Mathematica Moravica
Vol. **18-2** (2014), 15–20

On Decompositions of Continuity and α-Continuity

Zbigniew Duszyński

Abstract. Several results concerning a decomposition of α-continuous, continuous and complete continuous functions are offered.

1. Preliminaries

Throughout the paper, by (X, τ) and (Y, σ) we denote topological spaces (briefly: spaces) on which no separation axioms are assumed. For a subset S of a space (X, τ), $\mathrm{cl}\,(S)$ and $\mathrm{int}\,(S)$ denote the closure and the interior of S, respectively. A set $S \subset X$ is called *regular open* (resp. *regular closed*) if $S = \mathrm{int}\,(\mathrm{cl}\,(S))$ (resp. $S = \mathrm{cl}\,(\mathrm{int}\,(S))$). An $S \subset X$ is said to be *α-open* [9] (resp. *preopen* [8], *semi-open* [7], *semi-preopen* [1]) if $S \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S)))$ (resp. $S \subset \mathrm{int}\,(\mathrm{cl}\,(S))$, $S \subset \mathrm{cl}\,(\mathrm{int}\,(S))$, $S \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S)))$).

The family τ^α of all α-open subsets of a space (X, τ) is always a topology on X [9], such that $\tau^\alpha \supset \tau$ (the inclusion is proper, in general). Crossley and Hildebrand [3] investigated *semi-closed* subsets of (X, τ): S is semi-closed if $S \supset \mathrm{int}\,(\mathrm{cl}\,(S))$. They have obtained that S is semi-closed in (X, τ) if and only if $\mathrm{int}\,(\mathrm{cl}\,(S)) = \mathrm{int}\,(S)$. Using this identity, Tong [12] introduced the so-called *t-sets*. He proved that each regular open set is a t-set [12, Proposition 2]. The family of all regular open (resp. regular closed, closed, semi-open, semi-closed, preopen, semi-preopen) subsets of (X, τ) is denoted as $\mathrm{RO}\,(X, \tau)$ (resp. $\mathrm{RC}\,(X, \tau)$, $\mathrm{c}(X, \tau)$, $\mathrm{SO}\,(X, \tau)$, $\mathrm{SC}\,(X, \tau)$, $\mathrm{PO}(X, \tau)$, $\mathrm{SPO}(X, \tau)$). The following inclusions (proper in general) are known:

- $\mathrm{RO}\,(X, \tau) \subset \tau \subset \tau^\alpha \subset \mathrm{PO}(X, \tau) \subset \mathrm{SPO}(X, \tau)$,
- $\tau^\alpha \subset \mathrm{SO}\,(X, \tau) \subset \mathrm{SPO}(X, \tau)$.

The families $\mathrm{PO}(X, \tau)$ and $\mathrm{SO}\,(X, \tau)$ are, in general, independent of each other in the sense of inclusion [10].

2010 *Mathematics Subject Classification.* Primary: 54A05.

Key words and phrases. Preopen set, α-open set, semi-open set, α-precontinuity, continuity, complete continuity.

Lemma 1.1 ([10] for "semi-open", [4] for "semi-closed"). *If either* $S_1 \in \mathrm{SO}(X,\tau) \cup \mathrm{SC}(X,\tau)$ *or* $S_2 \in \mathrm{SO}(X,\tau) \cup \mathrm{SC}(X,\tau)$, *then*

$$\mathrm{int}(\mathrm{cl}(S_1 \cap S_2)) = \mathrm{int}(\mathrm{cl}(S_1)) \cap \mathrm{int}(\mathrm{cl}(S_2)).$$

Lemma 1.2 ([10]). *In any space* (X,τ), $\tau^\alpha = \mathrm{PO}(X,\tau) \cap \mathrm{SO}(X,\tau)$.

2. Continuity

We will need the following classes of subsets of a space (X,τ).

