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On Fuzzy Differential Subordination

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Abstract. The theory of differential subordination was introduced by S.S.Miller and P.T.Mocanu in [2], then developed in many papers. In [1] the authors investigate various subordination results for some subclasses of analytic functions in the unit disc. G.I.Oros and G.Oros define the notion of fuzzy subordination and in [3, 4, 5] they define the notion of fuzzy differential subordination. In this paper, we determine sufficient conditions for a multivalent function to be a dominant of the fuzzy differential subordination.

1. Introduction

We introduce some basic notions and results that are used in the sequel.

Definition 1.1 ([6]). Let X be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset. An alternate definition, more precise, would be the following: A pair (A, F_A), where $F_A : X \to [0,1]$ and

$$A = \{x \in X : 0 < F_A(x) \leq 1\} = supp(A, F_A),$$

is called fuzzy subset.

Proposition 1.1 ([3]). *If $(M, F_M) = (N, F_N)$, then we have $M = N$, where $M = supp(M, F_M), N = supp(N, F_N)$.*

Proposition 1.2 ([3]). *If $(M, F_M) \subseteq (N, F_N)$, then we have $M \subseteq N$, where $M = supp(M, F_M), N = supp(N, F_N)$.*

We also need the following notations and results from the classical complex analysis [5].

For $D \subset \mathbb{C}$, we denote by $\mathcal{H}(D)$ the class of holomorphic functions on D, and by $\mathcal{H}_n(D)$ the class of holomorphic and univalent functions on D.
In this paper, we denote by $\mathcal{H}(U)$ the set of holomorphic functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with $\partial U = \{z \in \mathbb{C} : |z| = 1\}$ the boundary of the unit disc.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we denote

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$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., z \in U\},$$
$$A_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + ..., z \in U\} \text{ with } A_1 = A,$$
$$\text{and } S = \{f \in A : f \text{ a univalent function in } U\}.$$

Let $\mathcal{B} = \{\varphi \in \mathcal{H}(U) : \varphi(0) = 0, |\varphi(z)| < 1, z \in U\}$ denote the class of Schwarz functions.

Definition 1.2 ([4]). Let $f, g \in \mathcal{H}(U)$. We say that the function f is subordinated to g, written $f \prec g$ or $f(z) \prec g(z)$ if there exists a function $w \in \mathcal{H}(U)$ with $w(0) = 0$ and $|w(z)| < 1, z \in U$, (which means $w \in \mathcal{B}$) such that $f(z) = g(w(z)), z \in U$.

Let $D \subset \mathbb{C}$ and $f, g \in \mathcal{H}(D)$ holomorphic functions. We denote by

$$f(D) = \{f(z) | 0 < F_{f(D)} f(z) \leq 1, z \in D\} = supp(f(D), F_{f(D)})$$

and

$$g(D) = \{g(z) | 0 < F_{g(D)} g(z) \leq 1, z \in D\} = supp(g(D), F_{g(D)}).$$

Definition 1.3 ([5]). Let $D \subseteq \mathbb{C}$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write
$f \prec_{\boldsymbol{F}} g$ or $f(z) \prec_{\boldsymbol{F}} g(z)$, if

1. $f(z_0) = g(z_0)$,
2. $F_{f(D)} f(z) \leq F_{g(D)} g(z), z \in D$.

Proposition 1.3 ([5]). *Let Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. If $f(z) \prec_{\mathbf{F}} g(z), z \in D$, then*

1. $f(z_0) = g(z_0)$,
2. $f(D) \subseteq g(D)$, *where* $f(D) = supp(f(D), F_{f(D)})$, $g(D) = supp(g(D), F_{g(D)})$.

The equality occurs if and only if $F_{f(D)} f(z) = F_{g(D)} g(z)$. *Denoted by*

$$S^* = \{f \in A : Re\frac{zf'(z)}{f(z)} > 0, z \in U\}$$

the class of normalized starlike functions in U,

$$K = \{f \in A : Re\frac{zf''(z)}{f'(z)} + 1 > 0, z \in U\}$$

the class of normalized convex functions in U and by

$$C = \{f \in A : \exists \varphi \in K, Re\frac{f'(z)}{\varphi'(z)} > 0, z \in U\}$$

the class of normalized close-to-convex functions in U [5].

