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A Quadruple Fixed Point Theorem for Contractive Type Condition by Using ICS Mapping and Application to Integral Equation

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Abstract. In this paper, we obtain a Quadruple fixed point theorem for $\psi - \phi$ contractive condition in partially ordered partial metric spaces by using ICS mapping. We are also given an example and an application to integral equation which supports our main theorem.

1. Introduction

The notion of partial metric space was introduced by Matthews [8] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation and domain theory in computer science.

First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.1. (See [8, 9]) A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leq p(x,y), p(y,y) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a partial metric space (PMS).

Clearly $p(x,y) = 0$ implies $x = y$ and $x \neq y$ implies $p(x,y) > 0$.
If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$ given by $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a metric on X.

Example 1.1 (See e.g. [9]). Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_p(x,y) = |x - y|$.

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Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon),\ x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

We now state some basic topological notions (such as convergence, completeness) on partial metric spaces (see e.g. [1, 2, 8, 9].)

Definition 1.2.

(1) A sequence $\{x_n\}$ in the PMS (X,p) converges to the limit x if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
(2) A sequence $\{x_n\}$ in the PMS (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(3) A PMS (X,p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

We need the following lemmas in PMS([8, 9]).

Lemma 1.1.

(1) *A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).*
(2) *A PMS (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 1.2. *Assume $x_n \to z$ as $n \to \infty$ in a PMS (X,p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

In 2009, K.P. Chi [3] introduced the concept of ICS mapping as follows.

Definition 1.3 ([3,7])**.** Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be ICS if T is injective, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.

Now we introduce the notion of Quadruple fixed point as follows.

Definition 1.4. An element $(x,y,z,w) \in X^4$ is called a Quadruple fixed point of $F : X^4 \to X$ if $F(x,y,z,w) = x, F(y,z,w,x) = y, F(z,w,x,y) = z$ and $F(w,x,y,z) = w$.

Definition 1.5 ([5,6])**.** Let $(X,\preceq)$ be a partial ordered set and $F : X^4 \to X$. We say that F has the mixed monotone property if $F(x,y,z,w)$ is monotone non decreasing in x and z, and is monotone non - increasing in y and w, that is, for any $x, y, z, w \in X$.

$$\begin{aligned} x_1, x_2 \in X, x_1 \preceq x_2 &\quad\Rightarrow\quad F(x_1, y, z, w) \preceq F(x_2, y, z, w), \\ y_1, y_2 \in X, y_1 \preceq y_2 &\quad\Rightarrow\quad F(x, y_1, z, w) \succeq F(x, y_2, z, w), \\ z_1, z_2 \in X, z_1 \preceq z_2 &\quad\Rightarrow\quad F(x, y, z_1, w) \preceq F(x, y, z_2, w), \\ w_1, w_2 \in X, w_1 \preceq w_2 &\quad\Rightarrow\quad F(x, y, z, w_1) \succeq F(x, y, z, w_2). \end{aligned}$$

2. Main Results

Let Ψ denote the set of all continuous and monotonically increasing functions $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$.

Let Φ denote the set of all lower semi continuous functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$.

Let $(X, \preceq)$ be a partial ordered set. We consider the following partial order on the product space $X^4 = X \times X \times X \times X$. $(x, y, z, w) \preceq (u, v, r, t)$ iff $x \preceq u$, $y \succeq v$, $z \preceq r$ and $w \succeq t$ where (x, y, z, w) and $(u, v, r, t) \in X^4$.

Theorem 2.1. *Let $(X, p, \preceq)$ be a complete ordered partial metric space. Suppose $T : X \to X$ is an ICS mapping and $F : X^4 \to X$ is such that F has the mixed monotone property. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\begin{aligned} &\psi\left(p(TF(x, y, z, w), TF(u, v, r, t))\right) \\ &\le \psi\left(\max\left\{\begin{array}{c} p(Tx, Tu), p(Ty, Tv), \\ p(Tz, Tr), p(Tw, Tt) \end{array}\right\}\right) \\ &\qquad - \phi\left(\max\left\{\begin{array}{c} p(Tx, Tu), p(Ty, Tv), \\ p(Tz, Tr), p(Tw, Tt) \end{array}\right\}\right), \end{aligned} \tag{2.1.1}$$

for all $x, y, z, w, u, v, r, t \in X$ for which $x \preceq u$, $y \succeq v$, $z \preceq r$ and $w \succeq t$.

