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Convolutions Involving the Exponential Function and the Exponential Integral

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Abstract. The exponential integral $\mathrm{ei}(\lambda x)$ and its associated functions $\mathrm{ei}_+(\lambda x)$ and $\mathrm{ei}_-(\lambda x)$ are defined as locally summable functions on the real line and their derivatives are found as distributions. The convolutions $x^r \mathrm{ei}_+(x) * x^s e^x_+$ and $x^r \mathrm{ei}_+(x) * x^s e^x$ are evaluated.

1. Introduction and Results

The *exponential integral* $\mathrm{ei}(x)$ is defined for $x > 0$ by

$$\mathrm{ei}(x) = \int_x^\infty u^{-1} e^{-u} \,\mathrm{d}\, u, \tag{1}$$

see Sneddon [8], the integral diverging for $x \leq 0$. It was pointed out in [1] that equation (1) can be rewritten in the form

$$\mathrm{ei}(x) = \int_x^\infty u^{-1}[e^{-u} - H(1-u)] \,\mathrm{d}\, u - H(1-x) \ln |x|, \tag{2}$$

where H denotes Heaviside's function. The integral in this equation is convergent for all x and so we use equation (2) to define $\mathrm{ei}(x)$ on the real line.

More generally, see [1], if $\lambda \neq 0$, we define $\mathrm{ei}(\lambda x)$ in the obvious way by

$$\mathrm{ei}(\lambda x) = \int_{\lambda x}^\infty u^{-1}[e^{-u} - H(1-u)] \,\mathrm{d}\, u - H(1-\lambda x) \ln |\lambda x|. \tag{3}$$

Further, we define the functions $\mathrm{ei}_+(\lambda x)$ and $\mathrm{ei}_-(\lambda x)$ by

$$\mathrm{ei}_+(\lambda x) = H(x)\, \mathrm{ei}(\lambda x), \quad \mathrm{ei}_-(\lambda x) = H(-x)\, \mathrm{ei}(\lambda x)$$

so that

$$\mathrm{ei}(\lambda x) = \mathrm{ei}_+(\lambda x) + \mathrm{ei}_-(\lambda x). \tag{4}$$

In particular, if $\lambda > 0$, we have

$$\mathrm{ei}(\lambda x) = \int_x^\infty u^{-1}[e^{-\lambda u} - H(1-\lambda u)] \,\mathrm{d}\, u - H(1-\lambda x) \ln |\lambda x|, \tag{5}$$

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$$\text{(6)} \qquad \mathrm{ei}_+(\lambda x) = \int_x^\infty u^{-1} e^{-\lambda u} \, \mathrm{d}\, u, \quad x > 0,$$

$$\text{(7)} \qquad \mathrm{ei}_-(\lambda x) = -\gamma(\lambda) + \int_x^0 u^{-1}(e^{-\lambda u} - 1) \, \mathrm{d}\, u - \ln x_-, \quad x < 0,$$

where

$$\gamma(\lambda) = \gamma + \ln|\lambda|$$

and

$$\gamma = -\int_0^\infty u^{-1}[e^{-\lambda u} - H(1 - \lambda u)] \, \mathrm{d}\, u$$

is Euler's constant.

The derivatives of these functions are given by

$$\begin{aligned} [\mathrm{ei}(\lambda x)]' &= -e^{-\lambda x} x^{-1}, \\ [\mathrm{ei}_+(\lambda x)]' &= -e^{-\lambda x} x_+^{-1} - \gamma(\lambda)\delta(x), \\ [\mathrm{ei}_-(\lambda x)]' &= e^{-\lambda x} x_-^{-1} + \gamma(\lambda)\delta(x)), \end{aligned} \tag{8}$$

for all $\lambda \neq 0$.