Let $J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)}), z \in U$, *for* α *real number and* $f \in A_p$ [2].

Let $\Omega = supp(\Omega, F_\Omega) = \{z \in \mathbb{C} : 0 < F_\Omega(z) \leq 1\}$,
$\Delta = supp(\Delta, F_\Delta) = \{z \in \mathbb{C} : 0 < F_\Delta(z) \leq 1\}, p(U) = supp(p(U), F_{P(U)})$
$= \{f(z) : 0 < F_{P(U)}(f(z)) \leq 1\}, z \in U\}$ *and*
$\psi(\mathbb{C}^3 \times U) = supp(\psi(\mathbb{C}^3 \times U), F_{\psi(\mathbb{C}^3 \times U)})$
$= \{\psi(p(z), zp'^2 p''(z); z) : 0 < F_{\psi(\mathbb{C}^3 \times U)}(\psi(p(z), zp'^2 p''(z), z)) \leq 1, z \in U\}$
[4].

Definition 1.4 ([4]). Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) fuzzy differential subordination

$$F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp'(z), z^2p''(z); z) \leq F_{h(U)}h(z) \ (1)$$

i.e. $\psi(p(z), zp'(z), z^2p''(z); z) \prec_{\boldsymbol{F}} h(z),\ z \in U$, then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $p(z) \prec_{\boldsymbol{F}} g(z), z \in U$, for all p satisfying (1). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_{\boldsymbol{F}} q(z), z \in U$, for all fuzzy dominant q of (1) is said to be the fuzzy best dominant of (1).

Theorem 1.1 ([5]). *Let h be analytic in U, let ϕ be analytic in domain D containing $h(U)$ and suppose*
a) $Re\phi[h(z)] > 0, z \in U$ and
b) $h(z)$ is convex.

If p is analytic in U , with $p(0) = h(0), p(U) \subset D$ and $\psi(\mathbb{C}^2 \times U) \to \mathbb{C}, \psi(p(z), zp'(z)) = p(z) + zp'(z).\phi[p(z)]$ is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z)) \leq F_{h(U)}h(z),$$

implies

$$F_{p(U)}p(z) \leq F_{h(U)}h(z), z \in U,$$

where

$$\psi(\mathbb{C}^2 \times U) = supp(\mathbb{C}^2 \times U, F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z))$$

$$= \{z \in \mathbb{C} : 0 < F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z)) \leq 1\},$$

$$h(U) = supp(U, F_{h(U)}h(z)) = \{z \in \mathbb{C} : 0 < F_{h(U)}h(z) \leq 1\}.$$

Theorem 1.2 ([5]). *Let h be convex in U and let $P : U \to \mathbb{C}$, with $ReP(z) > 0$. If p is analytic in U and $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,*

$$\psi(p(z), zp'(z)) = p(z) + P(z)zp'(z)$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}[p(z) + P(z)zp'(z)] \leq F_{h(U)}h(z),$$

implies

$$F_{p(U)}P(z) \leq F_{h(U)}h(z), z \in U.$$

Theorem 1.3 ([5]). *(Hallenbeck and Ruscheweyh) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ with $p(0) = a$ and $\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi(p(z) + zp'(z)) = p(z) + \frac{1}{\gamma}zp'(z)$ is analytic in U, then*

$$F_{\psi(\mathbb{C}^2\times U)}[p(z) + \tfrac{1}{\gamma}zp'(z)] \leq F_{h(U)}h(z),$$

implies

$$F_{p(U)}p(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), z \in U,$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\frac{\gamma}{n}-1}\, \mathrm{d}t.$$

The function q is convex and is the fuzzy best (a, n)-dominant.

2. MAIN RESULTS

Proposition 2.1. *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z).\phi[q(z)]$ and $h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

(i) Q is starlike, or

(ii) h is convex.

In addition, assume that

(iii) $Re(\frac{zh'(z)}{Q(z}) = Re(\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}) > 0$.