Suppose X has the following property (A):

I. *If non - decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
II. *if non - increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n.*

Suppose there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} x_0 &\preceq F(x_0, y_0, z_0, w_0), \\ y_0 &\succeq F(y_0, z_0, w_0, x_0), \\ z_0 &\preceq F(z_0, w_0, x_0, y_0), \\ w_0 &\succeq F(w_0, x_0, y_0, z_0). \end{aligned}$$

Then there exist $x, y, z, w \in X$ such that

$$\begin{aligned} F(x, y, z, w) = x, &\quad F(y, z, w, x) = y, \\ F(z, w, x, y) = z, &\quad F(w, x, y, z) = w, \end{aligned}$$

that is, F has a quadruple fixed point.

Proof. Let $x_0, y_0, z_0, w_0 \in X$ be such that

$$\begin{aligned} x_0 &\preceq F(x_0, y_0, z_0, w_0), \\ y_0 &\succeq F(y_0, z_0, w_0, x_0), \\ z_0 &\preceq F(z_0, w_0, x_0, y_0), \\ w_0 &\succeq F(w_0, x_0, y_0, z_0). \end{aligned}$$

Set

$$\begin{aligned} x_1 &= F(x_0, y_0, z_0, w_0) \succeq x_0, \\ y_1 &= F(y_0, z_0, w_0, x_0) \preceq y_0, \\ z_1 &= F(z_0, w_0, x_0, y_0) \succeq zo, \\ w_1 &= F(w_0, x_0, y_0, z_0) \preceq w_0 \end{aligned}$$

and by the mixed monotone property of F, for $n \geq 1$, inductively we get

$$\begin{aligned} x_n &= F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \succeq x_{n-1} \succeq \cdots \succeq x_0, \\ y_n &= F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) \preceq y_{n-1} \preceq \cdots \preceq y_0, \\ z_n &= F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) \succeq z_{n-1} \succeq \cdots \succeq z_0, \\ w_n &= F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}) \preceq w_{n-1} \preceq \cdots \preceq w_0. \end{aligned}$$

Assume for some $n \in N$, $x_{n+1} = x_n$, $y_{n+1} = y_n$, $z_{n+1} = z_n$ and $w_{n+1} = w_n$.

Then (x_n, y_n, z_n, w_n) is a quadruple fixed point of F. Hence the theorem.

Now assume that $x_{n+1} \neq x_n$ or $y_{n+1} \neq y_n$ or $z_{n+1} \neq z_n$ or $w_{n+1} \neq w_n$ for any $n \in N$.

Since T is injective, we have

$$a_n = \max\{p(Tx_{n+1}, Tx_n), p(Ty_{n+1}, Ty_n), p(Tz_{n+1}, Tz_n), p(Tw_{n+1}, Tw_n)\} > 0.$$

$$\begin{aligned} \psi\left(p(Tx_{n+1}, Tx_n)\right) &= \psi\left(p(TF(x_n, y_n, z_n, w_n), TF(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}))\right) \\ &\leq \psi\left(\max\left\{\begin{array}{c} p(Tx_n, Tx_{n-1}), p(Ty_n, Ty_{n-1}), \\ p(Tz_n, Tz_{n-1}), p(Tw_n, Tw_{n-1}) \end{array}\right\}\right) \\ &\quad - \phi\left(\max\left\{\begin{array}{c} p(Tx_n, Tx_{n-1}), p(Ty_n, Ty_{n-1}), \\ p(Tz_n, Tz_{n-1}), p(Tw_n, Tw_{n-1}) \end{array}\right\}\right) \\ &= \psi\left(a_{n-1}\right) - \phi\left(a_{n-1}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} \psi\left(p(Ty_{n+1}, Ty_n)\right) &\leq \psi\left(a_{n-1}\right) - \phi\left(a_{n-1}\right), \\ \psi\left(p(Tz_{n+1}, Tz_n)\right) &\leq \psi\left(a_{n-1}\right) - \phi\left(a_{n-1}\right), \\ \psi\left(p(Tw_{n+1}, Tw_n)\right) &\leq \psi\left(a_{n-1}\right) - \phi\left(a_{n-1}\right). \end{aligned}$$

Hence

$$\psi(a_n) = \psi\left(\max\left\{\begin{array}{l} p(Tx_{n+1},Tx_n), p(Ty_{n+1},Ty_n), \\ p(Tz_{n+1},Tz_n), p(Tw_{n+1},Tw_n) \end{array}\right\}\right)$$
$$= \max\left\{\begin{array}{l} \psi(p(Tx_{n+1},Tx_n)), \psi(p(Ty_{n+1},Ty_n)), \\ \psi(p(Tz_{n+1},Tz_n)), \psi(p(Tw_{n+1},Tw_n)) \end{array}\right\}$$
$$\leq \psi(a_{n-1}) - \phi(a_{n-1}).$$

