 0, u, v \to +\infty$, we know that A_u^- is a Cauchy sequence. Then $\lim\limits_{u\to+\infty} A_u^-$ exists.

By $\sup\limits_{t\in R} \frac{|x_u(t)-x_0(t)|}{\sigma(t)} \to 0, u \to +\infty$, we get that

$$\lim_{t\to-\infty} x_0(t) = \lim_{t\to-\infty}\lim_{u\to+\infty} \frac{x_u(t)}{\sigma(t)} = \lim_{u\to+\infty}\lim_{t\to-\infty} \frac{x_u(t)}{\sigma(t)} = \lim_{u\to+\infty} A_u^-.$$

Hence $\lim\limits_{t\to-\infty} x_0(t)$ exists. Similarly we can prove that $\lim\limits_{t\to+\infty} x_0(t)$, $\lim\limits_{t\to-\infty} y_0(t)$, $\lim\limits_{t\to+\infty} y_0(t)$ exist.

Step 3. Prove that $y_0(t) = \frac{\rho(t)[\sigma(t)x_0(t)]'}{\Phi^{-1}(\tau(t))}$.

We have for $t > \xi$ and some $c_u \in \mathbb{R}$ that

$$\begin{aligned}
&\left|x_u(t) - \sum_{\xi\le t_s<t} \Delta x_u(t_s) - x_u(\xi) - \int_\xi^t \frac{\Phi^{-1}(\tau(s))y_0(s)}{\rho(s)}ds\right| \\
&= \left|\int_\xi^t x_u'(s)ds - \int_\xi^t \frac{\Phi^{-1}(\tau(s))y_0(s)}{\rho(s)}ds\right| \\
&= \int_\xi^t \left|x_u'(s) - \frac{\Phi^{-1}(\tau(s))y_0(s)}{\rho(s)}\right| ds \le \int_\xi^t \frac{\Phi^{-1}(\tau(s))}{\rho(s)}ds \sup_{t\in R}\left|\frac{\rho(t)x_u'(t)}{\Phi^{-1}(\tau(t))} - y_0(t)\right| \\
&\le \int_\xi^t \frac{\Phi^{-1}(\tau(s))}{\rho(s)}ds \sup_{t\in R}\left|\frac{\rho(t)x_u'(t)}{\Phi^{-1}(\tau(t))} - y_0(t)\right| \to 0 \text{ as } u \to +\infty.
\end{aligned}$$

So

$$\lim_{u\to+\infty}\left(x_u(t) - \sum_{\xi\le t_s<t} \Delta x_u(t_s) - x_u(\xi)\right) = \int_\xi^t \frac{\Phi^{-1}(\tau(s))y_0(s)}{\rho(s)}ds.$$

Then

$$\sigma(t)x_0(t) - \sum_{\xi\le t_s<t} \Delta\sigma(t_s)x_0(t_s) - \sigma(\xi)c_0 = \int_\xi^t \frac{\Phi^{-1}(\tau(s))y_0(s)}{\rho(s)}ds.$$

It follows that $\frac{\Phi^{-1}(\tau(s))y_0(t)}{\rho(t)} = [\sigma(t)x_0(t)]$. That is $y_0(t) = \frac{\rho(t)[\sigma(t)x_0(t)]'}{\Phi^{-1}(\tau(t))}$. Similarly we can get $y_0(t) = \frac{\rho(t)[\sigma(t)x_0(t)]'}{\Phi^{-1}(\tau(t))}$ for $t < \xi$. So $x_0 \in X$ with $x_u \to x_0$ as $u \to +\infty$. It follows that X is a Banach space. ■

Lemma 2.2. *Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied:*

(i) *both $\{t \to \frac{x(t)}{\sigma(t)} : x \in M\}$ and $\{t \to \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))} : x \in M\}$ are uniformly bounded,*

(ii) *both $\{t \to \frac{x(t)}{\sigma(t)} : x \in M\}$ and $\{t \to \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))} : x \in M\}$ are equicontinuous in $(t_s, t_{s+1}](s \in N)$,*

(iii) *both $\{t \to \frac{x(t)}{\sigma(t)} : x \in M\}$ and $\{t \to \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))} : x \in M\}$ are equi-convergent as $t \to \pm\infty$.*

Proof. " $\Leftarrow$ ". From Lemma 2.1, we know X is a Banach space. In order to prove that the subset M is relatively compact in X, we only need to show M is totally bounded in X, that is for all $\epsilon > 0$, M has a finite ϵ-net.

