

Stability for Nonlinear Neutral Integro-Differential Equations with Variable Delay

ABDELOUAHEB ARDJOUNI AND AHCENE DJOUDI

ABSTRACT. In this paper we use the contraction mapping principle to obtain asymptotic stability results of a nonlinear neutral integro-differential equation with variable delay. An asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some previous results due to Burton [7], Becker and Burton [6] and Jin and Luo [17]. In the end we provide an example to illustrate our claim.

1. INTRODUCTION

Incontestably, Lyapunov's direct method has been, for more than 100 years, the main tool for investigating the stability properties of a wide variety of ordinary, functional, partial differential and integro-differential equations. Nevertheless, the application of this method to problems of stability in differential and integro-differential equations with delays has encountered serious obstacles if the delays are unbounded or if the equation has unbounded terms [8]–[10]. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Becker and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1]–[21], [23]). The fixed point theory does not only solve the problem on stability but has other significant advantage over Lyapunov's. The conditions of the former are often averages but those of the latter are usually pointwise (see [8]).

In this paper we consider the nonlinear neutral integro-differential equation with variable delay

$$(1) \quad \frac{d}{dt}x(t) = - \int_{t-\tau(t)}^t a(t,s)x(s) \, ds + \frac{d}{dt}Q(t, x(t-\tau(t))),$$

with the initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(0), 0],$$

2010 *Mathematics Subject Classification.* 34K20, 34K30, 34K40.

Key words and phrases. Fixed points, Stability, Integro-differential equation, Variable delay.

where $\psi \in C([m(0), 0], \mathbb{R})$ and $m(0) = \inf\{t - \tau(t), t \geq 0\}$.

Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \rightarrow S_2$ with the supremum norm $\|\cdot\|$. Throughout this paper we assume that $a \in C(\mathbb{R}^+ \times [m(0), \infty), \mathbb{R})$ and $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t - \tau(t) \rightarrow \infty$ as $t \rightarrow \infty$. The function $Q(t, x)$ is globally Lipschitz continuous in x . That is, there is positive constant L such that

$$(2) \quad |Q(t, x) - Q(t, y)| \leq L \|x - y\|.$$

We also assume that

$$(3) \quad Q(t, 0) = 0.$$

Special cases of equation (1) have been investigated by many authors. For example, Burton in [7], Becker and Burton in [6], Jin and Luo in [17] have studied the equation

$$(4) \quad x'(t) = - \int_{t-\tau(t)}^t a(t, s) x(s) \, ds,$$

and have respectively proved the following theorems.

Theorem 1.1 (Burton [7]). *Suppose that $\tau(t) = r$ and there exists a constant $\alpha < 1$ such that*

$$(5) \quad 2 \int_{t-r}^t |A(t, s)| \, ds \leq \alpha \text{ for all } t \geq 0,$$

and

$$(6) \quad \int_0^t A(s, s) \, ds \rightarrow \infty \text{ as } t \rightarrow \infty,$$

where

$$A(t, s) = \int_{t-s}^r a(u + s, s) \, du \text{ with } A(t, t) = \int_0^r a(u + t, t) \, du.$$

Then the zero solution of (4) is asymptotically stable.

Theorem 1.2 (Becker and Burton [6]). *Suppose that τ is differentiable, $t - \tau(t)$ is strictly increasing, and there exist constants $k \geq 0$, $\alpha \in (0, 1)$ such that for $t \geq 0$,*

$$(7) \quad - \int_0^t G(s, s) \, ds \leq k,$$

and

$$(8) \quad \int_{t-\tau(t)}^t |G(t, s)| \, ds + \int_0^t e^{-\int_s^t G(u, u) \, du} |G(s, s)| \left(\int_{s-\tau(s)}^s |G(s, u)| \, du \right) \, ds \leq \alpha,$$

with

$$G(t, s) = \int_t^{f(s)} a(u, s) \, d u,$$

$$G(t, t) = \int_t^{f(t)} a(u, t) \, d u,$$

where f is the inverse function of $t - \tau(t)$. Then for each continuous initial function $\psi : [m(0), 0] \rightarrow \mathbb{R}$, there is a unique continuous function $x : [m(0), \infty) \rightarrow \mathbb{R}$ with $x(t) = \psi(t)$ on $[m(0), 0]$ that satisfies (4) on $[0, \infty)$. Moreover, x is bounded on $[m(0), \infty)$. Furthermore, the zero solution of (4) is stable at $t = 0$. If, in addition,

$$(9) \quad \int_0^t G(s, s) \, d s \rightarrow \infty \text{ as } t \rightarrow \infty,$$

then $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Theorem 1.3 (Jin and Luo [17]). *Let τ be differentiable. Suppose that there exist constants $k \geq 0$, $\alpha \in (0, 1)$ and a function $h \in C(\mathbb{R}^+, \mathbb{R})$ such that for $t \geq 0$,*

$$(10) \quad - \int_0^t h(s) \, d s \leq k,$$

and

$$(11) \quad \int_{t-\tau(t)}^t |h(s) + B(t, s)| \, d s$$

$$+ \int_0^t e^{-\int_s^t h(u) \, d u} |h(s)| \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| \, d u \right) \, d s$$

$$+ \int_0^t e^{-\int_s^t h(u) \, d u} |h(s - \tau(s)) + B(s, s - \tau(s))| |1 - \tau'(s)| \leq \alpha,$$

where

$$(12) \quad B(t, s) = \int_t^s a(u, s) \, d u, \quad \text{with}$$

$$B(t, t - \tau(t)) = \int_t^{t-\tau(t)} a(u, t - \tau(t)) \, d u.$$

Then for each continuous initial function $\psi : [m(0), 0] \rightarrow \mathbb{R}$, there is an unique continuous function $x : [m(0), \infty) \rightarrow \mathbb{R}$ with $x(t) = \psi(t)$ on $[m(0), 0]$ that satisfies (4) on $[0, \infty)$. Moreover, x is bounded on $[m(0), \infty)$. Furthermore, the zero solution of (4) is stable at $t = 0$. If, in addition,

$$(13) \quad \int_0^t h(s) \, d s \rightarrow \infty \text{ as } t \rightarrow \infty,$$

then $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

In a recent work, we have studied the linear neutral equation

$$(14) \quad x'(t) = - \int_{t-\tau(t)}^t a(t, s) x(s) \, ds + c(t) x'(t - \tau(t)),$$

and have obtained the following result.

