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On the Theorem of Wan for K-Quasiconformal Hyperbolic Harmonic Self Mappings of the Unit Disk

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Abstract. We give a new glance to the theorem of Wan (Theorem 1.1) which is related to the hyperbolic bi-Lipschicity of the K-quasiconformal, $K \geqslant 1$, hyperbolic harmonic mappings of the unit disk $\mathbb{D}$ onto itself. Especially, if f is such a mapping and $f(0) = 0$, we obtained that the following double inequality is valid $2|z|/(K+1) \leqslant |f(z)| \leqslant \sqrt{K}|z|$, whenever $z \in \mathbb{D}$.

1. Introduction

Suppose that ρ is a positive function defined and of the class C^2 in some subdomain (open and connected) Ω of the complex plane $\mathbb{C}$ and let $z_0 \in \Omega$. Recall that the Gaussian curvature of the conformal metric $ds^2 = \rho(z)|dz|^2$ at the point z_0 is defined as

$$K_\rho(z_0) = -\frac{1}{2}\frac{(\triangle \log \rho)(z_0)}{\rho(z_0)}, \tag{1}$$

where $\triangle$ is the Laplace second order differential operator (the Laplacian). Also, the ρ-length of a rectifiable curve $\gamma : [0,1] \to \Omega$ is given by $|\gamma|_\rho = \int_\gamma \sqrt{\rho(z)}|dz|$. Otherwise, the ρ-distance between the points z_1 and z_2 in Ω is defined as $d_\rho(z_1, z_2) = \inf |\gamma|_\rho$, where the infimum is taken over all rectifiable curves γ in Ω that join the points z_1 and z_2.

Example 1.1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$. Consider a conformal metric $ds^2 = \lambda(z)|dz|^2$ on $\mathbb{D}$, where the corresponding density

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function λ is defined as

$$\lambda(z) = \left(\frac{2}{1-|z|^2}\right)^2, \; z \in \mathbb{D}. \tag{2}$$

Since, for arbitrary $z \in \mathbb{D}$, we have

$$\begin{aligned}(\triangle \log \lambda)(z) &= 4(\log \lambda)_{z\bar{z}}(z) = -8(\log(1-|z|^2))_{z\bar{z}}(z)\\ &= 8\left(\frac{\bar{z}}{1-|z|^2}\right)_{\bar{z}}(z) = \frac{8}{(1-|z|^2)^2},\end{aligned}$$

i.e. $(\triangle \log \lambda)(z) = 2\lambda(z)$, then $K_\lambda(z) = -1$, for all $z \in \mathbb{D}$. So, the conformal metric $ds^2 = \lambda(z)|dz|^2$ has the constant and negative Gaussian curvature on $\mathbb{D}$. On the other hand, it is easy to verify that the corresponding distance function induced by this metric on $\mathbb{D}$ is given by the formula

$$d_\lambda(z_1, z_2) = \log \frac{1 + \left|\frac{z_1 - z_2}{1-\bar{z}_2 z_1}\right|}{1 - \left|\frac{z_1 - z_2}{1-\bar{z}_2 z_1}\right|}, \; z_1,\, z_2 \in \mathbb{D}. \tag{3}$$

Definition 1.1. The hyperbolic metric on the unit disk is a conformal metric $ds^2 = \lambda(z)|dz|^2$, where the density function λ is given by (2). The function d_λ is called the hyperbolic distance on the unit disk $\mathbb{D}$.

For further properties of the hyperbolic metric we refer to [2] and [9].

Let Ω and Ω' be some subdomains of the complex plane $\mathbb{C}$.

Definition 1.2. For a mapping $f : \Omega \to \Omega'$, which is of the class C^2 in Ω, we say that it is harmonic with respect to a conformal metric $ds^2 = \rho(w)|dw|^2$ defined on Ω' (ρ is a positive function and of the class C^2 in Ω') if

$$f_{z\bar{z}}(z) + \frac{\rho_w(f(z))}{\rho(f(z))} f_z(z) f_{\bar{z}}(z) = 0, \tag{4}$$

for all $z \in \Omega$, where f_z and $f_{\bar{z}}$ are the partial derivatives of f in Ω related to the variables z and $\bar{z}$, respectively. Here by $f_{z\bar{z}}$ we denoted the second order partial derivative of the mapping f in Ω ($f_{z\bar{z}} = (f_z)_{\bar{z}} = (f_{\bar{z}})_z$).

It is obvious that in the presence of the Euclidean metric on the image subdomain Ω', i.e. in the case when the density function $\rho \equiv 0$ on Ω', the relation (4) defines a harmonic function, or Euclidean harmonic mapping, since $(\triangle f)(z) = 4 f_{z\bar{z}}(z) = 0$, for all $z \in \Omega$.

Definition 1.3. A sense preserving C^1 diffeomorphism $f : \Omega \to \Omega' = f(\Omega)$ is said to be regular K-quasiconformal (or just K-quasiconformal) if there is a constant $K \geqslant 1$ such that $|f_z(z)|^2 + |f_{\bar{z}}(z)|^2 \leqslant \frac{1}{2}\left(K + \frac{1}{K}\right) J_f(z)$, for all $z \in \Omega$, where $J_f : z \mapsto J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2$, $z \in \Omega$, is the Jacobian of the mapping f.

Note that if $K = 1$ the mapping f is a conformal mapping, since in that case $f_{\bar{z}} \equiv 0$ on Ω.

