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The strongly generalized triple difference Γ^3 sequence spaces defined by a modulus

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Abstract. In this paper we introduce the strongly generalized difference sequence spaces using non-negative four dimensional matrix of complex numbers. We also give natural relationship between strongly generalized difference $V_{3\Gamma^3}{}^{\lambda_3}[A, \Delta^m, p, f]-$ summable sequences with respect to f.

We examine some topological properties of $V_{3\Gamma^3}{}^{\lambda_3}[A, \Delta^m, p, f]-$ spaces and investigate some inclusion relations between these spaces.

1. Introduction

Throughout w, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers.Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$is said to be convergent if and only if the triple sequence $(S_{mnk})$is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}, \quad (m, n, k = 1, 2, 3, \dots).$$

A sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

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$$(1) \qquad d(x,y) = \sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,\dots \right\},$$

forall $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}\, in\, \Gamma^3$. Let $\phi = \{\text{finite sequences}\}$.

2. Definitions and Preliminaries

Consider a triple sequence $x = (x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \Im_{ijq}$ for all $m,n,k \in \mathbb{N}$; where $\Im_{ijq}$ denotes the triple sequence whose only non zero term is a 1 in the $(i,j,k)^{th}$ place for each $i,j,q \in \mathbb{N}$.

If X is a sequence space, we give the following definitions:

(i) X' is continuous dual of X;

(ii) $X^\alpha = \left\{ a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} |a_{mnk} x_{mnk}| < \infty, \text{ for each } x \in X \right\}$;

(iii) $X^\beta = \left\{ a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} a_{mnk} x_{mnk} \text{ is convergent, for each } x \in X \right\}$;

(iv) $X^\gamma = \left\{ a = (a_{mn}) : \sup_{m,n \geq 1} \left| \sum_{m,n,k=1}^{M,N,K} a_{mnk} x_{mnk} \right| < \infty, \text{ for each } x \in X \right\}$;

(v) Let X be an FK-space $\supset \phi$; then $X^f = \left\{ f(\Im_{mnk}) : f \in X' \right\}$;

(vi) $X^\delta = \left\{ a = (a_{mnk}) : \sup_{m,n,k} |a_{mnk} x_{mnk}|^{1/m+n+k} < \infty, \text{ for each } x \in X \right\}$;

X^α, X^β, X^γ are called α- (or Köthe-Toeplitz)dual of X, β-(or generalized-Köthe-Toeplitz)dual of X, γ-dual of X, δ-dual of X respectively. X^α is defined by Gupta and Kamptan [10]. It is clear that $X^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\alpha \subset X^\gamma$ does not hold. A sequence $x = (x_{mnk})$ is said to be triple analytic if

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The vector space of all triple analytic sequences is usually denoted by Λ^3 and is defined by $\sup_{mnk} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty$. A sequence $x = (x_{mnk})$ is called triple entire sequence if $|x_{mnk}|^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$. The vector space of triple entire sequences is usually denoted by Γ^3.

Throughout the article w^3, $\Gamma^3(\Delta)$, $\Lambda^3(\Delta)$ denote the spaces of all, triple entire difference sequence spaces and triple analytic difference sequence spaces respectively.

For a triple sequence $x \in w^3$, we define the sets

$$\Gamma^3(\Delta) = \Big\{ x \in w^3 : |\Delta x_{mnk}|^{1/m+n+k} \to 0 \text{ as } m, n, k \to \infty \Big\},$$

$$\Lambda^3(\Delta) = \left\{ x \in w^3 : \sup_{m,n,k} |\Delta x_{mnk}|^{1/m+n+k} < \infty \right\}.$$

The space $\Lambda^3(\Delta)$ is a metric space with the metric

$$d(x, y) = \sup_{m,n,k} \Big\{ |\Delta x_{mnk} - \Delta y_{mnk}|^{1/m+n} : \ m, n, k = 1, 2, \ldots \Big\}$$

for all $x = (x_{mnk})$ and $y = (y_{mnk})$ in $\Lambda^3(\Delta)$.