Theorem 1.4 (Ardjouni, Djoudi and Soualhia [5]). *Suppose that τ is twice continuously differentiable with $\tau'(t) \neq 1$ for all $t \in \mathbb{R}^+$, c is continuously differentiable on \mathbb{R}^+ , and there exist continuous function $h : [m(0), \infty) \rightarrow \mathbb{R}$ and a constant $\alpha \in (0, 1)$ such that for $t \geq 0$*

$$(15) \quad \liminf_{t \rightarrow \infty} \int_0^t h(s) \, ds > -\infty,$$

and

$$(16) \quad \begin{aligned} & \left| \frac{c(t)}{1 - \tau'(t)} \right| + \int_{t-\tau(t)}^t |h(s) + B(t, s)| \, ds \\ & + \int_0^t e^{-\int_s^t h(u) \, du} \left| [h(s - \tau(s)) \right. \\ & \quad \left. + B(s, s - \tau(s))] (1 - \tau'(s)) - r(s) \right| \, ds \\ & + \int_0^t e^{-\int_s^t h(u) \, du} u h(s) \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| \, du \right) \, ds \leq \alpha \end{aligned}$$

where B is given by (12) and

$$(17) \quad r(t) = \frac{[c(t)h(t) + c'(t)](1 - \tau'(t)) + c(t)\tau''(t)}{(1 - \tau'(t))^2}.$$

Then the zero solution of (14) is asymptotically stable if and only if

$$(18) \quad \int_0^t h(s) \, ds \rightarrow \infty, \text{ as } t \rightarrow \infty.$$

Remark 1.1. The result of Becker and Burton obtained in *Theorem 1.2* requires that $t - \tau(t)$ is strictly increasing. However, in *Theorem 1.3*, this condition is clearly removed. Also, the conditions of stability in *Theorem 1.3* are less restrictive than *Theorem 1.2*. That is, *Theorem 1.3* improves *Theorems 1.1* and *1.2*. Moreover, if we let $c = 0$ in (14) then the equation reduces to (4). Consequently, *Theorem 1.4* is a generalization of both *theorems 1.1, 1.2* and *1.3*.

Note that in our consideration the neutral term $\frac{d}{dt}Q(t, x(t - \tau(t)))$ of (1) produces nonlinearity in the derivative term $\frac{d}{dt}x(t - \tau(t))$. The neutral term $\frac{d}{dt}x(t - \tau(t))$ in [5] enters linearly. So, the analysis made here is different form that in [5].

Our objective here is to improve *Theorem 1.3* and extend it to investigate a wide class of nonlinear integro-differential equation with variable delay of

neutral type presented in (1). To do this we define a suitable continuous function h (see Theorem 2.1 below) and find conditions for h , with no need of further assumptions on the inverse of the delay $t - \tau(t)$, so that for a given continuous initial function ψ a mapping P for (1) is constructed in such a way to map a complete metric space S_ψ in itself and in which P possesses a fixed point. This procedure will enable us to establish and prove an asymptotic stability theorem for the zero solution of (1) with a necessary and sufficient condition and with less restrictive conditions. The results obtained in this paper improve and generalize the main results in [6, 7, 17]. We provide an example to illustrate our claim.

2. MAIN RESULTS

For each $\psi \in C([m(0), 0], \mathbb{R})$, a solution of (1) through $(0, \psi)$ is a continuous function $x : [m(0), \sigma] \rightarrow \mathbb{R}$ for some positive constant $\sigma > 0$ such that x satisfies (1) on $[0, \sigma]$ and $x = \psi$ on $[m(0), 0]$. We denote such a solution by $x(t) = x(t, 0, \psi)$. We define $\|\psi\| = \max\{|\psi(t)| : m(0) \leq t \leq 0\}$. Stability definitions may be found in [8], for example.

Our purpose here is to extend Theorem 1.3 by giving a necessary and sufficient condition for asymptotic stability of the zero solution of equation (1). But, to reach this end, one crucial step in the investigation of the stability of an equation using fixed point technic involves the construction of a suitable fixed point mapping. This can, in so many cases, be an arduous task. So, to construct our mapping, we begin by transforming (1) to a more tractable, but equivalent, equation, which we then invert to obtain an equivalent integral equation from which we derive a fixed point mapping. After then, we define prudently a suitable complete space, depending on the initial condition, so that the mapping is a contraction. Using Banach's contraction mapping principle, we obtain a solution for this mapping, and hence a solution for (1), which is asymptotically stable.

First, we have to transform (1) into an equivalent equation that possesses the same basic structure and properties to which we apply the variation of parameters to define a fixed point mapping.