Thus

$$\begin{aligned} \psi(a_n) &\leq \psi(a_{n-1}) - \phi(a_{n-1}) \\ &< \psi(a_{n-1}). \end{aligned} \tag{1}$$

Since ψ is increasing, we have

$$a_n < a_{n-1}, \ \forall\ n = 1,2,3,\cdots$$

Thus $\{a_n\}$ is a positive decreasing sequence of real numbers. Hence there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} a_n = r$. Suppose $r > 0$.
Letting $n \to \infty$ in (1), we obtain that

$$\psi(r) \leq \psi(r) - \phi(r) < \psi(r).$$

a contradiction. Hence $r = 0$.

Thus

$$\lim_{n\to\infty} \max\left\{\begin{array}{l} p(Tx_{n+1},Tx_n), p(Ty_{n+1},Ty_n), \\ p(Tz_{n+1},Tz_n), p(Tw_{n+1},Tw_n) \end{array}\right\} = 0. \tag{2}$$

From (p_2), we have

$$\lim_{n\to\infty} \max\{p(Tx_n,Tx_n), p(Ty_n,Ty_n), p(Tz_n,Tz_n), p(Tw_n,Tw_n)\} = 0. \tag{3}$$

From definition of d_p and from (2) and (3), it follows that

$$\lim_{n\to\infty} \max\left\{\begin{array}{l} d_p(Tx_{n+1},Tx_n), d_p(Ty_{n+1},Ty_n), \\ d_p(Tz_{n+1},Tz_n), d_p(Tw_{n+1},Tw_n) \end{array}\right\} = 0. \tag{4}$$

Now we shall prove that $\{Tx_n\}, \{Ty_n\}, \{Tz_n\}$ and $\{Tw_n\}$ are Cauchy sequences in the metric space (X, d_p). Assume on the contrary that $\{Tx_n\}$ or $\{Ty_n\}$ or $\{Tz_n\}$ or $\{Tw_n\}$ is not a Cauchy sequence in (X, d_p).

Then there exists $\epsilon > 0$ for which we can find subsequences of integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$\max\left\{\begin{array}{l} d_p(Tx_{m_k},Tx_{n_k}), d_p(Ty_{m_k},Ty_{n_k}), \\ d_p(Tz_{m_k},Tz_{n_k}), d_p(Tw_{m_k},Tw_{n_k}) \end{array}\right\} \geq \epsilon \tag{5}$$

Further, corresponding to m_k, we may choose n_k such that it is the smallest integer satisfying (5) and $n_k > m_k$. Then

$$\max\left\{\begin{array}{l} d_p(Tx_{m_k},Tx_{n_k-1}), d_p(Ty_{m_k},Ty_{n_k-1}), \\ d_p(Tz_{m_k},Tz_{n_k-1}), d_p(Tw_{m_k},Tw_{n_k-1}) \end{array}\right\} < \epsilon. \tag{6}$$

We have

$$
\begin{aligned}
d_p(Tx_{m_k}, Tx_{n_k}) &\leq d_p(Tx_{m_k}, Tx_{m_k-1}) \\
&\quad + d_p(Tx_{m_k-1}, Tx_{n_k-1}) + d_p(Tx_{n_k-1}, Tx_{n_k}) \qquad (7)\\
&\leq d_p(Tx_{m_k}, Tx_{m_k-1}) + d_p(Tx_{m_k-1}, Tx_{m_k}) \\
&\quad d_p(Tx_{m_k}, Tx_{n_k-1}) + d_p(Tx_{n_k-1}, Tx_{n_k}) \\
&< 2d_p(Tx_{m_k}, Tx_{m_k-1}) + \epsilon + d_p(Tx_{n_k-1}, Tx_{n_k}), \text{ from (6).} \qquad (8)
\end{aligned}
$$

Letting $k \to \infty$ in (7) and (8) and using (4), we get

$$
\lim_{k\to\infty} d_p(Tx_{m_k}, Tx_{n_k}) \leq \lim_{k\to\infty} d_p(Tx_{m_k-1}, Tx_{n_k-1}) \leq \epsilon. \qquad (9)
$$