The triple sequence $\lambda_3 = \{(\beta_r, \mu_s, \eta_t)\}$ is called triple λ_3 sequence if there exist three non-decreasing sequences of positive numbers tending to infinity such that $\beta_{r+1} \leq \beta_r + 1$, $\beta_1 = 1$, $\mu_{s+1} \leq \mu_s + 1$, $\mu_1 = 1$ and $\beta_{r+1} \leq \eta_t + 1$, $\eta_1 = 1$. The generalized double de Vallee-Poussin mean is defined by

$$t_{rst} = t_{rst}(x_{mnk}) = \frac{1}{\lambda_{rst}} \sum_{(m,n,k) \in I_{rst}} x_{mnk},$$

where $\lambda_{rst} = \beta_r \cdot \mu_s \cdot \eta_t$ and

$$I_{rst} = \{(mnk) : r - \beta_r + 1 \leq m \leq r, s - \mu_s + 1 \leq n \leq s, t - \eta_t + 1 \leq k \leq t\}.$$

A triple number sequence $x = (x_{mnk})$ is said to be $(V_3, \lambda_3)-$ summable to a number L if $P - lim_{rst} p_{rst} = L$. If $\lambda_{rst} = rst$, then then $(V_3, \lambda_3)-$ summability is reduced to $(C, 1, 1, 1)-$ summability. Let $A = \Big(a_{i(k,\ell)}^{i(m,n)} \Big)$ is an infinite four dimensional matrix of complex numbers and $p = \left(p_{i(mnk)}\right)$ be a triple analytic sequence of positive real numbers such that $0 < h = \inf_i p_{i(mnk)} \leq \sup_i p_{i(mnk)} = H < \infty$ and f be a modulus. We define

$$V_{3\Gamma^3}{}^{\lambda_2}\left[A, \Delta^m, p, f\right] = \Big\{x = (x_{mnk}) \in w^3 :$$
$$\lim_{r,s,t\to\infty} \lambda_{rst}^{-1} \sum_{mnk\in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}} = 0\Big\},$$
$$V_{3\Lambda^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right] = \Big\{x = (x_{mnk}) \in w^3 :$$
$$\sup_{r,s,t} \lambda_{rst}^{-1} \sum_{mnk\in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}} < \infty\Big\},$$

where $A_i\left(\Delta^m x\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{i(k,\ell)}^{i(mnk)} \Delta^m x_{mnk}$.

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We examine some topological properties of $V_{3\Gamma^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]$ spaces and investigate some inclusion relations between these spaces.

3. Main Results

Theorem 3.1. *Let f be a modulus function. Then $V_{3\Gamma^2}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]$ is a linear space over the complex field $\mathbb{C}$.*

Proof. Let $x, y \in V_{3\Gamma^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]$ and $\alpha, \mu \in \mathbb{C}$. Then there exists integers D_α and D_μ such that $|\alpha|^{\frac{1}{m+n+k}} \leq D_\alpha$ and $|\mu|^{\frac{1}{m+n+k}} \leq D_\mu$. The properties of modulus f, we have

$$\lambda_{rst}^{-1} \sum_{mnk\in I_{rst}} \left[f\left(\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{i(k,\ell)}^{i(mnk)} \left(\Delta^m\left(\alpha x_{mnk} + \mu x_{mnk}\right)\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}} \leq$$
$$DD_\alpha^H \lambda_{rst}^{-1} \sum_{mnk\in I_{rst}} \left[f\left(\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \alpha^{\frac{1}{m+n+k}} a_{i(k,\ell)}^{i(mnk)} \left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}} +$$
$$DD_\mu^H \lambda_{rs}^{-1} \sum_{mnk\in I_{rst}}$$
$$\left[f\left(\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \alpha^{\frac{1}{m+n+k}} a_{i(k,\ell)}^{i(mnk)} \left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}} \to 0 \text{ as } r, s, t \to \infty$$

This proves that $V_{3\Gamma^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]$ is linear.

This completes the proof. □

Theorem 3.2. *Let f be a modulus function. Then the inclusions $V_{3\Gamma^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right] \subset V_{3\Lambda^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]$ hold.*