For any given $\epsilon > 0$, by **(i)-(iii)**, there exist constants $M > 0, \delta > 0$, $t_{s_0} > 0$ and $t_{-s_0} < 0$ such that

$$\sup_{t\in R} \frac{|x(t)|}{\sigma(t)},\ \sup_{t\in R} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\tau(t))} \leq M, x \in M,$$

$$\left|\frac{x(w_1)}{\sigma(w_1)} - \frac{x(w_2)}{\sigma(w_2)}\right| \leq \frac{\epsilon}{3}, w_1, w_2 \in (t_s, t_{s+1}], |w_1 - w_2| < \delta,$$

$$s = -s_0, -s_0 + 1, \cdots, s_0 - 1, x \in M,$$

$$\left|\frac{\rho(w_1)x'(w_1)}{\Phi^{-1}(\tau(w_1))} - \frac{\rho(w_2)x'(w_2)}{\Phi^{-1}(tau(w_2))}\right| < \frac{\epsilon}{3},$$

$$w_1, w_2 \in (t_s, t_{s+1}], |w_1 - w_2| < \delta, s = -s_0, -s_0 + 1, \cdots, s_0 - 1, x \in M,$$

$$\left|\frac{x(w_1)}{\sigma(w_1)} - \frac{x(w_2)}{\sigma(w_2)}\right| \leq \frac{\epsilon}{3}, w_1, w_2 \leq t_{-s_0} \text{ or } w_1, w_2 \geq t_{s_0}, x \in M,$$

$$\left|\frac{\rho(w_1)x'(w_1)}{\Phi^{-1}(\tau(w_1))} - \frac{\rho(w_2)x'(w_2)}{\Phi^{-1}(tau(w_2))}\right| < \frac{\epsilon}{3}, w_1, w_2 \leq t_{-s_0} \text{ or } w_1, w_2 \geq t_{s_0}, x \in M.$$

Define

$$X|_{(-t_{s_0}, t_{s_0}]} = \left\{ x : \begin{array}{c} x, \rho(t)x' \in C(t_s, t_{s+1}], s = -s_0, -s_0 + 1, \cdots, s_0 - 1, \\ \lim_{t\to t_s^+} \frac{x(t)}{\sigma(t)} \text{ exist}, s = -s_0, -s_0 + 1, \cdots, s_0 - 1, \\ \lim_{t\to t_s^+} \frac{\rho(t)x'(t)}{\Phi^1(\tau(t))} \text{ exist}, s = -s_0, -s_0 + 1, \cdots, s_0 - 1 \end{array} \right\}.$$

For $x \in X|_{(-t_{s_0}, t_{s_0}]}$, define

$$||x||_{s_0} = \max\left\{ \max_{t\in(-t_{s_0}, t_{s_0}]} \frac{|x(t)|}{\sigma(t)},\ \max_{t\in(-t_{s_0}, t_{s_0}]} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\tau(t))} \right\}.$$

Similarly to Lemma 2.1, we can prove that $X_{(-t_{s_0}, t_{s_0}]}$ is a Banach space.

Let $M|_{(-t_{s_0}, t_{s_0}]} = \{t \to x(t), t \in (-t_{s_0}, t_{s_0}] : x \in M\}$. Then $M|_{(-t_{s_0}, t_{s_0}]}$ is a subset of $X|_{(-t_{s_0}, t_{s_0}]}$. By **(i)** and **(ii)**, and Ascoli-Arzela theorem, we can know that $M|_{(-t_{s_0}, t_{s_0}]}$ is relatively compact. Thus, there exist $x_1, x_2, \cdots, x_k \in M$ such that, for any $x \in M$, we have that there exists some $i = 1, 2, \cdots, k$ such that

$$||x - x_i||_{s_0} = \max\left\{ \sup_{t\in(-t_{s_0}, t_{s_0}]} \frac{|x(t)-x_i(t)|}{\sigma(t)},\ \sup_{t\in(-t_{s_0}, t_{s_0}]} \frac{\rho(t)|x'(t)-x_i'(t)|}{\Phi^{-1}(\tau(t))} \right\} < \frac{\epsilon}{3}.$$

Therefore, for $x \in M$, we have that

$$||x - x_i||_X = \max\left\{ \sup_{t\in(-t_{s_0}, t_{s_0}]} \frac{|x(t)-x_i(t)|}{\sigma(t)},\ \sup_{t\in(-t_{s_0}, t_{s_0}]} \frac{\rho(t)|x'(t)-x_i'(t)|}{\Phi^{-1}(\tau(t))}, \right.$$

$$\sup_{t\ge t_{s_0}}\frac{|x(t)-x_i(t)|}{\sigma(t)},\ \sup_{t\ge t_{s_0}}\frac{\rho(t)|x'(t)-x_i'(t)|}{\Phi^{-1}(\tau(t))}\ \sup_{t\le t_{-s_0}}\frac{|x(t)-x_i(t)|}{\sigma(t)},\ \sup_{t\le t_{-s_0}}\frac{\rho(t)|x'(t)-x_i'(t)|}{\Phi^{-1}(\tau(t))}\Bigg\}$$

