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The Fuzzy Stability of a Pexiderized Functional Equation

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Abstract. In this paper, Hyers-Ulam-Rassias Stability of the Pexiderized functional equation $f(x+y) = g(x) + h(y)$ is concerned in fuzzy Banach spaces.

1. Introduction

In 1940 stability problem of a functional equation was initiated by Ulam [14] concerning the stability of group homomorphism. In the next year, Hyers [6] gave answer for Cauchy functional equation in Banach spaces. T. Aoki [15] and Th. M. Rassias [16] generalized Hyers's theorem for additive mappings and linear mappings by considering an unbounded Cauchy difference respectively. Gavruta [10] generalized Rassias theorem by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias's approach. F. Skof [7] generalized Hyers-Ulam stability theorem for the function $f : X \to Y$, where X is a normed linear space and Y is a Banach space. Afterwards, the result of Skof was extended by P. W. Cholewa [11] and S. Czerwik [13].

Fuzzy set theory was initiated by Zadeh [8] and after the introduction of the notion of fuzzy norm on a linear space by Katsaras [2], many authors [12, 17], gave various ideas of fuzzy norm. Thereafter various notions of Banach spaces have been generalized in fuzzy Banach spaces. In fact, the notion of the Hyers-Ulam-Rassias stability for various functional equations are being generalized in fuzzy Banach Spaces by several authors [3,5,9,18,19].

In this paper, we investigate the generalized Hyers-Ulam-Rassias stability for the functional equation $f(x+y) = g(x) + h(y)$ in fuzzy Banach spaces.

2. Preliminaries

In the sequel, we need some definitions which are given bellow.

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Definition 2.1 ([4]). A binary operation $* : [0\ ,\ 1] \times [0\ ,\ 1] \longrightarrow [0\ ,\ 1]$ is continuous t - norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a \quad \forall a \in [0, 1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Again we assume that $a * a = a \forall a \in [0, 1]$.

Definition 2.2 ([17]). The 3 - tuple $(X, N, *)$ is called a fuzzy normed linear space if X is a real linear space, $*$ is a continuous t - norm and N is a fuzzy set in $X \times (0, \infty)$ satisfying the following conditions:

(i) $N(x, t) > 0$,
(ii) $N(x, t) = 1$ if and only if $x = 0$,
(iii) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$,
(iv) $N(x, s) * N(y, t) \leqslant N(x + y, s + t)$,
(v) $N(x, \cdot) : (0, \infty) \to (0, 1]$ is continuous

for all $x, y \in X$ and $t, s > 0$.

Note that $N(x, t)$ can be thought of as the degree of nearness between x and null vector 0 with respect to t.

Example 2.1. Let $X = [0, \infty)$, $a * b = ab$, for every $a, b \in [0, 1]$, and $\|\cdot\|$ be the usual metric defined on X. Define $N(x, t) = e^{-\frac{\|x\|}{t}}$ for all $x \in X$. Then clearly $(X, N, *)$ is a fuzzy normed linear space.

Example 2.2. Let $(X, \|\cdot\|)$ be a normed linear space, and let $a * b = ab$ or $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let $N(x, t) = \frac{t}{t + \|x\|}$ for all $x \in X$ and $t > 0$. Then $(X, N, *)$ is a fuzzy normed linear space and this fuzzy norm N induced by $\|\cdot\|$ is called the standard fuzzy norm.

Note 2.1. According to George and Veeramani [1], it can be proved that every fuzzy normed linear space is a metrizable topological space. In fact, also it can be proved that if $(X, \|\cdot\|)$ is a normed linear space, then the topology generated by $\|\cdot\|$ coincides with the topology generated by the fuzzy norm N of Example 2.2. As a result, we can say that an ordinary normed linear space is a special case of fuzzy normed linear space.

Remark 2.1. In fuzzy normed linear space $(X, N, *)$, for all $x \in X, N(x, \cdot)$ is non-decreasing with respect to the variable t and $\lim\limits_{t \to \infty} N(x, t) = 1$.

Definition 2.3. [17] Let $(X, N, *)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim\limits_{n \to \infty} N(x_n - x, t) = 1$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim\limits_{n \to \infty} x_n = x$.