Proof. Let $x \in V_{3\Gamma^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]$ such that $x \to \left(V_{3\Gamma^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]\right)$, we have

$$\sup_{rst} \lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}} =$$

$$\sup_{rst} \lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}} - 0 + 0\right|\right)\right]^{p_{i(mnk)}} \leq$$

$$D \sup_{rst} \lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}} - 0\right|\right)\right]^{p_{i(mnk)}} +$$

$$D \sup_{rst} \lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(|0|\right)\right]^{p_{i(mnk)}} \leq$$

$$D \sup_{rst} \lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}} - 0\right|\right)\right]^{p_{i(mnk)}} +$$

$$T \max\left\{f\left(|0|\right)^h, f\left(|0|\right)^H\right\} < \infty.$$

Hence $x \in V_{3\Lambda^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]$. Therefore the inclusion $V_{\chi^3}^{\lambda}\left[A, \Delta^m, p, f\right] \subset V_{3\Lambda^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]$ holds. This completes the proof. □

Theorem 3.3. *Let $p = \left(p_{i(mnk)}\right) \in \Lambda^3$. Then $V_{3\Gamma^3}{}^{\lambda_3}\left[A, \Delta^m, p, f\right]$ is a paranormed space*

$$g(x) = \sup_{rst} \left(\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}}\right)^{\frac{1}{M}}$$

where $M = \max\left(1, \sup_{ijq} p_{ijq}\right)$.

Proof. Clearly $g(-x) = g(x)$. It is trivial that $\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}} = 0$ for $x_{mnk} = 0$. Hence we get $g(0) = 0$. Since $\frac{p_i}{M} \leq 1$ and $M \geq 1$, using Minkowski's inequality and definition of modulus f, for each (r, s, t), we have

$$\left(\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m\left(x_{mnk} + y_{mnk}\right)\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}}\right)^{\frac{1}{M}} \leq$$

$$\left(\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}}\right)^{\frac{1}{M}} +$$

$$\left(\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(\left|A_i\left(\Delta^m y_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}}\right)^{\frac{1}{M}}.$$

Now it follows that g is subadditive. Let us take any complex number α. By definition of modulus f, we have

$$g(\alpha x) = \sup_{rst}\left(\lambda_{rst}^{-1}\sum_{mnk\in I_{rst}}\left[f\left(\left|A_i\left(\Delta^m \alpha x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}}\right)^{\frac{1}{M}} \le K^{\frac{H}{M}} g(x),$$

where $K = 1 + \left[|\alpha|^{\frac{1}{m+n+k}}\right]$ ($[|t|]$ denotes the integer part of t). Since f is modulus, we have $x \to 0$ implies $g(\alpha x) \to 0$.

Similarly $x \to 0$ and $\alpha \to 0$ implies $g(\alpha x) \to 0$.

Finally, we have x fixed and $\alpha \to 0$ implies $g(\alpha x) \to 0$. This completes the proof. □

Theorem 3.4. *Let f be a modulus function. Then $V_{3\Gamma^3}{}^{\lambda_3}[A, \Delta^m, p] \subset V_{3\Gamma^3}{}^{\lambda_3}[A, \Delta^m, p, f]$.*

Proof. Let $x \in V_{3\Gamma^3}{}^{\lambda_3}[A, \Delta^m, p]$. We can choose $0 < \delta < 1$ such that $f(t) < \epsilon$ for every $t \in [0, \infty)$ with $0 \le t \le \delta$. Then, we can write

$$\lambda_{rst}^{-1}\sum_{mnk\in I_{rst}}\left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}} - 0\right|\right)\right]^{p_{i(mnk)}} =$$

$$\lambda_{rst}^{-1}\sum_{mnk\in I_{rst},\left|A_i(\Delta^m x_{mnk})^{\frac{1}{m+n+k}}\right|\le\delta}\left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}} +$$

$$\lambda_{rst}^{-1}\sum_{mnk\in I_{rst},\left|A_i(\Delta^m x_{mnk})^{\frac{1}{m+n+k}}\right|>\delta}\left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}} \le$$

$$\max\left\{f(\epsilon)^h, f(\epsilon)^H\right\} + \max\left\{1, \left(3f(1)\delta^{-1}\right)^H\right\}\lambda_{rst}^{-1}\cdot$$

$$\cdot\sum_{mnk\in I_{rst}\left|A_i(\Delta^m x_{mnk})^{\frac{1}{m+n+k}}\right|>\delta}\left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{p_{i(mnk)}}.$$

Therefore $x \in V_{3\Gamma^3}{}^{\lambda_3}[A, \Delta^m, p, f]$. This completes the proof. □