Similarly,

$$
\lim_{k\to\infty} d_p(Ty_{m_k}, Ty_{n_k}) \leq \lim_{k\to\infty} d_p(Ty_{m_k-1}, Ty_{n_k-1}) \leq \epsilon, \qquad (10)
$$

$$
\lim_{k\to\infty} d_p(Tz_{m_k}, Tz_{n_k}) \leq \lim_{k\to\infty} d_p(Tz_{m_k-1}, Tz_{n_k-1}) \leq \epsilon, \qquad (11)
$$

$$
\lim_{k\to\infty} d_p(Tw_{m_k}, Tw_{n_k}) \leq \lim_{k\to\infty} d_p(Tw_{m_k-1}, Tw_{n_k-1}) \leq \epsilon, \qquad (12)
$$

Using (5) and (9) - (12), we have

$$
\begin{aligned}
\epsilon &\leq \lim_{k\to\infty} \max \left\{ \begin{array}{c} d_p(Tx_{m_k}, Tx_{n_k}), d_p(Ty_{m_k}, Ty_{n_k}), \\ d_p(Tz_{m_k}, Tz_{n_k}), d_p(Tw_{m_k}, Tw_{n_k}) \end{array} \right\} \\
&\leq \lim_{k\to\infty} \max \left\{ \begin{array}{c} d_p(Tx_{m_k-1}, Tx_{n_k-1}), d_p(Ty_{m_k-1}, Ty_{n_k-1}), \\ d_p(Tz_{m_k-1}, Tz_{n_k-1}), d_p(Tw_{m_k-1}, Tw_{n_k-1}) \end{array} \right\} \qquad (13)\\
&\leq \epsilon.
\end{aligned}
$$

Now using (13) and (3), we obtain

$$
\epsilon = \lim_{k\to\infty} \max \left\{ \begin{array}{c} 2p(Tx_{m_k}, Tx_{n_k}) - p(Tx_{m_k}, Tx_{m_k}) - p(Tx_{n_k}, Tx_{n_k}) \\ 2p(Ty_{m_k}, Ty_{n_k}) - p(Ty_{m_k}, Ty_{m_k}) - p(Ty_{n_k}, Ty_{n_k}) \\ 2p(Tz_{m_k}, Tz_{n_k}) - p(Tz_{m_k}, Tz_{m_k}) - p(Tz_{n_k}, Tz_{n_k}) \\ 2p(Tw_{m_k}, Tw_{n_k}) - p(Tw_{m_k}, Tw_{m_k}) - p(Tw_{n_k}, Tw_{n_k}) \end{array} \right\},
$$

$$
\frac{\epsilon}{2} = \lim_{k\to\infty} \max \left\{ \begin{array}{c} p(Tx_{m_k}, Tx_{n_k}), p(Ty_{m_k}, Ty_{n_k}), \\ p(Tz_{m_k}, Tz_{n_k}), p(Tw_{m_k}, Tw_{n_k}) \end{array} \right\}. \qquad (14)
$$

Similarly from (13) and (14), we obtain

$$
\frac{\epsilon}{2} = \lim_{k\to\infty} \max \left\{ \begin{array}{c} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}), \\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1}) \end{array} \right\}. \qquad (15)
$$

Now using (2.1.1), we have

$$\begin{aligned}
&\psi\left(p(Tx_{m_k}, Tx_{n_k})\right)\\
&= \psi\left(p(F(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}, w_{m_k-1}), F(x_{n_k-1}, y_{n_k-1}, z_{n_k-1}, w_{n_k-1}))\right)\\
&\le \psi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right)\\
&\qquad - \phi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\psi\left(p(Ty_{m_k}, Ty_{n_k})\right)\\
&\le \psi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right)\\
&\qquad - \phi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right),
\end{aligned}$$

$$\begin{aligned}
&\psi\left(p(Tz_{m_k}, Tz_{n_k})\right)\\
&\le \psi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right)\\
&\qquad - \phi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\left(p(Tw_{m_k}, Tw_{n_k})\right)\\
&\le \psi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right)\\
&\qquad - \phi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\psi\big(\max\big\{p(Tx_{m_k}, Tx_{n_k}), p(Ty_{m_k}, Ty_{n_k}),\\
&\qquad\qquad p(Tz_{m_k}, Tz_{n_k}), p(Tw_{m_k}, Tw_{n_k})\big\}\big)\\
&= \max\big\{\psi\left(p(Tx_{m_k}, Tx_{n_k})\right), \psi\left(p(Ty_{m_k}, Ty_{n_k})\right),\\
&\qquad\qquad \psi\left(p(Tz_{m_k}, Tz_{n_k})\right), \psi\left(p(Tw_{m_k}, Tw_{n_k})\right)\big\}\\
&\le \psi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right)\\
&\qquad - \phi\left(\max\left\{\begin{array}{l} p(Tx_{m_k-1}, Tx_{n_k-1}), p(Ty_{m_k-1}, Ty_{n_k-1}),\\ p(Tz_{m_k-1}, Tz_{n_k-1}), p(Tw_{m_k-1}, Tw_{n_k-1})\end{array}\right\}\right).
\end{aligned}$$