Lemma 2.1. *Equation (1) is equivalent to*

$$(19) \quad \begin{aligned} \frac{d}{dt}x(t) &= B(t, t - \tau(t)) (1 - \tau'(t)) x(t - \tau(t)) \\ &+ \frac{d}{dt} \int_{t-\tau(t)}^t B(t, s) x(s) ds + \frac{d}{dt} Q(t, x(t - \tau(t))), \end{aligned}$$

where

$$B(t, s) = \int_t^s a(u, s) du \text{ and } B(t, t - \tau(t)) = \int_t^{t-\tau(t)} a(u, t - \tau(t)) du.$$

Proof. Differentiating the integral term in (19), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{t-\tau(t)}^t B(t, s) x(s) ds &= B(t, t) x(t) \\ &\quad - B(t, t - \tau(t)) (1 - \tau'(t)) x(t - \tau(t)) + \int_{t-\tau(t)}^t \frac{\partial}{\partial t} B(t, s) x(s) ds. \end{aligned}$$

Substituting this into (19), it follows that (19) is equivalent to (1) provided B satisfies the following conditions

$$(20) \quad B(t, t) = 0 \text{ and } \frac{\partial}{\partial t} B(t, s) = -a(t, s).$$

Now (20) implies

$$(21) \quad B(t, s) = - \int_0^t a(u, s) du + \phi(s),$$

for some function ϕ and $B(t, s)$ must satisfy

$$B(t, t) = - \int_0^t a(u, t) du + \phi(t) = 0.$$

Consequently,

$$\phi(t) = \int_0^t a(u, t) du.$$

Substituting this into (21), we obtain

$$B(t, s) = - \int_0^t a(u, s) du + \int_0^s a(u, s) du = \int_t^s a(u, s) du.$$

This definition of B satisfies (20). Consequently, (1) is equivalent to (19). \square

Now, we have to invert (1) into an integral equation from which a fixed point mapping can be constructed. We remind that ψ denotes any real-valued continuous function defined on $[m(0), 0]$.

Lemma 2.2. *If x is a solution of (1) on an interval $[0, T]$ and satisfies the initial condition $x = \psi$ on $[m(0), 0]$, then x is a solution of the integral*

equation

$$\begin{aligned}
 (22) \quad x(t) = & \left(\psi(0) - Q(0, \psi(-\tau(0))) \right. \\
 & - \int_{-\tau(0)}^0 [h(s) + B(0, s)] \psi(s) \, ds \Big) e^{-\int_0^t h(u) \, du} \\
 & + Q(t, x(t - \tau(t))) + \int_{t-\tau(t)}^t [h(s) + B(t, s)] x(s) \, ds \\
 & + \int_0^t e^{-\int_s^t h(u) \, du} \left\{ [h(s - \tau(s)) \right. \\
 (23) \quad & \left. + B(s, s - \tau(s))] (1 - \tau'(s)) \right\} x(s - \tau(s)) \, ds \\
 & - \int_0^t e^{-\int_s^t h(u) \, du} h(s) Q(s, x(s - \tau(s))) \, ds
 \end{aligned}$$

$$(24) \quad - \int_0^t e^{-\int_s^t h(u) \, du} h(s) \left(\int_{s-\tau(s)}^s [h(u) + B(s, u)] x(u) \, du \right) \, ds.$$

on $[0, T]$, where $h : [m(0), \infty) \rightarrow \mathbb{R}$ is an arbitrary continuous function. Conversely, if a continuous function x is equal to ψ on $[m(0), 0]$ and is a solution of (24) on an interval $[0, \sigma)$, then x is a solution of (1) on $[0, \sigma)$.

Proof. Use Lemma 2.1 to rewrite (1) in the following equivalent form

$$\begin{aligned}
 (25) \quad & \frac{d}{dt} \{x(t) - Q(t, x(t - \tau(t)))\} \\
 & = B(t, t - \tau(t)) (1 - \tau'(t)) x(t - \tau(t)) \\
 & \quad + \frac{d}{dt} \int_{t-\tau(t)}^t B(t, s) x(s) \, ds.
 \end{aligned}$$

Multiplying both sides of (25) by $e^{\int_0^t h(u) \, du}$ and integrating with respect to s from 0 to t , we obtain

$$\begin{aligned}
 x(t) = & (\psi(0) - Q(0, \psi(-\tau(0)))) e^{-\int_0^t h(u) \, du} + Q(t, x(t - \tau(t))) \\
 & + \int_0^t e^{-\int_s^t h(u) \, du} h(s) x(s) \, ds \\
 & + \int_0^t e^{-\int_s^t h(u) \, du} \frac{d}{ds} \int_{s-\tau(s)}^s B(s, u) x(u) \, du \\
 & + \int_0^t e^{-\int_s^t h(u) \, du} B(s, s - \tau(s)) (1 - \tau'(s)) x(s - \tau(s)) \, ds \\
 & - \int_0^t e^{-\int_s^t h(u) \, du} h(s) Q(s, x(s - \tau(s))) \, ds.
 \end{aligned}$$

Performing an integration by parts, we get

$$\begin{aligned}
x(t) &= (\psi(0) - Q(0, \psi(-\tau(0)))) e^{-\int_0^t h(u) \, d u} + Q(t, x(t - \tau(t))) \\
&+ \int_0^t e^{-\int_s^t h(u) \, d u} \, d \left(\int_{s-\tau(s)}^s [h(u) + B(s, u)] x(u) \, d u \right) \\
&+ \int_0^t e^{-\int_s^t h(u) \, d u} [h(s - \tau(s)) + B(s, s - \tau(s))] (1 - \tau'(s)) x(s - \tau(s)) \, d s \\
&- \int_0^t e^{-\int_s^t h(u) \, d u} h(s) Q(s, x(s - \tau(s))) \, d s \\
&= \left(\psi(0) - Q(0, \psi(-\tau(0))) - \int_{-\tau(0)}^0 [h(s) + B(0, s)] \psi(s) \, d s \right) e^{-\int_0^t h(u) \, d u} \\
&+ Q(t, x(t - \tau(t))) + \int_{t-\tau(t)}^t [h(s) + B(t, s)] x(s) \, d s \\
&+ \int_0^t e^{-\int_s^t h(u) \, d u} \{ [h(s - \tau(s)) + B(s, s - \tau(s))] (1 - \tau'(s)) \} x(s - \tau(s)) \, d s \\
&- \int_0^t e^{-\int_s^t h(u) \, d u} h(s) Q(s, x(s - \tau(s))) \, d s \\
&- \int_0^t e^{-\int_s^t h(u) \, d u} h(s) \left(\int_{s-\tau(s)}^s [h(u) + B(s, u)] x(u) \, d u \right) \, d s,
\end{aligned}$$