Theorem 3.5. *Let $0 < p_{i(mnk)} < q_{i(mnk)}$ and $\left\{\frac{q_{i(mnk)}}{p_{i(mnk)}}\right\}$ be bounded. Then $V3^{\lambda}_{\Gamma^3}[A, \Delta^m, q, f] \subset V_{3\Gamma^3}{}^{\lambda_3}[A, \Delta^m, p, f]$.*

Proof. Let

$$x \in V_{3\Gamma^3}{}^{\lambda_3}[A, \Delta^m, q, f], \tag{2}$$

$$\lambda_{rst}^{-1}\sum_{mnk\in I_{rst}}\left[f\left(\left|A_i\left(\Delta^m x_{mnk}\right)^{\frac{1}{m+n+k}}\right|\right)\right]^{q_{i(mnk)}} \to 0 \text{ as } r, s, t \to \infty. \tag{3}$$

Let

$$t_i = \lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f \left(\left| A_i \left(\Delta^m x_{mnk} \right)^{\frac{1}{m+n+k}} \right| \right) \right]^{q_{i(mnk)}} \to 0 \text{ as } r, s, t \to \infty$$

and

$$\gamma_{i(mnk)} = \frac{p_{i(mnk)}}{q_{i(mnk)}}.$$

Since $p_{i(mnk)} \leq q_{i(mnk)}$, we have $0 \leq \gamma_{i(mnk)} \leq 1$. Take $0 < \gamma < \gamma_{i(mnk)}$.

Define

$$\begin{array}{ll} u_i = t_i \quad (t_i \geq 1); & u_i = 0 \quad (t_i < 1); \\ v_i = 0 \quad (t_i \geq 1); & v_i = t_i \quad (t_i < 1); \end{array}$$

$$t_i = u_i + v_i; \qquad t_i^{\gamma_{i(mnk)}} = u_i^{\gamma_{i(mnk)}} + v_i^{\gamma_{i(mnk)}}.$$

Now it follows that

$$u_i^{\gamma_{i(mnk)}} \leq u_i \leq t_i \quad \text{and} \quad v_i^{\gamma_{i(mnk)}} \leq v_i^{\gamma}, \tag{4}$$

i.e.

$$t_i^{\gamma_{i(mnk)}} = u_i^{\gamma_{i(mnk)}} + v_i^{\gamma_{i(mnk)}};$$
$$t_i^{\gamma_{i(mnk)}} \leq t_i + v_i^{\lambda}$$

by (4)

$$\left(\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f \left(\left| A_i \left(\Delta^m x_{mnk} \right)^{\frac{1}{m+n+k}} \right| \right) \right]^{q_{i(mnk)}} \right)^{\gamma_{i(mnk)}} \leq$$
$$\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f \left(\left| A_i \left(\Delta^m x_{mnk} \right)^{\frac{1}{m+n+k}} \right| \right) \right]^{q_{i(mn)}},$$

$$\left(\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f \left(\left| A_i \left(\Delta^m x_{mnk} \right)^{\frac{1}{m+n+k}} \right| \right) \right]^{q_{i(mnk)}} \right)^{\frac{p_{i(mn)}}{q_{i(mnk)}}} \leq$$
$$\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f \left(\left| A_i \left(\Delta^m x_{mnk} \right)^{\frac{1}{m+n+k}} \right| \right) \right]^{q_{i(mn)}},$$

$$\left(\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f \left(\left| A_i \left(\Delta^m x_{mnk} \right)^{\frac{1}{m+n+k}} \right| \right) \right]^{p_{i(mnk)}} \right) \leq$$
$$\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f \left(\left| A_i \left(\Delta^m x_{mnk} \right)^{\frac{1}{m+n+k}} \right| \right) \right]^{q_{i(mn)}}.$$

But

$$\left(\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f \left(\left| A_i \left(\Delta^m x_{mnk} \right)^{\frac{1}{m+n+k}} \right| \right) \right]^{q_{i(mnk)}} \right) \to 0 \text{ as } r, s, t \to \infty$$

by (3). Therefore

$$\left(\lambda_{rst}^{-1} \sum_{mnk \in I_{rst}} \left[f\left(\left| A_i \left(\Delta^m x_{mnk} \right)^{\frac{1}{m+n+k}} \right| \right) \right]^{p_{i(mnk)}} \right) \to 0 \text{ as } r,s,t \to \infty.$$