Definition 2.1. ($\tau 0$) $\mathcal{B}^0_\tau(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau,\ C \in \mathrm{RO}(X,\tau)\}$;

($\tau 1$) $\mathcal{B}^1_\tau(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau,\ C \in \mathrm{RC}(X,\tau)\}$ $(= \mathcal{A}(X,\tau)$ [11, Definition 3.1] and [12, p. 31]);

($\tau 2$) $\mathcal{B}^2_\tau(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau,\ C \in \mathrm{c}(X,\tau)\}$ $(= \mathrm{LC}(X,\tau)$ [5]);

($\tau 3$) $\mathcal{B}^3_\tau(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau,\ C \in \mathrm{c}(X,\tau^\alpha)\}$;

($\tau 4$) $\mathcal{B}^4_\tau(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau,\ C \in \mathrm{SC}(X,\tau)\}$ $(= \mathcal{B}(X,\tau)$ [12, Definition 2]).

The following (obvious) inclusions, proper in general, hold:

$$\mathcal{B}^1_\tau(X,\tau) \subset \mathcal{B}^2_\tau(X,\tau) \subset \mathcal{B}^3_\tau(X,\tau) \subset \mathcal{B}^4_\tau(X,\tau).$$

In Theorem 2.1 we recall some results that have been so far obtained.

Theorem 2.1. *Let* (X,τ) *be a topological space.*

(a) $\tau = \mathrm{PO}(X,\tau) \cap \mathcal{A}(X,\tau)$ *[6, case (iv), p. 31];*

(b) $\tau = \mathrm{PO}(X,\tau) \cap \mathrm{LC}(X,\tau)$ *[6, Theorem 2(3)];*

(c) $\tau = \mathrm{PO}(X,\tau) \cap \mathcal{B}(X,\tau)$ *[12, Proposition 9].*

We complete these decompositions of τ in the theorem below.

Theorem 2.2. *Let* (X,τ) *be a topological space.*

(d) $\tau = \mathrm{PO}(X,\tau) \cap \mathcal{B}^0_\tau(X,\tau)$;

(e) $\tau = \mathrm{PO}(X,\tau) \cap \mathcal{B}^3_\tau(X,\tau)$.

Proof. Let $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}^0_\tau(X,\tau)$. Then $S \in \mathrm{PO}(X,\tau)$ and $S \in \mathcal{B}^0_\tau(X,\tau) \subset \mathcal{B}(X,\tau)$ since $\mathrm{RO}(X,\tau) \subset \mathrm{SC}(X,\tau)$. By [12, Proposition 9] we have $S \in \tau$. That $S \in \tau$ follows from $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}^3_\tau(X,\tau)$ is analogous because $\mathrm{c}(X,\tau^\alpha) \subset \mathrm{SC}(X,\tau)$.

Let now $S \in \tau$. Then $S \in \mathrm{PO}(X,\tau)$ and $S = S \cap X$, where $X \in \mathrm{RO}(X,\tau)$. For (e), observe that $X \in \mathrm{c}(X,\tau^\alpha)$. □

Definition 2.2. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called $\mathcal{B}^i_\tau$*-continuous* on (X,τ), $i = 0,1,2,3,4$, if $f^{-1}(V) \in \mathcal{B}^i_\tau(X,\tau)$ for any $V \in \sigma$.

For $i = 1,2,4$ these types of continuity were introduced earlier: $\mathcal{A}$-*continuity* ($\equiv \mathcal{B}^1_\tau$-continuity) [11], LC-*continuity* ($\equiv \mathcal{B}^2_\tau$-continuity) [6], $\mathcal{B}$-*continuity* ($\equiv \mathcal{B}^4_\tau$-continuity) [12].

By Theorems 2.1 and 2.2 we get the following decomposition results.

Theorem 2.3. *Let* $f\colon (X,\tau) \to (Y,\sigma)$ *be a function. Then, for each* $i = 0,1,2,3,4$,

(a_i) f *is continuous if and only if* f *is precontinuous and* $\mathcal{B}^i_\tau$*-continuous.*

Decompositions of continuity for cases (a_1), (a_2) and (a_4) were known before: [6, Theorem 4(v)], [6, Theorem 4(iv)] and [12, Proposition 11], respectively.

3. Complete continuity

Definition 3.1. Let (X,τ) be a topological space.

($r\tau1$) $\mathcal{B}^1_{r\tau}(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \mathrm{RO}(X,\tau),\ C \in \mathrm{RC}(X,\tau)\}$;
($r\tau2$) $\mathcal{B}^2_{r\tau}(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \mathrm{RO}(X,\tau),\ C \in \mathrm{c}(X,\tau)\}$;
($r\tau3$) $\mathcal{B}^3_{r\tau}(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \mathrm{RO}(X,\tau),\ C \in \mathrm{c}(X,\tau^\alpha)\}$;
($r\tau4$) $\mathcal{B}^4_{r\tau}(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \mathrm{RO}(X,\tau),\ C \in \mathrm{SC}(X,\tau)\}$.