Definition 2.4. [17] Let $(X, N, *)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

3. Stability of The Functional Equation

Throughout this section, X is assumed to be a real vector space and (Y, N) is assumed to be a fuzzy Banach space.

Theorem 3.1. *Let $\phi : X^2 \to \mathbb{R}^+$ be a mapping such that*

$$\widetilde{\phi}(x, y) := \sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i y)}{3^i} < \infty, \quad \text{for all } x, y \in X.$$

Let $f, g, h : X \to Y$ be mappings such that

$$\lim_{t \to \infty} N(f(x+y) - g(x) - h(x), t\phi(x, y)) = 1 \tag{3.1}$$

uniformly on X^2. Then there exists a unique mapping $A : X \to Y$ such that

$$A(x+y) = A(x) + A(y) \tag{3.2}$$

for all $x, y \in X$ and if for some $\delta > 0$, $\alpha > 0$

$$N(f(x+y) - g(x) - h(y), \delta\phi(x, y)) \geq \alpha \tag{3.3}$$

for all $x, y \in X$, then

$$\begin{aligned} N\Bigg(f(x) - A(x) - f(0), \frac{\delta}{3}\Bigg[&\widetilde{\phi}\left(\frac{x}{2}, \frac{-x}{2}\right) + \\ &\widetilde{\phi}\left(\frac{-x}{2}, \frac{x}{2}\right) + \widetilde{\phi}\left(\frac{x}{2}, \frac{x}{2}\right) + 2\widetilde{\phi}\left(\frac{-x}{2}, \frac{-x}{2}\right) + \\ &\widetilde{\phi}\left(\frac{-x}{2}, \frac{3x}{2}\right) + \widetilde{\phi}\left(\frac{3x}{2}, \frac{-x}{2}\right) + \widetilde{\phi}\left(\frac{3x}{2}, \frac{3x}{2}\right)\Bigg]\Bigg) \geq \alpha \end{aligned} \tag{3.4}$$

and

$$N - \lim_{n \to \infty} \frac{f(3^n x)}{3^n} = N - \lim_{n \to \infty} \frac{g(3^n x)}{3^n} = N - \lim_{n \to \infty} \frac{h(3^n x)}{3^n} = A(x)$$

for all $x \in X$.

Proof. Corresponding to a given $\epsilon > 0$ and (3.1), there exists some $t_0 > 0$ such that

$$N(f(x+y) - g(x) - h(y), t\phi(x, y)) \geq 1 - \epsilon \tag{3.5}$$

for all $x, y \in X$ and $t \geq t_0$. Let

$$\phi_1(x, y) = \phi\left(\frac{x}{2}, \frac{y}{2}\right) + \phi\left(\frac{y}{2}, \frac{x}{2}\right) + \phi\left(\frac{x}{2}, \frac{x}{2}\right) + \phi\left(\frac{y}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Now,

$$\begin{aligned}
&N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\phi_1(x,y)\right) \\
&\geq N\left(f\left(\frac{x+y}{2}\right) - g\left(\frac{x}{2}\right) - h\left(\frac{y}{2}\right), t\phi\left(\frac{x}{2}, \frac{y}{2}\right)\right) * \\
&\quad N\left(f\left(\frac{x+y}{2}\right) - g\left(\frac{y}{2}\right) - h\left(\frac{x}{2}\right), t\phi\left(\frac{y}{2}, \frac{x}{2}\right)\right) * \\
&\quad N\left(f(x) - g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), t\phi\left(\frac{x}{2}, \frac{x}{2}\right)\right) * \\
&\quad N\left(f(y) - g\left(\frac{y}{2}\right) - h\left(\frac{y}{2}\right), t\phi\left(\frac{y}{2}, \frac{y}{2}\right)\right) \\
&\geq 1 - \epsilon \quad [\text{by } (3.5)]
\end{aligned} \tag{3.6}$$

for all $x, y \in X$ and $t \geq t_0$.

Define a function $F : X \to Y$ by $F(x) = f(x) - f(0)$. Clearly F satisfies (3.6) and $F(0) = 0$. Putting $y = -x$ in (3.6), we get

$$N(-F(x) - F(-x), t\phi_1(x, -x)) \geq 1 - \epsilon \tag{3.7}$$

for all $x \in X$ and $t \geq t_0$.