$$\le \max\Bigg\{\frac{\epsilon}{3},\ \sup_{t\ge t_{s_0}}\left|\frac{x(t)}{\sigma(t)}-\frac{x(t_{s_0}^+)}{\sigma(t_{s_0}^+)}\right|+\sup_{t\ge t_{s_0}}\left|\frac{x(t_{s_0}^+)}{\sigma(t_{s_0}^+)}-\frac{x_i(t_{s_0}^+)}{\sigma(t_{s_0}^+)}\right|+\sup_{t\ge t_{s_0}^+}\left|\frac{x_i(t_{s_0}^+)}{\sigma(t_{s_0}^+)}-\frac{x_i(t)}{\sigma(t)}\right|,$$

$$\sup_{t\ge t_{s_0}}\left|\frac{\rho(t)x'(t)}{\Phi^1(\tau(t))}-\frac{\rho(t_{s_0}^+)x'(t_{s_0}^+)}{\Phi^{-1}(\tau(t_{s_0}^+))}\right|+\sup_{t\ge t_{s_0}}\left|\frac{\rho(t_{s_0}^+)x'(t_{s_0}^+)}{\Phi^{-1}(\tau((t_{s_0}^+))}-\frac{\rho(t_{s_0}^+)x_i'(t_{s_0}^+)}{\Phi^{-1}(\tau((t_{s_0}^+))}\right|$$

$$+\sup_{t\ge t_{s_0}}\left|\frac{\rho(t_{s_0}^+)x_i'(t_{s_0}^+)}{\Phi^{-1}(\tau((t_{s_0}^+))}-\frac{\rho(t)x_i'(t)}{\Phi^1(\tau(t))}\right|,$$

$$\sup_{t\le t_{-s_0}}\left|\frac{x(t)}{\sigma(t)}-\frac{x(t_{-s_0})}{\sigma(t_{-s_0})}\right|+\sup_{t\le t_{-s_0}}\left|\frac{x(t_{-s_0})}{\sigma(t_{-s_0})}-\frac{x_i(t_{-s_0})}{\sigma(t_{-s_0})}\right|+\sup_{t\le t_{-s_0}}\left|\frac{x_i(t_{-s_0})}{\sigma(t_{-s_0})}-\frac{x_i(t)}{\sigma(t)}\right|,$$

$$\sup_{t\le t_{-s_0}}\left|\frac{\rho(t)x'(t)}{\Phi^1(\tau(t))}-\frac{\rho(t_{-s_0})x'(t_{-s_0})}{\Phi^{-1}(\tau(t_{-s_0}))}\right|+\sup_{t\le t_{-s_0}}\left|\frac{\rho(t_{-s_0})x'(t_{-s_0})}{\Phi^{-1}(\tau(t_{-s_0}))}-\frac{\rho(t_{-s_0})x_i'(t_{-s_0})}{\Phi^{-1}(\tau(t_{-s_0}))}\right|$$

$$+\sup_{t\le t_{-s_0}}\left|\frac{\rho(t_{-s_0})x_i'(t_{-s_0})}{\Phi^{-1}(\tau(t_{-s_0}))}-\frac{\rho(t)x_i'(t)}{\Phi^{-1}(\tau(t))}\right|\Bigg\}\le\epsilon.$$

So, for any $\epsilon>0$, M has a finite ϵ-net $\{U_{x_1},U_{x_2},\cdots,U_{x_k}\}$, that is, M is totally bounded in X. Hence M is relatively compact in X.

$\Rightarrow$. Assume that M is relatively compact, then for any $\epsilon>0$, there exists a finite ϵ-net of M. Let the finite ϵ-net be $\{U_{x_1},U_{x_2},\cdots,U_{x_k}\}$ with $x_i\subset M$. Then for any $x\in M$, there exists U_{x_i} such that $x\in U_{x_i}$ and $||x||\le||x-x_i||+||x_i||\le\epsilon+\max\{||x_i||:i=1,2,\cdots,k\}$. It follows that both $\{t\to\frac{x(t)}{\sigma(t)}:x\in M\}$ and $\{t\to\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}:x\in M\}$ are uniformly bounded. Then **(i)** holds.

Furthermore, there exists $t_{-s_0}<0$ and $t_{s_0}>0$ such that $|x_i(w_1)-x_i(w_2)|<\epsilon$ for all $w_1,w_2\ge t_{s_0}$ and all $w_1,w_2\le t_{-s_0}$ and $i=1,2,\cdots,k$. Then we have for $w_1,w_2\ge t_{s_0}$ and all $w_1,w_2\le t_{-s_0}$ that

$$\left|\frac{x(w_1)}{\sigma(w_1)}-\frac{x(w_2)}{\sigma(w_2)}\right|$$

$$\le\left|\frac{x(w_1)}{\sigma(w_1)}-\frac{x_i(w_1)}{\sigma(w_2)}\right|+\left|\frac{x_i(w_1)}{\sigma(w_1)}-\frac{x_i(w_2)}{\sigma(w_2)}\right|+\left|\frac{x_i(w_2)}{\sigma(w_2)}-\frac{x(w_2)}{\sigma(w_2)}\right|<3\epsilon, x\in M.$$