If p is analytic in U, with $p(0) = q(0), p(U) \subset D$ and $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z)) = p(z) + zp'(z).\phi(p(z))$ is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}[p(z) + zp'(z).\phi(p(z)] \leq F_{h(U)}h(z),$$

implies

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), z \in U, \text{ i.e.}$$

$p(z) <_{\mathbf{F}} q(z)$, and q is the best dominant, where
$\psi(\mathbb{C}^2 \times U) = supp(\mathbb{C}^2 \times U, F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z)))$
$= \{z \in \mathbb{C} : 0 < F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z)) \leq 1\}$, and
$h(U) = supp(U, F_{h(U)}h(z)) = \{z \in \mathbb{C} : 0 < F_{h(U)}h(z) \leq 1\}$.

Proof. The proof of Proposition is similar to Theorem 1.1[5]. □

Proposition 2.2. *Let $q \in \mathcal{H}[p, p]$ be univalent, $q(z) \neq 0$ and satisfies the following conditions.*

(i) $\frac{zq'(z)}{q(z)}$ is starlike,

(ii) $Re(\frac{q(z)}{\alpha} + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}) > 0$ for all $\alpha \neq 0$ and for all $z \in U$.

For $p \in \mathcal{H}[p, p]$ with $p(z) \neq 0$ in U and

$$\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi p(z), zp'(z)) = p(z) + \alpha \frac{zp'(z)}{p(z)}$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}[p(z) + \alpha \frac{zp'(z)}{p(z)}] \leq F_{\psi(\mathbb{C}^2\times U)}[q(z) + \alpha \frac{zq'(z)}{q(z)}] = F_{h(U)}h(z),$$

implies

$$F_{p(U)}p(z) \leq F_{q(U)}q(z) \text{ i.e. } p(z) <_{\mathbf{F}} q(z), z \in U,$$

and q is the best dominant.

Proof. Define the function θ and ϕ by $\theta(w) = w, \phi(w) = \frac{\alpha}{w}, D = \{w : w \neq 0\}$ in Proposition 2.1. Then the functions

$Q(z) = zq'(z)\phi[q(z)] = \alpha \frac{zq'(z)}{q(z)}$,

$h(z) = \theta[q(z)] + Q(z) = q(z) + \lambda \frac{zq'(z)}{q(z)}$.

Since $\frac{zq'(z)}{q(z)}$ is starlike, we obtain that Q is starlike in U and $Re(\frac{zh'(z)}{Q(z)}) > 0$ for all $z \in U$. It follows Proposition 2.1 and

$$F_{\psi(\mathbb{C}^2 \times U)}[p(z) + \alpha \frac{zp'(z)}{p(z)}] \leq F_{h(U)}h(z),$$

$$F_{p(U)}p(z) \leq F_{q(U)}q(z) \text{ i.e. } p(z) <_{\boldsymbol{F}} q(z), z \in U,$$

and q is the best dominant. □

Proposition 2.3. *Let $q \in \mathcal{H}[p,p]$ be univalent, $q(z) \neq 0$ and satisties the conditions:*

(i) $\frac{zq'(z)}{q(z)}$ is starlike,

(ii) $Re(\frac{q(z)}{\alpha} + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}) > 0$

for $\alpha \neq 0$ and for all $z \in U$. For $f \in A_p$ with

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)}), z \in U$$

and $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$$\psi(q(z), zq'(z)) = q(z) + \alpha \frac{zq'(z)}{q(z)}, \text{ then}$$

$$F_{\psi(\mathbb{C}^2 \times U)}(\frac{zf'(z)}{f(z)}) \leq F_{q(U)}q(z)$$

and q is the best dominant.

Proof. Let us put $p(z) = \frac{zf'(z)}{f(z)}, z \in U$, where p(0)=0.

Then we obtain that

$$p(z) + \alpha \frac{zp'(z)}{p(z)} = J(\alpha, f; z).$$

Using Proposition 2.1, we have

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), z \in U,$$

and q is the best dominant. □

Proposition 2.4. *Let $q \in \mathcal{H}[1,1]$ be univalent and satisfies the following conditions:*

(i) $q(z)$ is convex,

(ii) $Re[(\frac{1}{\alpha} + \rho) + \frac{zq''(z)}{q'(z)}] > 0$ $\rho \in \mathbb{N} = \{1,2,3,..\})$

for $\alpha \neq 0$ and for all $z \in U$. For $p \in \mathcal{H}[1,1]$ in U and

$\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$\psi(p(z), zp'(z)) = (1 - \alpha + \alpha\rho)p(z) + \alpha z p'(z)$ *is analytic in U, then*

$$F_{\psi(\mathbb{C}^2 \times U)}[(1 - \alpha + \alpha\rho)(p(z) + \alpha z p'(z)] \leq$$
$$F_{\psi(\mathbb{C}^2 \times U)}[(1 - \alpha + \alpha\rho)q(z) + \alpha z q'(z)] = F_{h(U)}h(z),$$

implies $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, and q is the best dominant.