Letting $k \to \infty$ and using (14) and (15), we obtain

$$\psi\left(\frac{\epsilon}{2}\right) \leq \psi\left(\frac{\epsilon}{2}\right) - \phi\left(\frac{\epsilon}{2}\right) < \psi\left(\frac{\epsilon}{2}\right),$$

a contradiction.

Thus $\{Tx_n\}, \{Ty_n\}, \{Tz_n\}$ and $\{Tw_n\}$ are Cauchy sequences in the metric space (X, d_p). That is

$$\lim_{m,n\to\infty} d_p(Tx_n, Tx_m) = 0, \qquad \lim_{m,n\to\infty} d_p(Ty_n, Ty_m) = 0,$$
$$\lim_{m,n\to\infty} d_p(Tz_n, Tz_m) = 0 \qquad \lim_{m,n\to\infty} d_p(Tw_n, Tw_m) = 0.$$

From the definition of d_p and from (3), we have

$$\left.\begin{array}{ll} \lim\limits_{m,n\to\infty} p(Tx_n, Tx_m) = 0, & \lim\limits_{m,n\to\infty} p(Ty_n, Ty_m) = 0, \\ \lim\limits_{m,n\to\infty} p(Tz_n, Tz_m) = 0, & \lim\limits_{m,n\to\infty} p(Tw_n, Tw_m) = 0. \end{array}\right\} \tag{16}$$

Thus $\{Tx_n\}, \{Ty_n\}, \{Tz_n\}$ and $\{Tw_n\}$ are Cauchy sequences in (X, p).

Since (X, p) is complete, the Cauchy sequences $\{Tx_n\}, \{Ty_n\}, \{Tz_n\}$ and $\{Tw_n\}$ are convergent.

Since T is an ICS mapping, there exist $x, y, z, w \in X$ such that

$$\lim_{n\to\infty} p(x_n, x) = p(x, x), \qquad \lim_{n\to\infty} p(y_n, y) = p(y, y),$$
$$\lim_{n\to\infty} p(z_n, z) = p(z, z), \qquad \lim_{n\to\infty} p(w_n, w) = p(w, w).$$

Since T is continuous, we have

$$\lim_{n\to\infty} p(Tx_n, Tx) = p(Tx, Tx), \qquad \lim_{n\to\infty} p(Ty_n, Ty) = p(Ty, Ty),$$
$$\lim_{n\to\infty} p(Tz_n, Tz) = p(Tz, Tz), \qquad \lim_{n\to\infty} p(Tw_n, Tw) = p(Tw, Tw).$$

These implies that $\{Tx_n\}, \{Ty_n\}, \{Tz_n\}$ and $\{Tw_n\}$ are convergent to Tx, Ty, Tz and Tw respectively. Using Lemma 1.1 (2) and from (16), it follows that

$$\left.\begin{array}{ll} \lim\limits_{n\to\infty} p(Tx_n, Tx) = 0, & \lim\limits_{n\to\infty} p(Ty_n, Ty) = 0, \\ \lim\limits_{n\to\infty} p(Tz_n, Tz) = 0 \quad \text{and} & \lim\limits_{n\to\infty} p(Tw_n, Tw) = 0. \end{array}\right\} \tag{17}$$

Suppose X has the property (A).

Since $\{x_n\}, \{z_n\}$ are non - decreasing with $x_n \to x, z_n \to z$ and also $\{y_n\}, \{w_n\}$ are non - increasing with $y_n \to y, w_n \to w$ then by the property (A), we have $x_n \preceq x, y_n \succeq y, z_n \preceq z$ and $w_n \succeq w$ for all n.