Conversely, suppose that a continuous function x is equal to ψ on $[m(0), 0]$ and satisfies (24) on some interval $[0, \sigma)$. Then it is differentiable on $[0, \sigma)$. Differentiating (24) we obtain (1). \square

Theorem 2.1. *Suppose (2) and (3) hold. Let τ be differentiable, and suppose that there exist continuous function $h : [m(0), \infty) \rightarrow \mathbb{R}$ and a constant $\alpha \in (0, 1)$ such that for $t \geq 0$*

$$(26) \quad \liminf_{t \rightarrow \infty} \int_0^t h(s) \, d s > -\infty,$$

and

$$\begin{aligned}
(27) \quad &L + \int_{t-\tau(t)}^t |h(s) + B(t, s)| \, d s \\
&+ \int_0^t e^{-\int_s^t h(u) \, d u} \\
&\cdot \left\{ \left| [h(s - \tau(s)) + B(s, s - \tau(s))] (1 - \tau'(s)) \right| + L |h(s)| \right\} \, d s \\
&+ \int_0^t e^{-\int_s^t h(u) \, d u} |h(s)| \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| \, d u \right) \, d s \leq \alpha,
\end{aligned}$$

where

$$B(t, s) = \int_t^s a(u, s) \, d u \text{ with } B(t, t - \tau(t)) = \int_t^{t-\tau(t)} a(u, t - \tau(t)) \, d u.$$

Then the zero solution of (1) is asymptotically stable if and only if

$$(28) \quad \int_0^t h(s) \, d s \rightarrow \infty \text{ as } t \rightarrow \infty.$$

Proof. First, suppose that (28) holds. We set

$$(29) \quad K = \sup_{t \geq 0} \left\{ e^{-\int_0^t h(s) \, d s} \right\}.$$

The set $C([m(0), \infty), \mathbb{R})$ of real valued bounded functions on $[m(0), \infty)$ is a Banach space when it is endowed with the supremum norm $\|\cdot\|$; that is, for $\phi \in C([m(0), \infty), \mathbb{R})$,

$$\|\phi\| := \sup \{ |\phi(t)| : t \in [m(0), \infty) \}.$$

Otherwise speaking, we carry out our investigations in the complete metric space $(C([m(0), \infty), \mathbb{R}), \rho)$ where ρ is supremum metric

$$\rho(x, y) := \sup_{t \geq m(0)} |x(t) - y(t)| = \|x - y\|, \quad \text{for } x, y \in C([m(0), \infty), \mathbb{R}).$$

Let $\psi \in C([m(0), 0], \mathbb{R})$ be fixed and define

$$S_\psi := \left\{ \varphi \in C([m(0), \infty), \mathbb{R}) : \varphi(t) = \psi(t) \text{ for } t \in [m(0), 0] \text{ and } \varphi(t) \rightarrow 0 \text{ as } t \rightarrow \infty \right\}.$$

Being closed in $C([m(0), \infty), \mathbb{R})$, S_ψ is itself a Banach space.

Now, use (24) to define the operator $P : S_\psi \rightarrow S_\psi$ by $(P\varphi)(t) = \psi(t)$ if $t \in [m(0), 0]$ and for $t \geq 0$ we let

$$(30) \quad \begin{aligned} (P\varphi)(t) &= \left(\psi(0) - Q(0, \psi(-\tau(0))) \right. \\ &\quad \left. - \int_{-\tau(0)}^0 [h(s) + B(0, s)] \psi(s) \, d s \right) e^{-\int_0^t h(u) \, d u} \\ &+ Q(t, \varphi(t - \tau(t))) + \int_{t-\tau(t)}^t [h(s) + B(t, s)] \varphi(s) \, d s \\ &+ \int_0^t e^{-\int_s^t h(u) \, d u} \left\{ \left[h(s - \tau(s)) \right. \right. \\ &\quad \left. \left. + B(s, s - \tau(s)) \right] (1 - \tau'(s)) \right\} \varphi(s - \tau(s)) \, d s \\ &- \int_0^t e^{-\int_s^t h(u) \, d u} h(s) Q(s, \varphi(s - \tau(s))) \, d s \\ &- \int_0^t e^{-\int_s^t h(u) \, d u} h(s) \left(\int_{s-\tau(s)}^s [h(u) + B(s, u)] \varphi(u) \, d u \right) \, d s. \end{aligned}$$

It is clear that $(P\varphi) \in C([m(0), \infty), \mathbb{R})$. We will show that $(P\varphi)(t) \rightarrow 0$ as $t \rightarrow \infty$. To this end, denote the five terms on the right hand side of (30) by I_1, I_2, \dots, I_6 , respectively. It is obvious that the first term I_1 tends to zero as $t \rightarrow \infty$, by condition (28). Also, due to the conditions (2) and (3) and the facts that $\varphi(t) \rightarrow 0$ and $t - \tau(t) \rightarrow \infty$ as $t \rightarrow \infty$, the second term I_2 in (30) tends to zero as $t \rightarrow \infty$. What is left to show is that each of the remaining terms in (30) go to zero at infinity.