Theorem 3.1. *For any topological space* (X,τ),

(a_{r4}) *a set* $S \in \mathrm{RO}(X,\tau)$ *if and only if* $S \in \mathrm{PO}(X,\tau)$ *and* $S \in \mathcal{B}^4_{r\tau}(X,\tau)$.

Proof. ($\Rightarrow$) Let $S \in \mathrm{RO}(X,\tau)$. Then $S \in \mathrm{PO}(X,\tau)$ and $S = S \cap X$ where $X \in \mathrm{SC}(X,\tau)$.

($\Leftarrow$) If $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}^4_{r\tau}(X,\tau)$, then $S \subset \mathrm{int}(\mathrm{cl}(U \cap C))$ where $U \in \mathrm{RO}(X,\tau)$ and $C \in \mathrm{SC}(X,\tau)$. Using Lemma 1.1 we get

$$(*) \qquad S \subset \mathrm{int}(\mathrm{cl}(U)) \cap \mathrm{int}(\mathrm{cl}(C)) = U \cap \mathrm{int}(C),$$

since $C \in \mathrm{SC}(X,\tau)$ (or by $U \in \mathrm{RO}(X,\tau)$). Then we have:

$$(**) \qquad S = (U \cap C) \cap U \subset (U \cap \mathrm{int}(C)) \cap U = U \cap \mathrm{int}(C).$$

Since $S \supset U \cap \mathrm{int}(C)$, we get $S = U \cap \mathrm{int}(C) = U \cap \mathrm{int}(\mathrm{cl}(C))$. But sets $U, \mathrm{int}(\mathrm{cl}(C)) \in \mathrm{RO}(X,\tau)$. Therefore S is an intersection of two regularly open sets — consequently, $S \in \mathrm{RO}(X,\tau)$. □

We introduce now four new types of continuity.

Definition 3.2. A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be $\mathcal{B}^i_{r\tau}$*-continuous* on (X,τ), $i = 1,2,3,4$, if $f^{-1}(V) \in \mathcal{B}^i_{r\tau}(X,\tau)$ for every $V \in \sigma$.

Definition 3.3. [2] A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *completely continuous* if $f^{-1}(V) \in \mathrm{RO}(X,\tau)$ for every $V \in \sigma$.

By Theorem 3.1 we obtain the following decomposition result:

Theorem 3.2. *Let* $f\colon (X,\tau) \to (Y,\sigma)$ *be a function. Then:*

(b_{r4}) f *is completely continuous if and only if it is both precontinuous and* $\mathcal{B}^4_{r\tau}$*-continuous.*

We can get more decompositions of complete continuity. Namely, one easily obtains

Theorem 3.3. *Let* (X,τ) *be an arbitrary space. Then for* $i = 1,2,3$,

(a_{ri}) *$S \in \mathrm{RO}(X,\tau)$ if and only if $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}^i_{r\tau}(X,\tau)$.*

Theorem 3.4. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function. Then, for $i = 1, 2, 3$, the following hold:*

(b_{ri}) *f is completely continuous if and only if f is precontinuous and $\mathcal{B}^i_{r\tau}$-continuous.*

4. α-CONTINUITY

Definition 4.1. Let (X,τ) be a topological space.

($\alpha 0$) $\mathcal{B}^0_\alpha(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau^\alpha,\ C \in \mathrm{RO}(X,\tau)\}$;
($\alpha 1$) $\mathcal{B}^1_\alpha(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau^\alpha,\ C \in \mathrm{RC}(X,\tau)\}$;
($\alpha 2$) $\mathcal{B}^2_\alpha(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau^\alpha,\ C \in \mathrm{c}(X,\tau)\}$;
($\alpha 3$) $\mathcal{B}^3_\alpha(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau^\alpha,\ C \in \mathrm{c}(X,\tau^\alpha)\}$;
($\alpha 4$) $\mathcal{B}^4_\alpha(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \tau^\alpha,\ C \in \mathrm{SC}(X,\tau)\}$.