In particular, we have

$$\begin{aligned} \mathrm{ei}(x) &= \int_x^\infty u^{-1}[e^{-u} - H(1-u)] \, \mathrm{d}\, u - H(1-x) \ln|x|, \\ \mathrm{ei}_+(x) &= \int_x^\infty u^{-1} e^{-u} \, \mathrm{d}\, u, \quad x > 0, \\ \mathrm{ei}_-(x) &= -\gamma + \int_x^0 u^{-1}(e^{-u} - 1) \, \mathrm{d}\, u - \ln x_-, \quad x < 0, \end{aligned} \tag{9}$$

where

$$\gamma = -\int_0^\infty u^{-1}[e^{-u} - H(1-u)] \, \mathrm{d}\, u$$

is Euler's constant.

The derivatives of these functions are given by

$$\begin{aligned} [\mathrm{ei}(x)]' &= -e^{-x} x^{-1}, \\ [\mathrm{ei}_+(x)]' &= -e^{-x} x_+^{-1} - \gamma\delta(x), \\ [\mathrm{ei}_-(x)]' &= e^{-x} x_-^{-1}. \end{aligned} \tag{10}$$

The classical definition of the convolution of two functions f and g is as follows:

Definition 1. Let f and g be functions. Then the *convolution* $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^\infty f(t) g(x - t) \, \mathrm{d}\, t$$

for all points x for which the integral exist.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f \tag{11}$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g'(\text{ or } f' * g). \tag{12}$$

Definition 1 can be extended to define the convolution $f * g$ of two distributions f and g in D' with the following definition, see Gel'fand and Shilov [7].

Definition 2. Let f and g be distributions in D'. Then the *convolution* $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi \rangle = \langle f(y), \langle g(x), \varphi(x + y) \rangle \rangle$$

for arbitrary φ in D, provided f and g satisfy either of the conditions

(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition then equations (10) and (11) are satisfied.

The locally summable functions e_+^x and e_-^x are defined by

$$e_+^x = H(x)e^x \quad e_-^x = H(-x)e^x.$$

In the following we need the following lemma, which is easily proved by induction.

Lemma 1.

$$\int_0^u t^k e^{-t} \,\mathrm{d}\, t = -\sum_{i=0}^{k} \frac{k!}{i!} u^i e^{-u} + k!,$$

$$\int_0^u t^k e^{-2t} \,\mathrm{d}\, t = -\sum_{i=0}^{k} \frac{k!}{2^{k-i+1} i!} u^i e^{-2u} + \frac{k!}{2^{k+1}},$$

for $k = 0, 1, 2, \ldots$.

We now prove the following theorem.

Theorem 1. *The convolution* $x^r\,\mathrm{ei}_+(x) * x^s e_+^x$ *exists and*

$$
\begin{aligned}
&x^r\,\mathrm{ei}_+(x) * x^s e_+^x = \\
&\quad = \sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\sum_{j=0}^{i-1}\frac{(i-1)!}{2^{i-j}i!j!}x^j e_+^{-x} - \frac{(i-1)!}{2^i i!}e_+^x\Big] \\
&\qquad + \sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}[e^x\,\mathrm{ei}_+(2x) - e^x\,\mathrm{ei}_+(x) + \ln 2\,e_+^x] \\
&\qquad - \sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\sum_{i=1}^{r+k}\frac{x^i}{i!} + (1-e^x)\Big]\,\mathrm{ei}_+(x),
\end{aligned}
\tag{13}
$$

for $r, s = 0, 1, 2, \ldots$ *and* r, s *not both zero.*

In particular,

$$
\begin{aligned}
x^r\,\mathrm{ei}_+(x) * e_+^x = r!\sum_{i=1}^{r}\Big[\sum_{j=0}^{i-1}\frac{(i-1)!}{2^{i-j}i!j!}x^j e_+^{-x} - \frac{(i-1)!}{2^i i!}e_+^x\Big]& \\
+ r![e^x\,\mathrm{ei}_+(2x) - e^x\,\mathrm{ei}_+(x) + \ln 2\,e_+^x]& \\
- r!\Big[\sum_{i=1}^{r}\frac{x^i}{i!} + (1-e^x)\Big]\,\mathrm{ei}_+(x),&
\end{aligned}
\tag{14}
$$

for $r = 1, 2, \ldots$ *and*

$$
\mathrm{ei}_+(x) * e_+^x = -\,\mathrm{ei}_+(x) + e^x\,\mathrm{ei}_+(2x) + \ln 2\,e_+^x. \tag{15}
$$