Let $\varphi \in S_\psi$ be fixed. For a given $\varepsilon > 0$, we choose $T_0 > 0$ large enough such that $t - \tau(t) \geq T_0$, implies $|\varphi(s)| < \varepsilon$ if $s \geq t - \tau(t)$. Therefore, the third term I_3 in (30) satisfies

$$\begin{aligned} |I_3| &= \left| \int_{t-\tau(t)}^t [h(s) + B(t, s)] \varphi(s) \, ds \right| \\ &\leq \int_{t-\tau(t)}^t |h(s) + B(t, s)| |\varphi(s)| \, ds \\ &\leq \varepsilon \int_{t-\tau(t)}^t |h(s) + B(t, s)| \, ds \leq \alpha\varepsilon < \varepsilon. \end{aligned}$$

Thus, $I_3 \rightarrow 0$ as $t \rightarrow \infty$. Now consider I_4 . For the given $\varepsilon > 0$, there exists a $T_1 > 0$ such that $s \geq T_1$ implies $|\varphi(s - \tau(s))| < \varepsilon$. Thus, for $t \geq T_1$, the term I_4 in (30) satisfies

$$\begin{aligned} |I_4| &= \left| \int_0^t e^{-\int_s^t h(u) \, du} \left\{ [h(s - \tau(s)) \right. \right. \\ &\quad \left. \left. + B(s, s - \tau(s))] (1 - \tau'(s)) \right\} \varphi(s - \tau(s)) \, ds \right| \\ &\leq \int_0^{T_1} e^{-\int_s^t h(u) \, du} \left| [h(s - \tau(s)) \right. \\ &\quad \left. + B(s, s - \tau(s))] (1 - \tau'(s)) \right| |\varphi(s - \tau(s))| \, ds \\ &\quad + \int_{T_1}^t e^{-\int_s^t h(u) \, du} \left| [h(s - \tau(s)) \right. \\ &\quad \left. + B(s, s - \tau(s))] (1 - \tau'(s)) \right| |\varphi(s - \tau(s))| \, ds \\ &\leq \sup_{\sigma \geq m(0)} |\varphi(\sigma)| \int_0^{T_1} e^{-\int_s^t h(u) \, du} \left| [h(s - \tau(s)) \right. \\ &\quad \left. + B(s, s - \tau(s))] (1 - \tau'(s)) \right| \, ds \\ &\quad + \varepsilon \int_{T_1}^t e^{-\int_s^t h(u) \, du} \left| [h(s - \tau(s)) + B(s, s - \tau(s))] (1 - \tau'(s)) \right| \, ds. \end{aligned}$$

By (28), we can find $T_2 > T_1$ such that $t \geq T_2$ implies

$$\begin{aligned} & \sup_{\sigma \geq m(0)} |\varphi(\sigma)| \int_0^{T_1} e^{-\int_s^t h(u) \, d u} \left| \left[h(s - \tau(s)) \right. \right. \\ & \quad \left. \left. + B(s, s - \tau(s)) \right] (1 - \tau'(s)) \right| \, d s \\ &= \sup_{\sigma \geq m(0)} |\varphi(\sigma)| e^{-\int_{T_2}^t h(u) \, d u} \times \int_0^{T_1} e^{-\int_s^{T_2} h(u) \, d u} \left| \left[h(s - \tau(s)) \right. \right. \\ & \quad \left. \left. + B(s, s - \tau(s)) \right] (1 - \tau'(s)) \right| \, d s \\ &< \varepsilon. \end{aligned}$$

Now, apply (27) to have $|I_4| < \varepsilon + \alpha\varepsilon < 2\varepsilon$. Thus, $I_4 \rightarrow 0$ as $t \rightarrow \infty$. Similarly, by using (2), (3) and (27), then, if $t \geq T_2$ then the terms I_5 and I_6 in (30) satisfy

$$\begin{aligned} |I_5| &= \left| \int_0^t e^{-\int_s^t h(u) \, d u} h(s) Q(s, \varphi(s - \tau(s))) \, d s \right| \\ &\leq \sup_{\sigma \geq m(0)} |\varphi(\sigma)| e^{-\int_{T_2}^t h(u) \, d u} \int_0^{T_1} e^{-\int_s^{T_2} h(u) \, d u} L |h(s)| \, d s \\ &\quad + \varepsilon \int_{T_1}^t e^{-\int_s^t h(u) \, d u} L |h(s)| \, d s \\ &< \varepsilon + \alpha\varepsilon < 2\varepsilon, \end{aligned}$$

and

$$\begin{aligned} |I_6| &= \left| \int_0^t e^{-\int_s^t h(u) \, d u} h(s) \left(\int_{s-\tau(s)}^s [h(u) + B(s, u)] \varphi(u) \, d u \right) \, d s \right| \\ &\leq \sup_{\sigma \geq m(0)} |\varphi(\sigma)| e^{-\int_{T_2}^t h(u) \, d u} \int_0^{T_1} e^{-\int_s^{T_2} h(u) \, d u} |h(s)| \\ & \quad \times \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| \, d u \right) \, d s \\ &\quad + \varepsilon \int_{T_1}^t e^{-\int_s^t h(u) \, d u} |h(s)| \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| \, d u \right) \, d s \\ &< \varepsilon + \alpha\varepsilon < 2\varepsilon. \end{aligned}$$

Thus, $I_5, I_6 \rightarrow 0$ as $t \rightarrow \infty$.