Replacing x and y by $-x$ and $3x$ respectively in (3.6), we get

$$N(2F(x) - F(-x) - F(3x), t\phi_1(-x, 3x)) \geq 1 - \epsilon \tag{3.8}$$

for all $x \in X$ and $t \geq t_0$. Now,

$$\begin{aligned}
&N(F(x) - 3^{-1}F(3x), t3^{-1}(\phi_1(x, -x) + \phi_1(-x, 3x))) \\
&\geq N(2F(x) - F(-x) - F(3x), t\phi_1(-x, 3x)) * \\
&\qquad N(-F(x) - F(-x), t\phi_1(x, -x)) \\
&\geq 1 - \epsilon \quad [\text{by } (3.7), (3.8)]
\end{aligned} \tag{3.9}$$

for all $x \in X$ and $t \geq t_0$.

Now we show for any positive integer n that

$$\begin{aligned}
&N\Big(3^{-n}F(3^n x) - F(x), \\
&\qquad t\sum_{i=0}^{n-1} 3^{-i-1}(\phi_1(3^i x, 3^i(-x)) + \phi_1(3^i(-x), 3^i(3x))\Big) \geq 1 - \epsilon
\end{aligned} \tag{3.10}$$

for all $x \in X$ and $t \geq t_0$.

(3.9) shows that (3.10) is true for $n = 1$. Let (3.10) be true for $n = k$. Now,

$$\begin{aligned}
&N\Big(3^{-k-1}F(3^{k+1}x) - F(x), \\
&\qquad t\sum_{i=0}^{k} 3^{-i-1}(\phi_1(3^i x, 3^i(-x)) + \phi_1(3^i(-x), 3^i(3x)))\Big) \geq
\end{aligned}$$

$$\geq N\Big(3^{-k-1}F(3^{k+1}x) - 3^{-k}F(3^kx),$$
$$t3^{-k-1}\left(\phi_1(3^kx, 3^k(-x)) + \phi_1(3^k(-x), 3^k(3x))\right)\Big)*$$
$$N\Big(3^{-k}F(3^kx) - F(x),$$
$$t\sum_{i=0}^{k-1}3^{-i-1}\left(\phi_1(3^ix, 3^i(-x)) + \phi_1(3^i(-x), 3^i(3x))\right)\Big)$$
$$\geq 1-\epsilon \quad \text{[by (3.9)]}.$$

This completes the proof of (3.10).

Putting $t = t_0$, replacing n by p and x by 3^nx in (3.10), we get

$$N\Big(3^{-p}F(3^{n+p}x) - F(3^nx), t_0\sum_{i=0}^{p-1}3^{-i-1}(\phi_1(3^{n+i}x, 3^i(-3^nx))+ \phi_1(3^i(-3^nx), 3^i(3^{n+1}x)))\Big) \geq 1-\epsilon \tag{3.11}$$

for all $x \in X, n \geq 0, p > 0$. Again,

$$\begin{aligned}&\sum_{i=0}^{p-1}3^{-i-1}(\phi_1(3^{n+i}x, 3^i(-3^nx)) + \phi_1(3^i(-3^nx), 3^i(3^{n+1}x)))\\ &= \sum_{i=n}^{n+p-1}3^{n-i-1}(\phi_1(3^ix, 3^i(-x)) + \phi_1(3^i(-x), 3^i(3x))).\end{aligned} \tag{3.12}$$

Since $\sum_{i=0}^{\infty}\frac{\phi(3^ix, 3^iy)}{3^i}$ converges for all $x, y \in X$, for a given $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{3}\sum_{i=n}^{n+p-1}3^{-i}\left(\phi_1(3^ix, 3^i(-x)) + \phi_1(3^i(-x), 3^i(3x))\right) < \delta \tag{3.13}$$

for all $x \in X, n \geq n_0$ and $p > 0$. Now,

$$N\left(\frac{F(3^nx)}{3^n} - \frac{F(3^{n+p}x)}{3^{n+p}}, \delta\right)$$
$$\geq N\Big(F(3^nx) - 3^{-p}F(3^{n+p}x), t_0\sum_{i=0}^{p-1}3^{-i-1}\left(\phi_1(3^{n+i}x, 3^i(-3^nx))+\right.$$
$$\left.\phi_1(3^i(-3^nx), 3^i(3^{n+1}x))\right)\Big) \quad \text{[by (3.12), (3.13)]}$$
$$\geq 1-\epsilon \quad \text{[by (3.11)]}$$

for all $x \in X$, $n \geq n_0$, $p > 0$.