Proof. For $\alpha \neq 0$ real number, we define the functions θ and ϕ by $\theta(w) = (1-\alpha+\alpha\rho)w$, $\phi(w) = \alpha$, $D = \{w : w \neq 0\}$ in Proposition 2.1. Then we have

(i) $Q(z) = zq'(z)\phi[q(z)] = \alpha zq'(z)$,

(ii) $h(z) = \theta[q(z) + Q(z)] = (1-\alpha+\mu\rho)q(z) + \alpha zq'(z)$.

By the (i)and (ii), we obtained that Q is starlike in U and $Re(\frac{zh'(z)}{Q(z)}) > 0$ for all $z \in U$. Since it satisfies preconditions of Proposition 2.1, it follows Proposition 2.1,

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), z \in U,$$

and q is the best dominant. □

Theorem 2.1. *Let $q \in \mathcal{H}[1,1]$ be univalent and satisfies the following conditions:*

(i) $q(z)$ is convex,

(ii) $Re[(\frac{1}{\alpha} + \rho) + \frac{zq''(z)}{q'(z)}] > 0$ $(\rho \in \mathbb{N} = \{1, 2, 3, ..\})$

for $\alpha \neq 0$ and for all $z \in U$. For $f \in A_p$ with

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)}), z \in U$$

and if $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$$\psi(q(z), zq'(z)) = (1-\alpha+\alpha\rho)q(z) + \mu zq'(z), \text{ then}$$

$$F_{\psi(\mathbb{C}^2\times U)}(\frac{f(z)}{z^p}) \leq F_{q(U)}q(z), z \in U$$

and q is the best dominant.

Proof. Let us put $p(z) = \frac{f(z)}{z^p}$, where $p(0) = 1$. Then we have

$$(1-\alpha+\alpha\rho)p(z) + \alpha zp'(z) = J_p(\alpha, f; z).$$

From the Proposition 2.4, we have

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), z \in U$$

and q is the best dominant. □

Corollary 2.1. *Let $q \in \mathcal{H}[1,1]$ be univalent and satisfies the following conditions:*

(i) $q(z)$ is convex,

(ii) $Re[(\frac{1}{\alpha} + 1) + \frac{zq''(z)}{q'(z)}] > 0$ $(\rho \in \mathbb{N} = \{1, 2, 3, ..\})$

for $\alpha \neq 0$ and for all $z \in U$. For $p \in \mathcal{H}[1,1]$ in U,

if $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$

$\psi(p(z), zp'(z)) = p(z) + \alpha zp'(z)$,

then $F_{\psi(\mathbb{C}^2\times U)}p(z) \leq F_{q(U)}q(z), z \in U$, and q is the best dominant.

Corollary 2.2. *Let $q \in \mathcal{H}[1,1]$ be univalent, $q(z)$ is convex for all $z \in U$. For $p \in \mathcal{H}[1,1]$ in U if*

$$\psi : \mathbb{C}^2 \times U \to \mathbb{C},\ \psi(p(z), zp'(z)) = p(z) + zp'(z),\ \textit{then}$$

$$F_{\psi(\mathbb{C}^2 \times U)} p(z) \le F_{\psi(\mathbb{C}^2 \times U)} q(z), z \in U,$$

and q is the best dominant.

Corollary 2.3. *Let $q \in \mathcal{H}[1,1]$ be univalent, $q(z)$ is convex for all $z \in U$. For $p \in \mathcal{H}[1,1]$ in U if*

$$\psi : \mathbb{C}^2 \times U \to \mathbb{C},\ \psi(p(z), zp'(z)) = \rho p(z) + zp'(z),\ (\rho \in \mathbb{N} = \{1, 2, 3, ..\}),$$

then

$$F_{\psi(\mathbb{C}^2 \times U)} p(z) \le F_{\psi(\mathbb{C}^2 \times U)} q(z), z \in U,$$

and q is the best dominant.

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