In conclusion $(P\varphi)(t) \rightarrow 0$ as $t \rightarrow \infty$, as required. Hence P maps S_ψ into S_ψ . Also, by condition (27), P is a contraction mapping with contraction constant α . Indeed, for $\phi, \eta \in S_\psi$ and $t > 0$

$$\begin{aligned}
& |(P\varphi)(t) - (P\eta)(t)| \\
& \leq L \|\varphi - \eta\| + \int_{t-\tau(t)}^t |h(s) + B(t, s)| |\varphi(s) - \eta(s)| \, ds \\
& \quad + \int_0^t e^{-\int_s^t h(u) \, du} \left| [h(s - \tau(s)) + B(s, s - \tau(s))] (1 - \tau'(s)) \right| \\
& \quad \quad \quad \times |\varphi(s - \tau(s)) - \eta(s - \tau(s))| \, ds \\
& \quad + \int_0^t e^{-\int_s^t h(u) \, du} L |h(s)| \|\varphi - \eta\| \, ds \\
& \quad + \int_0^t e^{-\int_s^t h(u) \, du} h(s) \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| |\varphi(u) - \eta(u)| \, du \right) \, ds \\
& \leq \left(L + \int_{t-\tau(t)}^t |h(s) + B(t, s)| \, ds + \int_0^t e^{-\int_s^t h(u) \, du} \left\{ \left[h(s - \tau(s)) \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. + B(s, s - \tau(s)) \right] (1 - \tau'(s)) \right\} + L |h(s)| \right) \, ds \\
& \quad + \int_0^t e^{-\int_s^t h(u) \, du} h(s) \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| \, du \right) \, ds \|\varphi - \eta\|.
\end{aligned}$$

By the condition (27), P is a contraction mapping with constant α . By the contraction mapping principle (Smart [22, p. 2]), P has a unique fixed point x in S_ψ which is a solution of (1) with $x(t) = \psi(t)$ on $[m(0), 0]$ and $x(t) = x(t, 0, \psi) \rightarrow 0$ as $t \rightarrow \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1) is stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying $2\delta K + \alpha\varepsilon < \varepsilon$. If $x(t) = x(t, 0, \psi)$ is a solution of (1) with $\|\psi\| < \delta$, then $x(t) = (Px)(t)$ defined in (30). We claim that $|x(t)| < \varepsilon$ for all $t \geq t_0$. Notice that $|x(s)| < \varepsilon$ on $[m(0), 0]$. If there exists $t^* > 0$ such that $|x(t^*)| = \varepsilon$ and $|x(s)| < \varepsilon$ for $m(0) \leq s < t^*$, then it follows from (30) that

$$\begin{aligned}
|x(t^*)| & \leq \|\psi\| \left(1 + L + \int_{-\tau(0)}^0 |h(s) + B(0, s)| \, ds \right) e^{-\int_0^{t^*} h(u) \, du} \\
& \quad + \varepsilon L + \varepsilon \int_{t^*-\tau(t^*)}^{t^*} |h(s) + B(t^*, s)| \, ds
\end{aligned}$$

$$\begin{aligned}
 & + \varepsilon \int_{t_0}^{t^*} e^{-\int_s^{t^*} h(u) \, d u} \left\{ \left| \left[h(s - \tau(s)) \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \left. + B(s, s - \tau(s)) \right] (1 - \tau'(s)) \right| + L |h(s)| \right\} \, d s \\
 & + \varepsilon \int_{t_0}^{t^*} e^{-\int_s^{t^*} h(u) \, d u} |h(s)| \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| \, d u \right) \, d s \\
 & \leq 2\delta K + \alpha\varepsilon < \varepsilon,
 \end{aligned}$$

which contradicts the definition of t^* . Thus, $|x(t)| < \varepsilon$ for all $t \geq 0$, and the zero solution of (1) is stable. This shows that the zero solution of (1) is asymptotically stable if (28) holds.

Conversely, suppose (28) fails. Then, by (26) there exists a sequence $\{t_n\}$, $t_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} \int_0^{t_n} h(u) \, d u = l$ for some $l \in \mathbb{R}$. We may also choose a positive constant J satisfying

$$-J \leq \int_0^{t_n} h(u) \, d u \leq J,$$

for all $n \geq 1$. To simplify our expressions, we define

$$\begin{aligned}
 \omega(s) & = \left| \left[h(s - \tau(s)) + B(s, s - \tau(s)) \right] (1 - \tau'(s)) \right| \\
 & \quad + |h(s)| \left(L + \int_{s-\tau(s)}^s |h(u) + B(s, u)| \, d u \right),
 \end{aligned}$$

for all $s \geq 0$. By (27), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} h(u) \, d u} \omega(s) \, d s \leq \alpha.$$

This yields

$$\int_0^{t_n} e^{\int_0^s h(u) \, d u} \omega(s) \, d s \leq \alpha e^{\int_0^{t_n} h(u) \, d u} \leq J.$$

The sequence $\left\{ \int_0^{t_n} e^{\int_0^s h(u) \, d u} \omega(s) \, d s \right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n \rightarrow \infty} \int_0^{t_n} e^{\int_0^s h(u) \, d u} \omega(s) \, d s = \gamma,$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s h(u) \, d u} \omega(s) \, d s < \delta_0/4K,$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies $2\delta_0 K e^J + \alpha \leq 1$.

By (26), K in (29) is well defined. We now consider the solution $x(t) = x(t, t_m, \psi)$ of (1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \leq \delta_0$ for $s \leq t_m$. We may choose ψ so that $|x(t)| \leq 1$ for $t \geq t_m$ and

$$\psi(t_m) - Q(t_m, \psi(t_m - \tau(t_m))) - \int_{t_m - \tau(t_m)}^{t_m} [h(s) + B(t_m, s)] \psi(s) \, ds \geq \frac{1}{2} \delta_0.$$