This shows that the sequence $\{3^{-n}F(3^nx)\}$ is a Cauchy sequence in Y. Since Y is a fuzzy Banach space, the sequence $\{3^{-n}F(3^nx)\}$ converges to some $A(x) \in Y$. So we can define a function $A : X \to Y$ by

$$A(x) := N - \lim_{n\to\infty} 3^{-n}F(3^n x) = N - \lim_{n\to\infty} 3^{-n}f(3^n x). \tag{3.14}$$

Replacing x and y by $3^n x$ in (3.5), we get

$$N\left(3^{-n}f(3^n 2x) - 3^{-n}g(3^n x) - 3^{-n}h(3^n x), 3^{-n}t\phi(3^n x, 3^n x)\right) \geq 1-\epsilon \tag{3.15}$$

for all $x \in X, t \geq t_0$. Since $\lim_{n\to\infty} 3^{-n}\phi(3^n x, 3^n x) = 0$, therefore for fixed $t > 0$ and $0 < \epsilon < 1$, there exists $n_0 \in \mathbb{N}$ such that

$$3^{-n}t_0\phi(3^n x, 3^n x) < \frac{t}{2} \tag{3.16}$$

for all $x \in X, n \geq n_0$. Now,

$$\begin{aligned} & N\left(3^{-n}g(3^n x) + 3^{-n}h(3^n x) - A(2x), t\right) \\ & \geq N\left(3^{-n}f(3^n 2x) - 3^{-n}g(3^n x) - 3^{-n}h(3^n x), 3^{-n}t_0\phi(3^n x, 3^n x)\right) * \\ & N\left(3^{-n}f(3^n 2x) - A(2x), \frac{t}{2}\right) \quad \text{[by (3.16)]}. \end{aligned}$$

The first term $\geq 1-\epsilon$ by (3.15) and last term tends to 1 as $n \to \infty$. Thus

$$\lim_{n\to\infty} N\left(3^{-n}g(3^n x) + 3^{-n}h(3^n x) - A(2x), t\right) = 1$$

for all $x \in X, t > 0$. Hence for all $x \in X$

$$N - \lim_{n\to\infty} \left(3^{-n}g(3^n x) + 3^{-n}h(3^n x)\right) = A(2x). \tag{3.17}$$

Again from definition of A we get

$$A(3^n x) = 3^n A(x), A(0) = 0, \tag{3.18}$$

$$\lim_{n\to\infty} N\left(A(x) - 3^{-n}f(3^n x), t\right) = 1 \tag{3.19}$$

for all $t > 0, x \in X$. Since $\lim_{n\to\infty} 3^{-n}\phi(3^n x, 3^n y) = 0$ for all $x, y \in X$, therefore for fixed $t > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$3^{-n}t_0\phi_1(3^{n+1}x, 3^n x) < \frac{t}{4} \tag{3.20}$$

for all $x \in X, n \geq n_1$. Replacing x and y by $3^{n+1}x$ and $3^n x$ respectively in (3.6) and for $t = t_0$, we get

$$N\left(2f(3^n 2x) - f(3^{n+1}x) - f(3^n x), t_0\phi_1(3^{n+1}x, 3^n x)\right) \geq 1-\epsilon \tag{3.21}$$

for all $x \in X$. Now,

$$\begin{aligned} & N(2A(2x) - 4A(x), t) \geq N\left(A(2x) - 3^{-n}f(3^n 2x), \frac{t}{8}\right) * \\ & N\left(A(3x) - 3^{-n}f(3^{n+1}x), \frac{t}{4}\right) * N\left(A(x) - 3^{-n}f(3^n x), \frac{t}{4}\right) * \\ & N\left(2f(3^n 2x) - f(3^{n+1}x) - f(3^n x), t_0\phi_1(3^{n+1}x, 3^n x)\right) \quad \text{[by (3.18), (3.20)]} \end{aligned}$$