Similarly we have for $w_1,w_2\ge t_{s_0}$ and all $w_1,w_2\le t_{-s_0}$ that

$$\left|\frac{\rho(w_1)x'(w_1)}{\Phi^{-1}(\tau(w_1))}-\frac{\rho(w_2)x'(w_2)}{\Phi^{-1}(\tau(w_2))}\right|<3\epsilon, x\in M.$$

Thus **(iii)** is valid. Similarly we can prove that **(ii)** holds. Consequently, the Lemma is proved. ■

For ease expression, we define $\sum\limits_{a\le t_s<b} k_s := -\sum\limits_{b\le t_s<a} k_s$ for $a > b$. For $x\in X$, define $(Tx)(t)$ by

$$(Tx)(t) = \int_{-\infty}^{+\infty} m(s)\phi(s,x(s),\rho(s)x'(s))ds + \sum_{\xi\le t_s<t} I(t_s,x(t_s),\rho(t_s)x'(t_s))$$

$$+\int_\xi^t \frac{1}{\rho(u)}\Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} n(s)\psi(s,x(s),\rho(s)x'(s))ds\right)\right.$$

$$\left.+\sum_{\eta\le t_s<u} J(t_s,x(t_s),\rho(t_s)x'(t_s)) - \int_\eta^u p(w)f(w,x(w),\rho(w)x'(w))dw\right)du.$$

Lemma 2.3. *Suppose that f,ϕ,ψ are weak Carathéodory functions and I,J are discrete Carathéodory functions and for each $r>0$, $f\left(t,\sigma(t)u,\Phi^{-1}(\tau(t))v\right)$ converges uniformly as $t\to\pm\infty$ on $[-r,r]\times[-r,r]$. Then $T: X\to X$ is well defined and is completely continuous, $x\in X$ is a solution of (1.1) if and only if $x\in X$ is a fixed point of T in X.*

Proof. From Lemma 2.1, X is a Banach space. For $x\in X$, we have $||x||\le r$ for some $r\ge 0$. Then there exists constants $M_{r,f}\ge 0$, $M_{r,J,s}\ge 0$ and $M_{r,I,s}\ge 0$ such that

$$|f(t,x(t),x'(t))| = \left|f\left(t,\sigma(t)\frac{x(t)}{\sigma(t)},\Phi^{-1}(\tau(t))\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\right)\right| \le M_{r,f}, a.e., t\in R,$$

$$|\phi(t,x(t),\rho(t)x'(t))| \le M_{r,\phi}, a.e., t\in R,$$

$$|\psi(t,x(t),\rho(t)x'(t))| \le M_{r,\psi}, a.e., t\in R,$$

$$|I(t_s,x(t_s),\rho(t_s)x'(t_s))| \le M_{r,I,s}, s\in Z, \sum_{s=-\infty}^{+\infty} M_{r,I,s} < +\infty,$$

$$|J(t_s,x(t_s),\rho(t_s)x'(t_s))| \le M_{r,J,s}, s\in Z, \sum_{s=-\infty}^{+\infty} M_{r,J,s} < +\infty.$$

By direct computation, we get

$$(2)\quad \begin{cases} [\Phi(\rho(t)(Tx)'(t))]' + p(t)f(t,x(t),\rho(t)x'(t)) = 0, \quad a.e.,\ t\in\mathbb{R}, \\ (Tx)(\xi) = \int_{-\infty}^{+\infty} m(s)\phi(s,x(s),\rho(s)x'(s))ds, \\ \rho(\eta)(Tx)'(\eta) = \int_{-\infty}^{+\infty} n(s)\psi(s,x(s),\rho(s)x'(s))ds, \\ \Delta(Tx)(t_i) = I(t_i,x(t_i),\rho(t_i)x'(t_i)), i\in\mathbb{Z}, \\ \Delta\Phi(\rho(t_i)(Tx)'(t_i)) = J(t_i,x(t_i),\rho(t_i)x'(t_i)), i\in\mathbb{Z}. \end{cases}$$

One can show that $Tx \in X$. Thus $T : X \to X$ is well defined and $x \in X$ is a solution of (1) if and only if $x \in X$ is a fixed point of T in X. Similarly to the proof of Lemma 2.2 in [18], we can prove that T is completely continuous. The proof is complete. ■

Lemma 2.4. *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $0 \in \Omega$ and let $T : X \to X$ be a completely continuous operator such that $||Tx|| \le ||x||$ for all $x \in \partial\Omega$. Then T has a fixed point in Ω.*

3. Main result

In this section, we prove the main result of this paper. We need the following assumption:

(A) there exist nonnegative constants $A_j, a_{ij}, b_{ij} \ge 0 (i = 1, 2, j = 0, 1, 2, \cdots, m)$, $\phi_s \ge 0, \psi_s \ge 0 (s \in \mathbb{Z})$ and $k_j, l_j \ge 0 (j = 1, 2, \cdots, m)$ with $k_j + l_j > 0$ and $\sum\limits_{s=-\infty}^{+\infty} \phi_s < +\infty$, $\sum\limits_{s=-\infty}^{+\infty} \psi_s < +\infty$ and

$$\left|f\left(t, \sigma(t)u, \Phi^{-1}(\tau(t))v\right)\right| \le \Phi\left(A_0 + \sum_{j=1}^{m} A_j |u|^{k_j} |v|^{l_j}\right),$$

$$\left|I\left(t_s, \sigma(t_s)u, \Phi^{-1}(\tau(t_s))v\right)\right| \le \phi_s \left[b_{10} + \sum_{j=1}^{m} b_{1j} |u|^{k_j} |v|^{l_j}\right],$$

$$\left|J\left(t_s, \sigma(t_s)u, \Phi^{-1}(\tau(t_s))v\right)\right| \le \psi_s \Phi\left(b_{20} + \sum_{j=1}^{m} b_{2j} |u|^{k_j} |v|^{l_j}\right),$$

$$|\phi(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)| \le a_{10} + \sum_{j=1}^{m} a_{1j} |u|^{k_j} \, |v|^{l_j},$$

$$|\psi(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)| \le a_{20} + \sum_{j=1}^{m} a_{2j} |u|^{k_j} \, |v|^{l_j},$$

hold for all $u, v \in R, a.e. t \in \mathbb{R}, s \in \mathbb{Z}$.

We denote

$$\sigma = \max\{k_j + l_j : \ j = 1, 2, \cdots, m\},$$

$$\overline{A}_1 = ||m||_1 a_{10} + \sum_{s=-\infty}^{+\infty} \phi_s b_{1,0} + c_l ||n||_1 a_{20} + c_l \Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right) b_{2,0} + c_l A_0,$$

$$\overline{B}_{1,j} = ||m||_1 a_{1j} + \sum_{s=-\infty}^{+\infty} \phi_s b_{1,j} + c_l ||n||_1 a_{2j} + c_l \Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right) b_{2,j} + c_l A_j,$$

and

$$\overline{A}_2 = c_l||n||_1 a_{10} + c_l \Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right) b_{2,0} + c_l A_0,$$

$$\overline{B}_{2,j} = c_l||n||_1 a_{1j} + c_l \Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right) b_{2j} + c_l A_j,$$

$$A = \max\{\overline{A}_1,\ \overline{A}_2\}, \quad B = \max\left\{\sum_{j=1}^{m} \overline{B}_{1,j},\ \sum_{j=1}^{m} \overline{B}_{2,j}\right\}.$$

Theorem 3.1. *Suppose that (A) holds, f, ϕ, ψ are weak Carathéodory functions and I, J are discrete Carathéodory functions and for each $r > 0$, $f\left(t, \sigma(t)u, \Phi^{-1}(\tau(t))v\right)$ converges uniformly as $t \to \pm\infty$ on $[-r, r] \times [-r, r]$. Then BVP(1) has at least one solution if*

(i) $\sigma \in (0, 1)$ *or* **(ii)** $\sigma = 1$ *and* $B < 1$ *or* **(iii)** $\sigma > 1$ *and* $B(A+B)^{\sigma-1} \leq \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

Proof. We will apply Lemma 2.4. Let X and T be defined in Section 2. From Lemma 2.3, $T : X \to X$ is well defined and is a completely continuous operator. We prove that T has a fixed point in X to get a solution of BVP(1). For $x \in X$, we have $||x|| \leq r < +\infty$. Then (A) implies that

$$|f(t, x(t), \rho(t)x'(t))| = \left|f\left(t, \sigma(t)\frac{x(t)}{\sigma(t)}, \Phi^{-1}(\tau(t))\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\right)\right|$$

$$\leq \Phi\left(A_0 + \sum_{j=1}^{m} A_j \left|\frac{x(t)}{\sigma(t)}\right|^{k_j} \left|\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\right|^{l_j}\right) \leq \Phi\left(A_0 + \sum_{j=1}^{m} A_j r^{k_j+l_j}\right), t \in \mathbb{R},$$

$$|I(t_s, x(t_s), \rho(t_s)x'(t_s))| \leq \phi_s\left[b_{10} + \sum_{j=1}^{m} b_{1j} r^{k_j+l_j}\right], s \in \mathbb{Z},$$

$$|J(t_s, x(t_s), \rho(t_s)x'(t_s))| \leq \psi_s \Phi\left(b_{20} + \sum_{j=1}^{m} b_{2j} r^{k_j+l_j}\right), s \in \mathbb{Z},$$

$$|\phi(t, x(t), \rho(t)x'(t))| \leq a_{10} + \sum_{j=1}^{m} a_{1j} r^{k_j+l_j}, a.e., t \in \mathbb{R},$$

$$|\psi(t, x(t), \rho(t)x'(t))| \leq a_{20} + \sum_{j=1}^{m} a_{2j} r^{k_j+l_j}, a.e., t \in \mathbb{R}.$$