Consider now

$$\begin{aligned}\psi\left(p(Tx_{n+1}, TF(x,y,z,w))\right) &= \psi\left(p(TF(x_n,y_n,z_n,w_n), TF(x,y,z,w))\right)\\ &\leq \psi\left(\max\left\{\begin{array}{c} p(Tx_n,Tx), p(Ty_n,Ty)\\ p(Tz_n,Tz), p(Tw_n,Tw)\end{array}\right\}\right)\\ &\quad - \phi\left(\max\left\{\begin{array}{c} p(Tx_n,Tx), p(Ty_n,Ty)\\ p(Tz_n,Tz), p(Tw_n,Tw)\end{array}\right\}\right).\end{aligned}$$

Taking $n \to \infty$ and using (17), we get

$$p(Tx, TF(x,y,z,w)) = 0$$

so that $Tx = TF(x,y,z,w)$.Since T is injective, we obtain $x = F(x,y,z,w)$.

Similarly we can show that $y = F(y,z,w,x), z = F(z,w,x,y), w = F(w,x,y,z)$.

Thus (x,y,z,w) is a quadruple fixed point of F. □

Theorem 2.2. *Let (X,p) be a complete partial metric space. Suppose $T : X \to X$ is an ICS mapping and $F : X^4 \to X$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\begin{aligned}&\psi\left(p(TF(x,y,z,w), TF(u,v,r,t))\right)\\ &\leq \psi\left(\max\left\{\begin{array}{c} p(Tx,Tu), p(Ty,Tv),\\ p(Tz,Tr), p(Tw,Tt)\end{array}\right\}\right)\\ &\quad - \phi\left(\max\left\{\begin{array}{c} p(Tx,Tu), p(Ty,Tv),\\ p(Tz,Tr), p(Tw,Tt)\end{array}\right\}\right),\end{aligned}\tag{2.2.1}$$

for all $x,y,z,w,u,v,r,t \in X$.

Then F has a quadruple fixed point of the form (x,x,x,x) where $x \in X$.

Proof. By proceeding the proof of Theorem 2.1, we get

$$\begin{aligned}F(x,y,z,w) = x, &\quad F(y,z,w,x) = y,\\ F(z,w,x,y) = z, &\quad F(w,x,y,z) = w,\end{aligned}$$

for some $x,y,z,w \in X$.

Now from (2.2.1), we have

$$\begin{aligned}\psi(p(Tx,Ty)) &= \psi\left(p(TF(x,y,z,w), TF(y,z,w,x))\right)\\ &\leq \psi\left(\max\left\{\begin{array}{c} p(Tx,Ty), p(Ty,Tz),\\ p(Tz,Tw), p(Tw,Tx)\end{array}\right\}\right)\\ &\quad - \phi\left(\max\left\{\begin{array}{c} p(Tx,Ty), p(Ty,Tz),\\ p(Tz,Tw), p(Tw,Tx)\end{array}\right\}\right),\end{aligned}$$

Similarly, we have

$$\psi(p(Ty,Tz)) \leq \psi\left(\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\}\right) - \phi\left(\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\}\right),$$

$$\psi(p(Tz,Tw)) \leq \psi\left(\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\}\right) - \phi\left(\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\}\right),$$

and

$$\psi(p(Tw,Tx)) \leq \psi\left(\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\}\right) - \phi\left(\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\}\right),$$

Now

$$\begin{aligned} &\psi\left(\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\}\right) \\ &= \max\left\{\begin{array}{c} \psi(p(Tx,Ty)),\psi(p(Ty,Tz)), \\ \psi(p(Tz,Tw)),\psi(p(Tw,Tx)) \end{array}\right\} \\ &\leq \psi\left(\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\}\right) \\ &\quad - \phi\left(\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\}\right), \end{aligned}$$

Hence

$$\max\left\{\begin{array}{c} p(Tx,Ty),p(Ty,Tz), \\ p(Tz,Tw),p(Tw,Tx) \end{array}\right\} = 0,$$

since $\phi(t) > 0$ for $t > 0$.

$\therefore \; Tx = Ty, Ty = Tz, Tz = Tw, Tw = Tx$.

Since T is injuctive, we have $x = y = z = w$.