Hence

$$x \in V_{3\Gamma^3}{}^{\lambda_3} \left[A, \Delta^m, p, f \right]. \tag{5}$$

From (2) and (5), we get $V_{3\Gamma^3}{}^{\lambda_3} [A, \Delta^m, q, f] \subset V_{3\Gamma^3}{}^{\lambda_3} [A, \Delta^m, p, f]$. This completes the proof. □

Theorem 3.6. (i) *Let* $0 < \inf p_i \leq p_i \leq 1$.
Then $V_{3\Gamma^3}{}^{\lambda_3} [A, \Delta^m, p, f] \subset V_{3\Gamma^3}{}^{\lambda_3} [A, \Delta^m, f]$
(ii) *Let* $1 \leq p_i \leq \sup p_i < \infty$.
Then $V_{3\Gamma^3}{}^{\lambda_3} [A, \Delta^m, f] \subset V_{3\Gamma^3}{}^{\lambda_3} [A, \Delta^m, p, f]$;
(iii) *Let* $0 < p_i \leq q_i < \infty$ *for each* i.
Then $V_{3\Gamma^3}{}^{\lambda_3} [A, \Delta^m, p, f] \subset V_{3\Gamma^3}{}^{\lambda_3} [A, \Delta^m, q, f]$.

Proof. The proof is a routine verification. □

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References

[1] T. Apostol, Mathematical Analysis, *Addison-Wesley , London*, **1**978.

[2] A. Esi, On some triple almost lacunary sequence spaces defined by Orlicz functions, *Research and Reviews:Discrete Mathematical Structures*, **1(2)** (2014), 16-25.

[3] A. Esi and M. Necdet Catalbas,Almost convergence of triple sequences, *Global Journal of Mathematical Analysis*, **2(1)** (2014), 6-10.

[4] A. Esi and E. Savażs, On lacunary statistically convergent triple sequences in probabilistic normed space,*Appl.Math.and Inf.Sci.*, **9 (5)**, (2015), 2529-2534.

[5] G.H. Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86-95.

[6] P.K. Kamthan and M. Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, In c., New York* , **1**981.

[7] I.J. Maddox, Sequence spaces defined by a modulus, *Math. Proc. Camb. Philos. Soc.*, **100** (1986), 161-166.

[8] W.H. Ruckle, FK spaces in which the sequence of coordinate vectors in bounded, *Canad. J. Math.*, **25** (1973), 973-978.

[9] A. Sahiner, M. Gurdal and F.K. Duden, Triple sequences and their statistical convergence, *Selcuk J. Appl. Math.*, Vol. **8**, No. **2** (2007), 49-55.

[10] N. Subramanian and U.K. Misra, Characterization of gai sequences via double Orlicz space, *Southeast Asian Bulletin of Mathematics*, (revised).

[11] N. Subramanian, B.C. Tripathy and C. Murugesan, The double sequence space of Γ^2, *Fasciculi Math.*, **40** (2008), 91-103.

[12] N. Subramanian, B.C. Tripathy and C. Murugesan, The Cesáro of double entire sequences, *International Mathematical Forum*, **4 no.2**(2009), 49-59.

[13] N. Subramanian and A. Esi, The generalized triple difference of χ^3 sequence spaces, *Global Journal of Mathematical Analysis*, **3 (2)** (2015), 54-60.

[14] N. Subramanian and A. Esi, The study on χ^3 sequence spaces, *Songklanakarin Journal of Science and Technology*, **under review.**

[15] N. Subramanian and A. Esi, Some New Semi-Normed Triple Sequence Spaces Defined by a Sequence Of Moduli, *Journal of Analysis* & *Number Theory*, **3 (2)** (2015), 79-88.

[16] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian , The Triple Almost Lacunary Γ^3 sequence spaces defined by a modulus function, *International Journal of Applied Engineering Research*, Vol. **10**, No. **80** (2015), 94-99.

[17] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian , The triple entire sequence defined by Musielak Orlicz functions over $p-$ metric space, *Asian Journal of Mathematics and Computer Research, International Press*, **5(4)** (2015), 196-203.

[18] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, The Random of Lacunary statistical on Γ^3 over metric spaces defined by Musielak Orlicz functions, *Modern Applied Science*, in press.

[19] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, The Triple Γ^3 of tensor products in Orlicz sequence spaces, *Mathematical Sciences International Research Journal*, Vol. **4**, No. **2** (2015), 162-166.

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