By the definition of T, we get

$$\frac{|(Tx)(t)|}{\sigma(t)} \le \frac{1}{1+\left|\int_\xi^t \frac{\Phi^{-1}(\tau(u))}{\rho(u)}du\right|}\left[\int_{-\infty}^{+\infty}|m(s)||\phi(s,x(s),\rho(s)x'(s))|ds\right.$$

$$+\left|\sum_{\xi\le t_s<t}|I(t_s,x(t_s),\rho(t_s)x'(t_s))|\right|$$

$$+\left|\int_\xi^t \frac{1}{\rho(u)}\Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty}|n(s)||\psi(s,x(s),\rho(s)x'(s))|ds\right)\right.\right.$$

$$+\sum_{\eta\le t_s<u}|J(t_s,x(t_s),\rho(t_s)x'(t_s))|$$

$$\left.\left.\left.+\left|\int_\eta^u p(w)|f(w,x(w),\rho(w)x'(w))|dw\right|\right)du\right|\right]$$

$$\le ||m||_1\left[a_{10}+\sum_{j=1}^m a_{1j}r^{k_j+l_j}\right]+\sum_{s=-\infty}^{+\infty}\phi_s\left[b_{10}+\sum_{j=1}^m b_{1j}r^{k_j+l_j}\right]$$

$$+\frac{\left|\int_\xi^t \frac{\Phi^{-1}(\tau(u))}{\rho(u)}du\right|}{1+\left|\int_\xi^t \frac{\Phi^{-1}(\tau(u))}{\rho(u)}du\right|}\Phi^{-1}\left(\Phi\left(||n||_1\left(a_{20}+\sum_{j=1}^m a_{2j}r^{k_j+l_j}\right)\right)\right.$$

$$\left.+\Phi\left(b_{20}+\sum_{j=1}^m b_{2j}r^{k_j+l_j}\right)\sum_{s=-\infty}^{+\infty}\psi_s+\Phi\left(A_0+\sum_{j=1}^m A_jr^{k_j+l_j}\right)\right).$$

One knows that

$$\Phi^{-1}(u+v)\le c_l[\Phi^{-1}(u)+\Phi^{-1}(v)],u,v\ge 0 \text{ with } c_l=\begin{cases}1,\ 1<l<2,\\ 2^{l-1},\ l\ge 2.\end{cases}$$

It follows that

$$\sup_{t\in\mathbb{R}}\frac{|(Tx)(t)|}{\sigma(t)}\le ||m||_1\left[a_{10}+\sum_{j=1}^m a_{1j}r^{k_j+l_j}\right]+\sum_{s=-\infty}^{+\infty}\phi_s\left[b_{1,0}+\sum_{j=1}^m b_{1,j}r^{k_j+l_j}\right]$$

$$+c_l\left(||n||_1\left(a_{20}+\sum_{j=1}^m a_{2j}r^{k_j+l_j}\right)\right)$$

$$+c_l\left(b_{20}+\sum_{j=1}^m b_{2j}r^{k_j+l_j}\right)\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty}\psi_s\right)+c_l\left(A_0+\sum_{j=1}^m A_jr^{k_j+l_j}\right)$$

$$= ||m||_1 a_{10} + \sum_{s=-\infty}^{+\infty} \phi_s b_{1,0} + c_l ||n||_1 a_{20} + c_l \Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right) b_{2,0} + c_l A_0$$

$$+ \sum_{j=1}^{m} \left(||m||_1 a_{1j} + \sum_{s=-\infty}^{+\infty} \phi_s b_{1,j} \right.$$

$$\left. + c_l ||n||_1 a_{2j} + c_l \Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right) b_{2,j} + c_l A_j \right) r^{k_j + l_j}.$$

Then

$$\sup_{t\in\mathbb{R}} \frac{|(Tx)(t)|}{\sigma(t)} \le \overline{A}_1 + \sum_{j=1}^{m} \overline{B}_{1,j} r^{k_j+l_j}. \tag{3}$$

On the other hand, we have

$$\frac{\rho(t)|(Tx)'(t)|}{\Phi^{-1}(\tau(t))} \le \frac{1}{\Phi^{-1}(\tau(t))} \left| \Phi^{-1}\left(\Phi\left(||n||_1 \left(a_{10} + \sum_{j=1}^{m} a_{1j} r^{k_j+l_j} \right)\right)\right.\right.$$

$$+ \sum_{s=-\infty}^{+\infty} \psi_s \Phi\left(b_{20} + \sum_{j=1}^{m} b_{2j} r^{k_j+l_j} \right)$$

$$\left.\left. + \left| \int_\eta^t p(w)dw \right| \Phi\left(A_0 + \sum_{j=1}^{m} A_j r^{k_j+l_j} \right)\right)\right|$$