From (3.19) and (3.21), we see that first three terms on RHS tend to 1 as $n \to \infty$ and last term $\geq 1 - \epsilon$. Therefore $N(2A(2x) - 4A(x), t) = 1$ for all $t > 0$. Thus for all $x \in X$

$$A(2x) = 2A(x). \tag{3.22}$$

Since $\lim_{n\to\infty} 3^{-n}\phi(3^n x, 3^n y) = 0$ for all $x, y \in X$, therefore for fixed $t > 0$ there exists $n_2 \in \mathbb{N}$ such that

$$3^{-n} t_0 \phi_1(3^n x, 3^n y) < \frac{t}{4} \tag{3.23}$$

for all $n \geq n_2$. Replacing x and y by $3^n x$ and $3^n y$ respectively in (3.6) and for $t = t_0$, we get

$$N\left(2f\left(\frac{3^n x + 3^n y}{2}\right) - f(3^n x) - f(3^n y), t_0\phi_1(3^n x, 3^n y)\right) \geq 1 - \epsilon \tag{3.24}$$

for all $x, y \in X$. Now,

$$N(A(x + y) - A(x) - A(y), t) \geq$$
$$N\left(A\left(\frac{x+y}{2}\right) - 3^{-n} f\left(3^n\left(\frac{x+y}{2}\right)\right), \frac{t}{8}\right) *$$
$$N\left(A(x) - 3^{-n} f(3^n x), \frac{t}{4}\right) * N\left(A(y) - 3^{-n} f(3^n y), \frac{t}{4}\right) *$$
$$N\left(2f\left(\frac{3^n x + 3^n y}{2}\right) - f(3^n x) - f(3^n y), t_0\phi_1(3^n x, 3^n y)\right) \quad [\text{by } (3.22), (3.23)].$$

From (3.19) and (3.24), we see that first three terms on RHS tend to 1 as $n \to \infty$ and last term $\geq 1 - \epsilon$. Therefore $N(A(x + y) - A(x) - A(y), t) = 1$ for all $t > 0$ that is, $A(x + y) = A(x) + A(y)$ for all $x, y \in X$.

Let (3.3) hold for some $\delta > 0, \alpha > 0$. Then by similar approach as in the beginning of proof we can deduce from (3.3) that

$$N\Bigg(3^{-n}F(3^n x) - F(x), \delta \sum_{i=0}^{n-1} 3^{-i-1}(\phi_1(3^i x, 3^i(-x)) + \phi_1(3^i(-x), 3^i(3x)))\Bigg) \geq \alpha \tag{3.25}$$

for all $x \in X, t \geq t_0, n \in \mathbb{N}$. Now for $t > 0$

$$N\left(F(x) - A(X), \delta \sum_{i=0}^{n-1} 3^{-i-1}(\phi_1(3^i x, 3^i(-x)) + \phi_1(3^i(-x), 3^i(3x))) + t\right) \geq$$

$$N\left(3^{-n}F(3^n x)-F(x),\delta\sum_{i=0}^{n-1}3^{-i-1}(\phi_1(3^i x,3^i(-x))+\phi_1(3^i(-x),3^i(3x)))\right)*$$
$$N\left(A(x)-3^{-n}F(3^n x),t\right).$$

Taking limit as $n\to\infty$, we get from (3.14) and (3.25)

$$N\left(F(x)-A(X),\delta\sum_{i=0}^{\infty}3^{-i-1}(\phi_1(3^i x,3^i(-x))+\phi_1(3^i(-x),3^i(3x)))+t\right)\geq\alpha.$$

Because of continuity of $N(x,\cdot)$ and taking limit as $t\to 0$, we get

$$N\left(F(x)-A(X),\delta\sum_{i=0}^{\infty}3^{-i-1}(\phi_1(3^i x,3^i(-x))+\phi_1(3^i(-x),3^i(3x)))\right)\geq\alpha,$$

it proves the result (3.4).