It follows from (30) with $x(t) = (Px)(t)$ that for $n \geq m$

$$\begin{aligned} & \left| x(t_n) - Q(t_n, x(t_n - \tau(t_n))) - \int_{t_n - \tau(t_n)}^{t_n} [h(s) + B(t_n, s)] x(s) \, ds \right| \\ & \geq \frac{1}{2} \delta_0 e^{-\int_{t_m}^{t_n} h(u) \, du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} h(u) \, du} \omega(s) \, ds \\ (31) \quad & = \frac{1}{2} \delta_0 e^{-\int_{t_m}^{t_n} h(u) \, du} - e^{-\int_0^{t_n} h(u) \, du} \int_{t_m}^{t_n} e^{\int_0^s h(u) \, du} \omega(s) \, ds \\ & = e^{-\int_{t_m}^{t_n} h(u) \, du} \left(\frac{1}{2} \delta_0 - e^{-\int_0^{t_m} h(u) \, du} \int_{t_m}^{t_n} e^{\int_0^s h(u) \, du} \omega(s) \, ds \right) \\ & \geq e^{-\int_{t_m}^{t_n} h(u) \, du} \left(\frac{1}{2} \delta_0 - K \int_{t_m}^{t_n} e^{\int_0^s h(u) \, du} \omega(s) \, ds \right) \\ & \geq \frac{1}{4} \delta_0 e^{-\int_{t_m}^{t_n} h(u) \, du} \geq \frac{1}{4} \delta_0 e^{-2J} > 0. \end{aligned}$$

On the other hand, if the zero solution of (1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \rightarrow 0$ as $t \rightarrow \infty$. Since $t_n - \tau(t_n) \rightarrow \infty$ as $n \rightarrow \infty$ and (27) holds, we have

$$\begin{aligned} & x(t_n) - Q(t_n, x(t_n - \tau(t_n))) \\ & \quad - \int_{t_n - \tau(t_n)}^{t_n} [h(s) + B(t_n, s)] x(s) \, ds \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which contradicts (31). Hence condition (28) is necessary for the asymptotic stability of the zero solution of (1). The proof is complete. \square

Remark 2.1. It follows from the first part of the proof of Theorem 2.1 that the zero solution of (1) is stable under (26) and (27). Moreover, Theorem 2.1 still holds if (27) is satisfied for $t \geq t_\sigma$ for some $t_\sigma \in \mathbb{R}^+$.

For the special case $Q(t, x) = 0$, we can get

Corollary 2.1. *Suppose that τ is differentiable and there exist continuous function $h : [m(0), \infty) \rightarrow \mathbb{R}$ and a constant $\alpha \in (0, 1)$ such that for $t \geq 0$*

$$(32) \quad \liminf_{t \rightarrow \infty} \int_0^t h(s) \, ds > -\infty,$$

and

$$\begin{aligned}
 & \int_{t-\tau(t)}^t |h(s) + B(t, s)| \, ds \\
 & + \int_0^t e^{-\int_s^t h(u) \, du} |h(s - \tau(s)) + B(s, s - \tau(s))| |1 - \tau'(s)| \, ds \\
 (33) \quad & + \int_0^t e^{-\int_s^t h(u) \, du} |h(s)| \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| \, du \right) \, ds \leq \alpha.
 \end{aligned}$$

Then the zero solution of (4) is asymptotically stable if and only if

$$(34) \quad \int_0^t h(s) \, ds \rightarrow \infty \text{ as } t \rightarrow \infty.$$

Remark 2.2. Obviously, Corollary 2.1 extends Theorem 1.3. Thus, Theorem 2.1 improves and generalizes Theorem 1.3.

3. AN EXAMPLE

In this section, we give an example to illustrate the applications of Theorem 2.1.

Example 3.1. Consider the following linear neutral integro-differential equation with variable delays

$$(35) \quad \frac{d}{dt}x(t) = - \int_{t-\tau(t)}^t a(t, s) x(s) \, ds + \frac{d}{dt}Q(t, x(t - \tau(t))),$$

where $\tau(t) = 0.477t$, $a(t, s) = 0.45/(s^2 + 1)$, $Q(t, x) = 0.112 \sin x$. Then the zero solution of (35) is asymptotically stable.

Proof. We have

$$B(t, s) = \int_t^s \frac{0.45}{s^2 + 1} \, du = \frac{0.45(s - t)}{s^2 + 1}.$$

Choosing $h(t) = 0.55t/(t^2 + 1)$ in Theorem 2.1, we have $L = 0.112$ and

$$\begin{aligned}
 & \int_{t-\tau(t)}^t |h(s) + B(t, s)| \, ds = \int_{0.523t}^t \left| \frac{s - 0.45t}{s^2 + 1} \right| \, ds \\
 & = \int_{0.523t}^t \frac{s - 0.45t}{s^2 + 1} \, ds = 0.45t [\arctan 0.523t - \arctan t] \\
 & \quad + \frac{1}{2} [\ln(t^2 + 1) - \ln(0.523^2 t^2 + 1)] \\
 & = \omega(t).
 \end{aligned}$$

Since the function ω is increasing in $[0, \infty)$ and

$$\lim_{t \rightarrow \infty} \omega(t) = 0.45 - 0.45/0.523 - \ln(0.523) \simeq 0.238,$$

then

$$\int_{t-\tau(t)}^t |h(s) + B(t, s)| \, ds < 0.238,$$

$$\int_0^t e^{-\int_s^t h(u) \, du} |h(s)| \left(\int_{s-\tau(s)}^s |h(u) + B(s, u)| \, du \right) \, ds < 0.238,$$

and

$$\begin{aligned} & \int_0^t e^{-\int_s^t h(u) \, du} \left\{ \left| \left[h(s - \tau(s)) \right. \right. \right. \\ & \qquad \qquad \qquad \left. \left. \left. + B(s, s - \tau(s)) \right] (1 - \tau'(s)) \right| + L |h(s)| \right\} \, ds \\ &= \int_0^t e^{-\int_s^t \frac{0.55u}{u^2+1} \, du} \left\{ \left| 0.523 \left(\frac{0.55 \times 0.523s}{0.523^2 s^2 + 1} - \frac{0.45 \times 0.477s}{0.523^2 s^2 + 1} \right) \right| \right. \\ & \qquad \qquad \qquad \left. + \frac{0.112 \times 0.55s}{s^2 + 1} \right\} \, ds \\ &\leq \frac{0.073}{0.523 \times 0.55} \int_0^t e^{-\int_s^t \frac{0.55u}{u^2+1} \, du} \frac{0.55s}{s^2 + 1/0.523^2} \, ds \\ & \qquad \qquad \qquad + 0.112 \int_0^t e^{-\int_s^t \frac{0.55u}{u^2+1} \, du} \frac{0.55s}{s^2 + 1} \, ds \\ &< \frac{0.073}{0.523 \times 0.55} + 0.112 < 0.366. \end{aligned}$$