Theorem 4.1. *Let (X,τ) be arbitrary. Then:*

($\mathrm{a}_{\alpha 4}$) *$S \in \tau^\alpha$ if and only if $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}^4_\alpha(X,\tau)$.*

Proof. ($\Rightarrow$) Let $S \in \tau^\alpha$. Then $S \in \mathrm{PO}(X,\tau)$ and $S = S \cap C$, where $C = X \in \mathrm{SC}(X,\tau)$.

($\Leftarrow$) Suppose $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}^4_\alpha(X,\tau)$. Then $S \in \mathrm{PO}(X,\tau)$ and $S = U \cap C$, where $U \in \tau^\alpha$, $C \in \mathrm{SC}(X,\tau)$. Utilizing Lemma 1.1 we obtain

$$(*') \quad \begin{aligned} &S \subset \mathrm{int}(\mathrm{cl}(U \cap C)) = \mathrm{int}(\mathrm{cl}(U)) \cap \mathrm{int}(\mathrm{cl}(C)) \subset \\ &\quad \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(U))))) \cap \mathrm{int}(C) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(U))) \cap \mathrm{int}(C). \end{aligned}$$

Hence

$$(**') \quad S = (U \cap C) \cap U \subset \big(\mathrm{int}(\mathrm{cl}(\mathrm{int}(U))) \cap \mathrm{int}(C)\big) \cap U = U \cap \mathrm{int}(C).$$

On the other hand, $S = U \cap C \supset U \cap \mathrm{int}(C)$. This shows that $S = U \cap \mathrm{int}(C) \in \tau^\alpha$. □

Observe that $X \in \mathrm{RO}(X,\tau) \cap \mathrm{RC}(X,\tau)$ and $\mathrm{RC}(X,\tau) \subset \mathrm{c}(X,\tau) \subset \mathrm{c}(X,\tau^\alpha) \subset \mathrm{SC}(X,\tau)$. Thus, by the proof of sufficiency of Theorem 4.1, one easily deduces the following result.

Theorem 4.2. *Let (X,τ) be arbitrary. Then for $i = 0, 1, 2, 3$ one has:*

($\mathrm{a}_{\alpha i}$) *$S \in \tau^\alpha$ if and only if $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}^i_\alpha(X,\tau)$.*

For a function $f\colon (X,\tau) \to (Y,\sigma)$ we may define the respective notions of $\mathcal{B}^i_\alpha$-*continuity* — see Definitions 2.2 or 3.2. Theorems 4.1 and 4.2 lead immediately to

Theorem 4.3. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function. For $i = 0, 1, 2, 3, 4$,*

($\mathrm{b}_{\alpha i}$) *f is α-continuous if and only if f is precontinuous and $\mathcal{B}^i_\alpha$-continuous.*

Definition 4.2. Let (X,τ) be a space.

($s0$) $\mathcal{B}^0_s(X,\tau) := \{S \subset X : S = U \cap C,\ U \in \mathrm{SO}(X,\tau),\ C \in \mathrm{RO}(X,\tau)\}$;

($s1$) $\mathcal{B}_s^1(X,\tau) := \{S \subset X : S = U \cap C,\, U \in \mathrm{SO}(X,\tau),\, C \in \mathrm{RC}(X,\tau)\}$;
($s2$) $\mathcal{B}_s^2(X,\tau) := \{S \subset X : S = U \cap C,\, U \in \mathrm{SO}(X,\tau),\, C \in \mathrm{c}(X,\tau)\}$;
($s3$) $\mathcal{B}_s^3(X,\tau) := \{S \subset X : S = U \cap C,\, U \in \mathrm{SO}(X,\tau),\, C \in \mathrm{c}(X,\tau^\alpha)\}$;
($s4$) $\mathcal{B}_s^4(X,\tau) := \{S \subset X : S = U \cap C,\, U \in \mathrm{SO}(X,\tau),\, C \in \mathrm{SC}(X,\tau)\}$.