Thus F has a quadruple fixed point of the form (x,x,x,x). □

The following example illustrates our Theorem 2.2

Example 2.1. Let $X = [0, 1]$, $p(x, y) = \max\{x, y\}$ and $T : X \to X$ be defined by $T(x) = \frac{x}{2}$. Let $F : X \times X \times X \times X \to X$ by $F(x, y, z, w) = \frac{x^2+y^2+z^2+w^2}{8}$ and $\psi,\ \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t,\ \phi(t) = \frac{t}{2}$ clearly all conditions of Theorem 2.2 are satisfied and

$$
\begin{aligned}
&\psi(p(TF(x, y, z, w), TF(u, v, r, t))) = p\left(\frac{F(x, y, z, w)}{2}, \frac{F(u, v, r, t)}{2}\right)\\
&= \max\left\{\frac{x^2 + y^2 + z^2 + w^2}{16}, \frac{u^2 + v^2 + r^2 + t^2}{16}\right\}\\
&= \frac{1}{16}\max\Big\{x^2 + y^2 + z^2 + w^2, u^2 + v^2 + r^2 + t^2\Big\}\\
&= \frac{1}{16}\Big[\max\left\{x^2, u^2\right\} + \max\left\{y^2, v^2\right\} + \max\left\{z^2, r^2\right\} + \max\left\{w^2, t^2\right\}\Big]\\
&\le \frac{1}{4}\max\Big\{\max\left\{x^2, u^2\right\}, \max\left\{y^2, v^2\right\}, \max\left\{z^2, r^2\right\}, \max\left\{w^2, t^2\right\}\Big\}\\
&\le \frac{1}{4}\max\Big\{\max\{x, u\}, \max\{y, v\}, \max\{z, r\}, \max\{w, t\}\Big\}\\
&\le \frac{1}{2}\max\Big\{\max\{Tx, Tu\}, \max\{Ty, Tv\}, \max\{Tz, Tr\}, \max\{Tw, Tt\}\Big\}\\
&= \psi\Big(\max\Big\{p(Tx, Tu), p(Ty, Tv), p(Tz, Tr), p(Tw, Tt)\Big\}\Big)\\
&\quad - \phi\Big(\max\Big\{p(Tx, Tu), p(Ty, Tv), p(Tz, Tr), p(Tw, Tt)\Big\}\Big).
\end{aligned}
$$

Clearly (0, 0, 0, 0) is quadruple fixed point of F.

3. Application

In this section, we study the existence of a unique solution to an initial value problem, as an application to Theorem 2.2.

Consider the initial value problem

$$
x^1(t_1) = l(t_1, x(t_1), x(t_1), x(t_1), x(t_1)),\ t_1 \in I = [0, 1],\ x(0) = x_0 \tag{18}
$$

where $l : I \times [\frac{x_0}{8}, \infty) \times [\frac{x_0}{8}, \infty) \times [\frac{x_0}{8}, \infty) \times [\frac{x_0}{8}, \infty) \to [\frac{x_0}{8}, \infty)$ and $x_0 \in \mathbb{R}^+$.

Theorem 3.1. *Consider the initial value problem 18 with*

$$
l \in C\Big(I \times [\frac{x_0}{8}, \infty) \times [\frac{x_0}{8}, \infty) \times [\frac{x_0}{8}, \infty) \times [\frac{x_0}{8}, \infty)\Big)
$$

and

$$\int_0^{t_1} l(s,x(s),y(s),z(s),w(s))\mathrm{d}s \leq$$

$$\leq \max\left\{\begin{array}{c} \frac{1}{2}\int_0^{t_1} l(s,x(s),x(s),x(s),x(s))\mathrm{d}s - \frac{x_0}{4}, \\ \frac{1}{2}\int_0^{t_1} l(s,y(s),y(s),y(s),y(s))\mathrm{d}s - \frac{x_0}{4}, \\ \frac{1}{2}\int_0^{t_1} l(s,z(s),z(s),z(s),z(s))\mathrm{d}s - \frac{x_0}{4}, \\ \frac{1}{2}\int_0^{t_1} l(s,w(s),w(s),w(s),w(s))\mathrm{d}s - \frac{x_0}{4} \end{array}\right\}.$$

Then there exists unique solution in $C(I,[\frac{x_0}{8},\infty))$ for the initial value problem 18.

Proof. The integral equation corresponding to initial value problem 18 is

$$x(t) = x_0 + \int_0^{t} l(s,x(s),x(s),x(s),x(s))\mathrm{d}s. \tag{19}$$

Let $X = C(I,[\frac{x_0}{8},\infty))$ and $p(x,y) = \max\{x-\frac{x_0}{8}, y-\frac{x_0}{8}\}$ for $x,y \in X$. Define ICS mapping $T : X \to X$ by $Tx = \frac{x}{4}$, $\psi,\phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t, \phi(t) = \frac{t}{2}$ and $F : X \times X \times X \to X$ by