It is easy to see that all the conditions of Theorem 2.1 hold for $\alpha = 0.112 + 0.238 + 0.366 + 0.238 = 0.954 < 1$. Thus, Theorem 2.1 implies that the zero solution of (35) is asymptotically stable. \square

REFERENCES

- [1] A. Ardjouni and A. Djoudi, *Stability in nonlinear neutral differential equations with infinite delay*, *Mathematica Moravica*, Vol. 18-2 (2014), 91–103.
- [2] A. Ardjouni and A. Djoudi, *Fixed points and stability in linear neutral differential equations with variable delays*, *Nonlinear Analysis* 74 (2011) 2062-2070.
- [3] A. Ardjouni and A. Djoudi, *Stability in nonlinear neutral differential with variable delays using fixed point theory*, *Electronic Journal of Qualitative Theory of Differential Equations* 2011, No. 43, 1-11.
- [4] A. Ardjouni and A. Djoudi, *Fixed point and stability in neutral nonlinear differential equations with variable delays*, *Opuscula Mathematica*, Vol. 32, No. 1, 2012, pp. 5-19.

-
- [5] A. Ardjouni, A. Djoudi and I. Soualhia, *Stability for linear neutral integro-differential equations with variable delays*, Electronic journal of Differential Equations, 2012 (2012), No. 172, 1-14.
- [6] L. C. Becker and T. A. Burton, *Stability, fixed points and inverse of delays*, Proc. Roy. Soc. Edinburgh 136A (2006) 245-275.
- [7] T. A. Burton, *Fixed points and stability of a nonconvolution equation*, Proceedings of the American Mathematical Society 132 (2004) 3679–3687.
- [8] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publications, New York, 2006.
- [9] T. A. Burton, *Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem*, Nonlinear Studies 9 (2001) 181–190.
- [10] T. A. Burton, *Stability by fixed point theory or Liapunov's theory: A comparison*, Fixed Point Theory 4 (2003) 15–32.
- [11] T. A. Burton and T. Furumochi, *A note on stability by Schauder's theorem*, Funkcialaj Ekvacioj 44 (2001) 73–82.
- [12] T. A. Burton and T. Furumochi, *Fixed points and problems in stability theory*, Dynamical Systems and Applications 10 (2001) 89–116.
- [13] T. A. Burton and T. Furumochi, *Asymptotic behavior of solutions of functional differential equations by fixed point theorems*, Dynamic Systems and Applications 11 (2002) 499–519.
- [14] T. A. Burton and T. Furumochi, *Krasnoselskii's fixed point theorem and stability*, Nonlinear Analysis 49 (2002) 445–454.
- [15] Y. M. Dib, M. R. Maroun and Y. N. Raffoul, *Periodicity and stability in neutral nonlinear differential equations with functional delay*, Electronic Journal of Differential Equations, Vol. 2005 (2005), No. 142, pp. 1-11.
- [16] A. Djoudi and R. Khemis, *Fixed point techniques and stability for neutral nonlinear differential equations with unbounded delays*, Georgian Mathematical Journal, Vol. 13 (2006), No. 1, 25-34.
- [17] C. H. Jin and J. W. Luo, *Stability of an integro-differential equation*, Computers and Mathematics with Applications 57 (2009) 1080-1088.
- [18] C. H. Jin and J. W. Luo, *Stability in functional differential equations established using fixed point theory*, Nonlinear Anal. 68 (2008) 3307-3315.
- [19] C. H. Jin and J. W. Luo, *Fixed points and stability in neutral differential equations with variable delays*, Proceedings of the American Mathematical Society, Vol. 136, Nu. 3 (2008) 909-918.
- [20] M. B. Mesmouli, A. Ardjouni and A. Djoudi, *Periodic solutions for a system of nonlinear neutral functional difference equations with two functional delays*, Mathematica Moravica, Vol. 19-1 (2015), 57–71.
- [21] Y. N. Raffoul, *Stability in neutral nonlinear differential equations with functional delays using fixed-point theory*, Math. Comput. Modelling 40 (2004) 691–700.

- [22] D. R. Smart, *Fixed point theorems*, Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, 1974.
- [23] B. Zhang, *Fixed points and stability in differential equations with variable delays*, *Nonlinear Anal.* 63 (2005) e233–e242.

ABDELOUAHEB ARDJOUNI

FACULTY OF SCIENCES AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS AND INFORMATICS
UNIV SOUK AHRAS
P.O. BOX 1553, SOUK AHRAS, 41000
ALGERIA

Secondary address:

APPLIED MATHEMATICS LAB
FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
UNIV ANNABA,
P.O. BOX 12, ANNABA 23000
ALGERIA

E-mail address: abd_ardjouni@yahoo.fr

AHCENE DJOUDI

APPLIED MATHEMATICS LAB
FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
UNIV ANNABA
P.O. BOX 12, ANNABA 23000
ALGERIA

E-mail address: adjoudi@yahoo.com