Theorem 4.4. *Let (X,τ) be a space. Then*

(a_{s4}) *$S \in \tau^\alpha$ if and only if $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}_s^4(X,\tau)$.*

Proof. Let $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}_s^4(X,\tau)$. Hence $S \subset \mathrm{int}(\mathrm{cl}(S))$ and $S = U \cap C$ for $U \in \mathrm{SO}(X,\tau)$ and $C \in \mathrm{SC}(X,\tau)$. We have what follows (using Lemma 1.1):

(∗”) $\quad S \subset \mathrm{int}(\mathrm{cl}(U \cap C)) = \mathrm{int}(\mathrm{cl}(U)) \cap \mathrm{int}(C) \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) \cap \mathrm{int}(C)$.

Since $\mathrm{SO}(X,\tau) \subset \mathrm{SPO}(X,\tau)$,

(∗∗”) $\quad S = (U \cap C) \cap U \subset \big(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) \cap \mathrm{int}(C)\big) \cap U = U \cap \mathrm{int}(C)$.

But $S = U \cap C \supset U \cap \mathrm{int}(C)$ and hence $S = U \cap \mathrm{int}(C) \in \mathrm{SO}(X,\tau)$, since the intersection of semi-open and open sets is always semi-open [7].

Thus we have proved that $\mathrm{PO}(X,\tau) \cap \mathcal{B}_s^4(X,\tau) \subset \mathrm{SO}(X,\tau)$. Since $\tau^\alpha \subset \mathrm{PO}(X,\tau) \cap \mathcal{B}_s^4(X,\tau)$ and, by Lemma 1.2, we have

$$\tau^\alpha \cap \mathrm{PO}(X,\tau) \subset \mathrm{PO}(X,\tau) \cap \mathcal{B}_s^4(X,\tau) \subset \mathrm{PO}(X,\tau) \cap \mathrm{SO}(X,\tau) = \tau^\alpha.$$

Therefore ($\tau^\alpha \subset \mathrm{PO}(X,\tau)$), one obtains $\tau^\alpha = \mathrm{PO}(X,\tau) \cap \mathcal{B}_s^4(X,\tau)$. □

Similarly to the observations in the lines preceding Theorem 4.2 one can formulate

Theorem 4.5. *Let (X,τ) be a space. Then for $i = 0,1,2,3$,*

(a_{si}) *$S \in \tau^\alpha$ if and only if $S \in \mathrm{PO}(X,\tau) \cap \mathcal{B}_s^i(X,\tau)$.*

For a function $f\colon (X,\tau) \to (Y,\sigma)$ one can define the respective notions of $\mathcal{B}_s^i$*-continuity* ($i = 0,1,2,3,4$) — we leave it to the reader. The following decomposition results follow from Theorems 4.4 and 4.5.

Theorem 4.6. *Let $f\colon (X,\tau) \to (Y,\sigma)$, be a function. Then for $i = 0,1,2,3,4$,*

(b_{si}) *f is α-continuous if and only if f is precontinuous and $\mathcal{B}_s^i$-continuous.*

References

[1] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24–32.

[2] S.P. Arya, R. Gupta, *On strongly continuous mappings*, Kyungpook Math. J., **14** (1974), 131–143.

[3] C.G. Crossley, S. K. Hildebrand, *Semi-closure*, Texas J. Sci., **22**(2-3) (1971), 99–112.

[4] Z. Duszyński, *Optimality of mappings and some separation axioms*, Rendiconti del Circolo Matematico di Palermo, **57** (2008), 213–228.

[5] M. Ganster, I. L. Reilly, *Locally closed sets and LC-continuous functions*, Internat. J. Math. Math. Sci., **3** (1989), 417–424.

[6] M. Ganster, I. L. Reilly, *A decomposition of continuity*, Acta Math. Hungar., **56**(3-4) (1990), 299–301.

[7] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.

[8] A.S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. and Phys. Soc. Egypt, **53** (1982), 47–53.

[9] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math., **15** (1965), 961–970.

[10] T. Noiri, *On α-continuous functions*, Čas. pěst. mat., **109** (1984), 118–126.

[11] J. Tong, *A decomposition of continuity*, Acta Math. Hungar., **48**(1-2) (1986), 11–15.

[12] J. Tong, *On decompositions of continuity in topological spaces*, Acta Math. Hungar., **51**(1-2) (1989), 51–55.

Zbigniew Duszyński
Casimirus the Great University
Department of Mathematics
Pl. Weyssenhoffa 11
85-072 Bydgoszcz
Poland
E-mail address: kayalnabin82@gmail.com