$$\le \Phi^{-1}\left(\Phi\left(||n||_1 \left(a_{10} + \sum_{j=1}^{m} a_{1j} r^{k_j+l_j} \right)\right)\right.$$

$$\left. + \sum_{s=-\infty}^{+\infty} \psi_s \Phi\left(b_{20} + \sum_{j=1}^{m} b_{2j} r^{k_j+l_j} \right) + \Phi\left(A_0 + \sum_{j=1}^{m} A_j r^{k_j+l_j} \right)\right)$$

$$\le c_l \left(||n||_1 \left(a_{10} + \sum_{j=1}^{m} a_{1j} r^{k_j+l_j} \right)\right)$$

$$+ c_l \Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right)\left(b_{20} + \sum_{j=1}^{m} b_{2j} r^{k_j+l_j} \right) + c_l \left(A_0 + \sum_{j=1}^{m} A_j r^{k_j+l_j} \right).$$

It follows that

$$\sup_{t\in\mathbb{R}} \frac{\rho(t)|(Tx)'(t)|}{\Phi^{-1}(\tau(t))} \le \overline{A}_2 + \sum_{j=1}^{m} \overline{B}_{2,j} ||x||^{k_j+l_j}. \tag{4}$$

It follows from (3) and (4) that

$$||Tx|| \le A + B \max\{||(x,y)||^\sigma,\ 1\} \le A + B + B||x||^\sigma.$$

(i) $\sigma \in (0,1)$.

Since $\sigma \in (0,1)$, change $r_0 > 0$ such that $A + B + Br_0^\sigma \le r_0$. Let $\Omega_0 = \{x \in X : ||x|| \le r_0\}$. Then we get $||Tx|| \le A + B + Br_0^\sigma \le r_0$. So $T\overline{\Omega_0} \subset \overline{\Omega_0}$. Then $||Tx|| \le ||x||$ for all $x \in \partial\Omega$. Thus Lemma 2.4 implies that the operator T has at least one fixed point in $\overline{\Omega_0}$. So BVP(1) has at least one solution.

(ii) $\sigma = 1$.

Let $r_0 = \frac{A+B}{1-B}$ such that $A+B+Br_0 = r_0$. Let $\Omega_0 = \{x \in X : ||x|| \le r_0\}$. Then we get $||Tx|| \le A + B + Br_0^\sigma \le r_0$. So $T\overline{\Omega_0} \subset \overline{\Omega_0}$. Then $||Tx|| \le ||x||$ for all $x \in \partial\Omega$. Thus Lemma 2.4 implies that the operator T has at least one fixed point in $\overline{\Omega_0}$. So BVP(1) has at least one solution.

(iii) $\sigma > 1$.

Let $r_0 = \left(\frac{A+B}{B(\sigma-1)}\right)^{\frac{1}{\sigma}}$. It is easy to show from $\frac{(A+B)^{\sigma-1}\sigma^\sigma}{(\sigma-1)^{\sigma-1}} \le \frac{1}{B}$ that $A + B + Br_0^\sigma \le r_0$. Let $\Omega_0 = \{x \in X : ||x|| \le r_0\}$. Then we get $||Tx|| \le A + B + Br_0^\sigma \le r_0$. So $T\overline{\Omega_0} \subset \overline{\Omega_0}$. Then $||Tx|| \le ||x||$ for all $x \in \partial\Omega$. Thus Lemma 2.4 implies that the operator T has at least one fixed point in $\overline{\Omega_0}$. So BVP(1) has at least one solution.

The proof of Theorem 3.1 is completed. ■

4. An example

In this section, we present an example to illustrate the main result.

Example 4.1. We consider the following BVP

$$(5)\quad \begin{cases} [t|x'(t)|x'(t)]' + f\left(t, x(t), \sqrt{|t|}x'(t)\right) = 0, \ \text{a.e. } t \in \mathbb{R}, \\ x(0) = \sqrt{\pi}, x'(0) = 2\sqrt{\pi}, \\ \Delta x(s) = \lim\limits_{t \to s^+} x(t) - x(s) = 2^{-|s|}, s \in \mathbf{Z}, \\ \Delta(\sqrt{|s|}x'(s))^3 = \lim\limits_{t \to s^+} \left(\sqrt{|t|}x'(t)\right)^3 - \left(\sqrt{|s|}x'(s)\right)^3 = 2^{-|s|}, s \in \mathbf{Z}, \end{cases}$$

where

$$f(t,u,v) = \frac{1}{1+|t|}\left(A_0 + \sum_{j=1}^m A_j \left|\frac{u}{1+\sqrt{t^2+|t|}-\ln(\sqrt{1+|t|}-\sqrt{|t|})}\right|^{k_j} \left|\frac{v}{\sqrt{1+|t|}}\right|^{l_j}\right)^2,$$

$A_j \ge 0, k_j, l_j \ge 0 (j = 0, 1, 2, \cdots, m)$.