By using a new approach and technique, in the article [5] we gave a new proof of Wan's result (see [12]) related to the bi-Lipschicity of the quasiconformal hyperbolic harmonic diffeomorphisms of the unit disk. More specifically, we constructed some conformal metrics on the unit disk $\mathbb{D}$ and, by understanding their properties and by calculating their Gaussian curvatures, we applied some versions of the results that are of the Ahlfors-Schwarz-Pick type to show that every K-quasiconformal mapping f of the unit disk $\mathbb{D}$ onto itself, which is also harmonic with respect to the hyperbolic metric $ds^2 = \lambda(w)|dw|^2$ on $\mathbb{D}$, is a quasi-isometry with respect to the hyperbolic metric. Moreover, such a mapping f is $(2/(K+1), \sqrt{K})$ bi-Lipschitz with respect to the hyperbolic metric, too.

Theorem 1.1 (Wan [12], KM [5]). *Let $f \in C^2(\mathbb{D})$ be a K-quasiconformal mapping of the unit disk $\mathbb{D}$ onto itself which is harmonic with respect to the hyperbolic metric $ds^2 = \lambda(w)|dw|^2$ on $\mathbb{D}$. Then f is a $(2/(K+1), \sqrt{K})$ bi-Lipschitz with respect to the hyperbolic metric. Since $\sqrt{K} \leqslant (K+1)/2$, then f is also a quasi-isometry with respect to the hyperbolic metric.*

For details about quasiconformal harmonic mappings we refer a interested reader to [7], [8], [9] and [10].

2. The main result

Suppose now that a given mapping f satisfies the conditions of the previous theorem. In addition, suppose that $f(0) = 0$. According to the Theorem 1.1, we have

$$d_\lambda(f(z), 0) \leqslant \sqrt{K} d_\lambda(z, 0) \quad \text{and} \quad d_\lambda(f(z), 0) \geqslant \frac{2}{K+1} d_\lambda(z, 0),$$

for all $z \in \mathbb{D}$, and since $d_\lambda(r, 0) = \ln \dfrac{1+r}{1-r}$, for all $0 \leqslant r < 1$, we get

$$(5) \qquad |f(z)| \leqslant \frac{(1+|z|)^{\sqrt{K}} - (1-|z|)^{\sqrt{K}}}{(1+|z|)^{\sqrt{K}} + (1-|z|)^{\sqrt{K}}}$$

and

$$(6) \qquad |f(z)| \geqslant \frac{(1+|z|)^{\frac{2}{K+1}} - (1-|z|)^{\frac{2}{K+1}}}{(1+|z|)^{\frac{2}{K+1}} + (1-|z|)^{\frac{2}{K+1}}},$$

for all $z \in \mathbb{D}$.

To obtain the main result of this paper, we have to prove the following lemma.

Lemma 2.1. *Let $\alpha > 0$, $\alpha \neq 1$, be a real number. Then the function*

$$a : x \mapsto a(x) = \frac{(1+x)^\alpha - (1-x)^\alpha}{(1+x)^\alpha + (1-x)^\alpha},\ 0 < x < 1,$$

is strictly increasing on the interval $(0,1)$. In addition, if $\alpha > 1$, then $a(x) < \alpha x$, for all $0 < x < 1$, whereas, if $0 < \alpha < 1$, then $a(x) > \alpha x$, for all $0 < x < 1$.

Proof. For the defined function a we easily get

$$a'(x) = \frac{4\alpha(1-x^2)^{\alpha-1}}{[(1-x)^\alpha + (1+x)^\alpha]^2} > 0,$$

for all $0 < x < 1$. On the other hand, for its second derivative we have,

$$a''(x) = \frac{8\alpha(1-x^2)^{\alpha-2}[(1+x)^\alpha(x-\alpha) + (1-x)^\alpha(\alpha+x)]}{[(1-x)^\alpha + (1+x)^\alpha]^3},\ 0 < x < 1.$$

Therefore, since for $\alpha > 1$, $\dfrac{\alpha - x}{\alpha + x} > \dfrac{1-x}{1+x} > \left(\dfrac{1-x}{1+x}\right)^\alpha$, for all $0 < x < 1$, we obtain that in this case the function a is concave on $(0,1)$. Otherwise, if $0 < \alpha < 1$, we have $\dfrac{\alpha - x}{\alpha + x} < \dfrac{1-x}{1+x} < \left(\dfrac{1-x}{1+x}\right)^\alpha$, whenever $0 < x < 1$, and the function a is then convex on $(0,1)$. Now, the statement easily follows from the fact that $a'_+(0) = \alpha$, where $a'_+(0)$ is the right derivative of the function a at the point $x = 0$. □

We are ready now to prove the main result.

Theorem 2.1. *Let $f \in \mathrm{C}^2(\mathbb{D})$ be a K-quasiconformal mapping of the unit disk $\mathbb{D}$ onto itself which is harmonic with respect to the hyperbolic metric $ds^2 = \lambda(w)|dw|^2$ on $\mathbb{D}$. Suppose, in addition, that $f(0) = 0$. Then, for all $z \in \mathbb{D}$ we have*

$$\frac{2}{K+1}|z| \leqslant |f(z)| \leqslant \sqrt{K}|z|, \tag{7}$$

for all $z \in \mathbb{D}$.

Proof. The proof is a trivial consequence of the inequalities (5) and (6), and of the Lemma 2.1. □

Remark 2.1. In [5] we obtained some version of the Theorem 1.1 that are related to the K-quasiconformal harmonic mappings f, which are harmonic with respect to some conformal metric defined on the image subdomain, and with the property that its Gaussian curvature is not greater then some negative constant $-a$, $a > 0$. Therefore, we could easily generalize the Theorem 2.1 in this case.

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