Proof. The convolution $x^r\,\mathrm{ei}_+(x) * x^s e_+^x = 0$ if $x < 0$ and so when $x > 0$, we have

$$
\begin{aligned}
x^r\,\mathrm{ei}_+(x) * x^s e_+^x &= \int_0^x t^r(x-t)^s e^{x-t}\int_t^\infty u^{-1}e^{-u}\,\mathrm{d}\,u\,\mathrm{d}\,t \\
&= \int_0^x u^{-1}e^{x-u}\int_0^u t^r(x-t)^s e^{-t}\,\mathrm{d}\,t\,\mathrm{d}\,u \\
&\quad + \int_x^\infty u^{-1}e^{x-u}\int_0^x t^r(x-t)^s e^{-t}\,\mathrm{d}\,t\,\mathrm{d}\,u \\
&= I_1 + I_2,
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
&\int_0^u t^r(x-t)^s e^{-t}\,\mathrm{d}\,t = \\
&\quad = \sum_{k=0}^{s}\binom{s}{k}(-1)^k x^{s-k}\int_0^u t^{r+k}e^{-t}\,\mathrm{d}\,t \\
&\quad = -\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\sum_{i=1}^{r+k}\frac{u^i}{i!}e^{-u} + (e^{-u}-1)\Big]
\end{aligned}
\tag{17}
$$

and in particular when $r = s = 0$,

$$\int_0^u e^{-t}\,\mathrm{d}\,t = -e^{-u} + 1, \tag{18}$$

on using the lemma.

Hence

$$\begin{aligned} I_1 = {} & -\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}e^x \cdot \\ & \cdot \int_0^x \Big[\sum_{i=1}^{r+k}\frac{u^{i-1}}{i!}e^{-2u} + u^{-1}(e^{-2u} - e^{-u})\Big]\,\mathrm{d}\,u \\ = {} & -\sum_{k=0}^{s}(r+k)!\sum_{i=1}^{r+k}\binom{s}{k}(-1)^k x^{s-k}e^x\int_0^x \frac{u^{i-1}}{i!}e^{-2u}\,\mathrm{d}\,u \\ & -\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}e^x\int_0^x u^{-1}(e^{-2u} - e^{-u})\,\mathrm{d}\,u. \end{aligned} \tag{19}$$

Further, we have

$$\int_0^x \frac{u^{i-1}}{i!}e^{-2u}\,\mathrm{d}\,u = -\sum_{j=0}^{i-1}\frac{(i-1)!}{2^{i-j}i!j!}x^j e^{-2x} + \frac{(i-1)!}{2^i i!}, \tag{20}$$

on using the lemma, and

$$\begin{aligned} & \int_0^x u^{-1}[e^{-u} - H(1-u)]\,\mathrm{d}\,u = \\ & = \int_0^\infty u^{-1}[e^{-u} - H(1-u)]\,\mathrm{d}\,u \\ & \quad - \int_x^\infty u^{-1}e^{-u}\,\mathrm{d}\,u + \int_x^\infty u^{-1}H(1-u)\,\mathrm{d}\,u \\ & = -\gamma - \mathrm{ei}_+(x) + \int_x^\infty u^{-1}H(1-u)\,\mathrm{d}\,u. \end{aligned} \tag{21}$$

Similarly

$$\int_0^x u^{-1}[e^{-2u} - H(1-2u)]\,\mathrm{d}\,u = -\gamma - \mathrm{ei}_+(2x) + \int_x^\infty u^{-1}H[1-2u]\,\mathrm{d}\,u. \tag{22}$$