$$F(x,y,z,w)(t_1) = x_0 + \int_0^{t_1} l(s,x(s),y(s),z(s),w(s))\mathrm{d}s.$$

Now

$$\begin{aligned} &\psi(p(TF(x,y,z,w)(t_1), F(u,v,r,t)(t_1))) \\ &= \max\left\{\frac{F(x,y,z,w)}{4} - \frac{x_0}{8}, \frac{F(u,v,r,t)}{4} - \frac{x_0}{8}\right\} \\ &= \frac{1}{4}\max\left\{\begin{array}{c} \frac{x_0}{2} + \int_0^{t_1} l(s,x(s),y(s),z(s),w(s))\mathrm{d}s, \\ \frac{x_0}{2} + \int_0^{t_1} l(s,u(s),v(s),r(s),t(s))\mathrm{d}s \end{array}\right\} \end{aligned}$$

$$\leq \frac{1}{4}\max\left\{\begin{array}{c} \frac{x_0}{2}+\max\left\{\begin{array}{c} \frac{1}{2}\int\limits_0^{t_1} l(s,x(s),x(s),x(s),x(s))\mathrm{d}s-\frac{x_0}{4}, \\ \frac{1}{2}\int\limits_0^{t_1} l(s,y(s),y(s),y(s),y(s))\mathrm{d}s-\frac{x_0}{4}, \\ \frac{1}{2}\int\limits_0^{t_1} l(s,z(s),z(s),z(s),z(s))\mathrm{d}s-\frac{x_0}{4}, \\ \frac{1}{2}\int\limits_0^{t_1} l(s,w(s),w(s),w(s),w(s))\mathrm{d}s-\frac{x_0}{4} \end{array}\right\}, \\ \frac{x_0}{2}+\max\left\{\begin{array}{c} \frac{1}{2}\int\limits_0^{t_1} l(s,u(s),u(s),u(s),u(s))\mathrm{d}s-\frac{x_0}{4}, \\ \frac{1}{2}\int\limits_0^{t_1} l(s,v(s),v(s),v(s),v(s))\mathrm{d}s-\frac{x_0}{4}, \\ \frac{1}{2}\int\limits_0^{t_1} l(s,r(s),r(s),r(s),r(s))\mathrm{d}s-\frac{x_0}{4}, \\ \frac{1}{2}\int\limits_0^{t_1} l(s,t(s),t(s),t(s),t(s))\mathrm{d}s-\frac{x_0}{4} \end{array}\right\} \end{array}\right\}$$

$$= \frac{1}{2}\max\left\{\begin{array}{c} \max\left\{\frac{x(t_1)}{4}-\frac{x_0}{8},\frac{y(t_1)}{4}-\frac{x_0}{8},\frac{z(t_1)}{4}-\frac{x_0}{8},\frac{w(t_1)}{4}-\frac{x_0}{8}\right\}, \\ \max\left\{\frac{u(t_1)}{4}-\frac{x_0}{8},\frac{v(t_1)}{4}-\frac{x_0}{8},\frac{r(t_1)}{4}-\frac{x_0}{8},\frac{t(t_1)}{4}-\frac{x_0}{8}\right\} \end{array}\right\}$$

$$= \frac{1}{2}\max\left\{\begin{array}{c} \max\left\{Tx(t_1)-\frac{x_0}{8},Tu(t_1)-\frac{x_0}{8}\right\}, \\ \max\left\{Ty(t_1)-\frac{x_0}{8},Tv(t_1)-\frac{x_0}{8}\right\}, \\ \max\left\{Tz(t_1)-\frac{x_0}{8},Tr(t_1)-\frac{x_0}{8}\right\}, \\ \max\left\{Tw(t_1)-\frac{x_0}{8},Tt(t_1)-\frac{x_0}{8}\right\} \end{array}\right\}$$

$$= \frac{1}{2}\max\left\{ p(Tx,Tu),p(Ty,Tv),p(Tz,Tr),p(Tw,Tt) \right\}$$

$$= \psi\left(\max\left\{\begin{array}{c} p(Tx,Tu),p(Ty,Tv), \\ p(Tz,Tr),p(Tw,Tt) \end{array}\right\}\right) - \phi\left(\max\left\{\begin{array}{c} p(Tx,Tu),p(Ty,Tv), \\ p(Tz,Tr),p(Tw,Tt) \end{array}\right\}\right)$$

Thus F satisfies the condition (2.2.1) of Theorem 2.2. From Theorem 2.2, we conclude that F has a quadruple fixed point. In particular $x(t)$ is the unique solution of the integral equation (19). □

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