Corresponding to BVP(1), we have $\Phi(x) = |s|s$ and $\Phi^{-1}(x) = |s|^{-\frac{1}{2}}s$, $\rho(t) = \sqrt{|t|}$, $p(t) = 1$, $\xi = \eta = 0$, $t_s = s, s \in \mathbf{Z}$, $\phi(t,u,v) = 1$ and

$\psi(t,u,v) = 2$, $m(t) = n(t) = e^{-t^2}$, $I(s,u,v) = J(s,u,v) = 2^{-|s|}(s \in Z)$. One sees that

(i) $\Phi(x) = |x|^{k-2}x$ with $k = 3 > 1$, the inverse of Φ is denoted by Φ^{-1} and $\Phi^{-1}(x) = |x|^{l-2}x$ with $l = \frac{3}{2}$,

(ii) $p : R \to [0,\infty)$ with $p \in L^1_{loc}(R)$ and $\int_{-\infty}^0 p(t)dt = \int_0^{+\infty} p(s)ds = +\infty$, we find that $\tau(t) = 1 + |t| + \frac{t^2}{2}$,

(iii) one sees that $\rho : R \to [0,\infty)$ and

$$\tau(t) = 1 + \left|\int_0^t p(s)ds\right| = 1 + |t|,$$

$$\sigma(t) = 1 + \left|\int_0^t \frac{\Phi^{-1}(\tau(u))}{\rho(u)}du\right|$$

$$= 1 + \left|\int_0^t \frac{\sqrt{1+|u|}}{\sqrt{|u|}}du\right| = 1 + \sqrt{t^2+|t|} - \ln(\sqrt{1+|t|} - \sqrt{|t|}).$$

with $\frac{\Phi^{-1}(\tau(\cdot))}{\rho(\cdot)} \in L^1_{loc}(R)$ and $\int_{-\infty}^0 \frac{\Phi^{-1}(\tau(u))}{\rho(u)}du = \int_0^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)}du = +\infty$, and f, ϕ, ψ are weak Carathéodory functions and I, J are discrete Carathéodory functions and for each $r > 0$,

$$f\left(t, \sigma(t)u, \Phi^{-1}(\tau(t))v\right) = \frac{1}{1+|t|}\left(A_0 + \sum_{j=1}^m A_j\,|u|^{k_j}\,|v|^{l_j}\right)^2 \to 0$$

uniformly as $t \to \pm\infty$ on $[-r,r] \times [-r,r]$.

(iv) $f, \phi, \psi : \mathbb{R}^3 \to \mathbb{R}$ are strong Carathéodory functions, $m, n \in L^1(\mathbb{R})$,

(v) $\{t_s : s \in \mathbf{Z}\}$ is a increasing sequence with $\lim_{s\to-\infty} t_s = -\infty$ and $\lim_{s\to+\infty} t_s = +\infty$,

(vi) $I, J : \{t_s : s \in Z\} \times \mathbb{R}^2 \to \mathbb{R}$ are discrete Carathéodory functions.

Choose $A_j (j = 0,1,2,\cdots,m)$ and

$$b_{1,0} = b_{2,0} = 1,\ \ b_{1,j} = b_{2,j} = 0,\ j = 1,2,\cdots,m,$$

$$a_{1,0} = 1,\ a_{2,0} = 2,\ \ a_{1,j} = a_{2,j} = 0,\ j = 1,2,\cdots,m.$$

Then (A) holds and $\sum_{s=-\infty}^{+\infty} \phi_s = \sum_{s=-\infty}^{+\infty} \psi_s = 3$. Denote $\sigma = \max\{k_j + l_j :\ j = 1,2,\cdots,m\}$ and by direct computation, we get

$$\overline{A}_1 = 3 + \sqrt{3} + 3\sqrt{\pi} + A_0,\ \ \overline{B}_{1,j} = A_j,$$

$$\overline{A}_2 = \sqrt{3} + \sqrt{\pi} + A_0,\ \ \overline{B}_{2,j} = A_j,$$

$$A = \max\{\overline{A}_1,\ \overline{A}_2\} = 3 + \sqrt{3} + 3\sqrt{\pi} + A_0,$$

$$B = \max\left\{\sum_{j=1}^{m} \overline{B}_{1,j},\ \sum_{j=1}^{m} \overline{B}_{2,j}\right\} = \sum_{j=1}^{m} A_j.$$

By Theorem 3.1, BVP(5) has at least one solution if

(i) $\sigma \in (0,1)$ or

(ii) $\sigma = 1$ and $\sum\limits_{j=1}^{m} A_j < 1$ or

(iii) $\sigma > 1$ and $\sum\limits_{j=1}^{m} A_j \left(3 + \sqrt{3} + 3\sqrt{\pi} + A_0 + \sum\limits_{j=1}^{m} A_j\right)^{\sigma-1} \leq \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

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