It follows from equations (21) and (22) that

$$\begin{aligned} & \int_0^x u^{-1}(e^{-u} - e^{-2u})\,\mathrm{d}\,u = \\ & = \mathrm{ei}_+(2x) - \mathrm{ei}_+(x) + \int_0^\infty u^{-1}[H(1-u) - H(1-2u)]\,\mathrm{d}\,u \\ & = \mathrm{ei}_+(2x) - \mathrm{ei}_+(x) + \ln 2. \end{aligned} \tag{23}$$

In particular, when $r = s = 0$, we have

$$I_1 = [\mathrm{ei}_+(2x) - \mathrm{ei}_+(x) + \ln 2]e_+^x. \tag{24}$$

Next, as in equation (17), we have

$$\begin{aligned}\int_0^x t^r(x-t)^s e^{-t}\,\mathrm{d}\,t &= \\ &= -\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\sum_{i=1}^{r+k}\frac{x^i}{i!}e^{-x} + (e^{-x}-1)\Big]\end{aligned} \tag{25}$$

and so

$$I_2 = -\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\sum_{i=1}^{r+k}\frac{x^i}{i!} + (1-e^x)\Big]\,\mathrm{ei}_+(x). \tag{26}$$

In particular, when $r = s = 0$, we have

$$I_2 = (e^x - 1)\int_x^\infty u^{-1}e^{-u}\,\mathrm{d}\,u = (e^x-1)\,\mathrm{ei}_+(x). \tag{27}$$

Equation (13) now follows from equations (20), (21), (22), (25) and (26).

Equation (14) follows on putting $s = 0$ in equation (13) and equation (15) follows on putting $r = 0$ in equation (14). □

In the corollary, the distribution x_+^{-2} is defined by $x_+^{-2} = (x_+^{-1})'$ and not as in Gel'fand and Shilov.

Corollary 1.1. *The convolutions* $(e^{-x}x_+^{-1}) * e_+^x$ *and* $(e^{-x}x_+^{-2}) * e_+^x$ *exist and*

$$(e^{-x}x_+^{-1}) * e_+^x = -e^x\,\mathrm{ei}_+(2x) - \gamma(2)e_+^x \tag{28}$$

$$(e^{-x}x_+^{-2}) * e_+^x = 2e^x\,\mathrm{ei}_+(2x) + 2\gamma(2)e_+^x - e^{-x}x_+^{-1}. \tag{29}$$

Proof. The convolution $(e^{-x}x_+^{-1}) * e_+^x$ exists by Definition 2, since $e^{-x}x_+^{-1}$ and e_+^x are both bounded on the left. From equation (12), we have

$$\begin{aligned}[\mathrm{ei}_+(x) * e_+^x]' &= -[e^{-x}x_+^{-1} + \gamma\delta(x)] * e_+^x \\ &= -(e^{-x}x_+^{-1}) * e_+^x - \gamma e_+^x \\ &= \mathrm{ei}_+(x) * [e_+^x + \delta(x)] \\ &= e^x\,\mathrm{ei}_+(2x) + \ln 2\, e_+^x\end{aligned}$$

and equation (28) follows.

From equations (12) and (28), we now have

$$\begin{aligned}[(e^{-x}x_+^{-1}) * e_+^x]' &= -(e^{-x}x_+^{-1} + e^{-x}x_+^{-2}) * e_+^x \\ &= e^x\,\mathrm{ei}_+(2x) + \gamma(2)e_+^x - (e^{-x}x_+^{-2}) * e_+^x \\ &= (e^{-x}x_+^{-1}) * [e_+^x + \delta(x)] \\ &= -e^x\,\mathrm{ei}_+(2x) - \gamma(2)e_+^x + e^{-x}x_+^{-1}\end{aligned}$$

and equation (29) follows. □

Theorem 2. *The convolution* $x^r\,\mathrm{ei}_+(x) * x^s e^x$ *exists and*

$$x^r\,\mathrm{ei}_+(x) * x^s e^x = -\sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^k(r+k)!}{2^i i!}x^{s-k}e^x + \sum_{k=0}^{s}\binom{s}{k}(-1)^k\ln 2(r+k)!x^{s-k}e^x, \tag{30}$$

for $r,s = 0,1,2,\dots$ *and* r,s *not both zero.*

In particular

$$x^r\,\mathrm{ei}_+(x) * e^x = -\sum_{i=1}^{r}\frac{r!}{2^i i!}e^x + \ln 2r!e^x, \tag{31}$$

for $r = 1,2,\dots$ *and*

$$\mathrm{ei}_+(x) * e^x = \ln 2\, e^x \tag{32}$$

$$\mathrm{ei}_+(x) * xe^x = \ln 2\, xe^x - \ln 2\, e^x + \frac{1}{2}e^x. \tag{33}$$

Proof. We have

$$\begin{aligned}
x^r\,\mathrm{ei}_+(x) * x^s e^x &= \int_0^\infty t^r(x-t)^s e^{x-t}\int_t^\infty u^{-1}e^{-u}\,\mathrm{d}\,u\,\mathrm{d}\,t \\
&= \int_0^\infty u^{-1}e^{x-u}\int_0^u t^r(x-t)^s e^{-t}\,\mathrm{d}\,t\,\mathrm{d}\,u \\
&= -\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}e^x\sum_{i=1}^{r+k}\int_0^\infty\frac{u^{i-1}}{i!}e^{-2u}\,\mathrm{d}\,u \\
&\quad -\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}e^x\int_0^\infty(u^{-1}e^{-2u}-u^{-1}e^{-u})\,\mathrm{d}\,u \\
&= -\sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^k(r+k)!}{2^i i}x^{s-k}e^x \\
&\quad +\sum_{k=0}^{s}\binom{s}{k}(-1)^k\ln 2(r+k)!x^{s-k}e^x
\end{aligned}$$

on making use of equation (17), the lemma and noting that

$$\begin{aligned}
\int_0^\infty u^{-1}(e^{-2u}-e^{-u})\,\mathrm{d}\,u &= \int_0^\infty \ln u(2e^{-2u}-e^{-u})\,\mathrm{d}\,u \\
&= \Gamma'(1) - \ln 2 - \Gamma'(1) = -\ln 2,
\end{aligned}$$

proving equation (30).

Equation (31) follows on putting $s = 0$ in equation (30) and equation (32) follows on putting $r = 0$ in equation (31).

Equation (31) follows on putting $r = 0$ and $s = 1$ in equation (30). □

Corollary 2.1. *The convolution* $(e^{-x}x_+^{-n}) * e^x$ *exists and*

$$e^{-x}x_+^{-n} * e^x = \frac{(-1)^n 2^{n-1}}{(n-1)!}\gamma(2)e^x \tag{34}$$

for $n = 1, 2, \dots$.

In particular,

$$e^{-x}x_+^{-1} * xe^x = -\gamma(2)xe^x - \frac{1}{2}e^x. \tag{35}$$

Proof. Differentiating equation (32), we get

$$[-e^{-x}x_+^{-1} - \gamma\delta(x)] * e^x = -(e^{-x}x_+^{-1}) * e^x - \gamma e^x = \ln 2e^x$$

and we see that equation (34) is true when $n = 1$.

Now assume that equation (34) is true for some n. Then differentiating equation (34), we get

$$(-e^{-x}x_+^{-n} - ne^{-x}x_+^{-n-1}) * e^x = (e^{-x}x_+^{-n}) * e^x.$$

It follows that

$$\begin{aligned} ne^{-x}x_+^{-n-1} * e^x &= -2(e^{-x}x_+^{-n}) * e^x \\ &= \frac{(-1)^{n+1}2^n}{(n-1)!}\gamma(2)e^x \end{aligned}$$

and so equation (33) is true for $n+1$. Equation (34) now follows by induction.

Differentiating equation (33), we get

$$[-e^{-x}x_+^{-1} - \gamma\delta(x)] * xe^x = \ln 2\, xe^x + \frac{1}{2}e^x$$

and equation (35) follows. □

For further results involving the exponential integral, see [2, 3, 4, 5] and [6].

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