To prove the uniqueness of A let us assume that A' be another mapping satisfying (3.2) and (3.4). Then for a given $\epsilon>0$, we can find some $t_0>0$ such that

$$\begin{aligned}N\Bigg(&f(x)-A(X)-f(0),\\ &t\sum_{i=0}^{\infty}3^{-i-1}(\phi_1(3^i x,3^i(-x))+\phi_1(3^i(-x),3^i(3x)))\Bigg)\geq 1-\epsilon,\end{aligned}\tag{3.26}$$

$$\begin{aligned}N\Bigg(&f(x)-A'(X)-f(0),\\ &t\sum_{i=0}^{\infty}3^{-i-1}(\phi_1(3^i x,3^i(-x))+\phi_1(3^i(-x),3^i(3x)))\Bigg)\geq 1-\epsilon\end{aligned}\tag{3.27}$$

for all $x\in X, t\geq t_0$. Since $\sum\limits_{i=0}^{\infty}\frac{\phi(3^i x,3^i y)}{3^i}$ converges for all $x,y\in X$, therefore for a fixed $c>0$ there exists $n_3\in\mathbb{N}$ such that

$$\frac{t_0}{3}\sum_{i=n}^{\infty}3^{-i}(\phi_1(3^i x,3^i(-x))+\phi_1(3^i(-x),3^i(3x)))<\frac{c}{2}\tag{3.28}$$

for all $n\geq n_3$. Again,

$$\begin{aligned}&\sum_{i=n}^{\infty}3^{n-i-1}(\phi_1(3^i x,3^i(-x))+\phi_1(3^i(-x),3^i(3x)))\\ &=\sum_{i=0}^{\infty}3^{-i-1}(\phi_1(3^{n+i}x,3^{n+i}(-x))+\phi_1(3^{n+i}(-x),3^{n+i}(3x))).\end{aligned}\tag{3.29}$$

Now, for $c>0$

$$N(A(x)-A'(x),c)\geq$$

$$N\Bigg(A(3^nx)-f(3^nx)+f(0),$$
$$t_0\sum_{i=0}^{\infty}3^{-i-1}(\phi_1(3^{n+i}x,3^{n+i}(-x))+\phi_1(3^{n+i}(-x),3^{n+i}(3x)))\Bigg)*$$
$$N\Bigg(A'(3^nx)-f(3^nx)+f(0),t_0\sum_{i=0}^{\infty}3^{-i-1}(\phi_1(3^{n+i}x,3^{n+i}(-x))+$$
$$\phi_1(3^{n+i}(-x),3^{n+i}(3x)))\Bigg)\quad [\text{by } (3.18),(3.28),(3.29)]$$
$$\geq 1-\epsilon\quad [\text{by } (3.26),(3.27)].$$

It implies that $A(x)=A'(x)$ for all $x\in X$. This proves that A is unique.

Replacing x and y by $3^{n+1}x$ and 3^nx respectively in (3.5), we get

$$N\Bigg(3^{-n}f(3^{n+1}x+3^nx)-3^{-n}g(3^{n+1}x)-$$
$$3^{-n}h(3^nx),3^{-n}t\phi(3^{n+1}x,3^nx)\Bigg)\geq 1-\epsilon \tag{3.30}$$

for all $x\in X,n\geq 0,t\geq t_0$. Again, replacing x and y by 3^nx and $3^{n+1}x$ respectively in (3.5), we get

$$N\Big(3^{-n}f(3^nx+3^{n+1}x)-3^{-n}g(3^nx)-$$
$$3^{-n}h(3^{n+1}x),3^{-n}t\phi(3^nx,3^{n+1}x)\Big)\geq 1-\epsilon \tag{3.31}$$

for all $x\in X,n\geq 0,t\geq t_0$. Since $\lim_{n\to\infty}3^{-n}\phi(3^nx,3^ny)=0$ for all $x,y\in X$, therefore for fixed $t>0$, there exists $m\in\mathbb{N}$ such that

$$3^{-n}t_0(\phi(3^{n+1}x,3^nx)+\phi(3^nx,3^{n+1}x))<t \tag{3.32}$$

for all $n\geq m$. Now, by (3.32),

$$N\Big(3^{-n}((g(3^{n+1}x)-h(3^{n+1}x))-(g(3^nx)-h(3^nx))),t\Big)\geq$$
$$N\Big(3^{-n}f(3^{n+1}x+3^nx)-3^{-n}g(3^{n+1}x)-$$
$$3^{-n}h(3^nx),3^{-n}t_0\phi(3^{n+1}x,3^nx)\Big)* \tag{3.33}$$
$$N\Big(3^{-n}f(3^nx+3^{n+1}x)-3^{-n}g(3^nx)-$$
$$3^{-n}h(3^{n+1}x),3^{-n}t_0\phi(3^nx,3^{n+1}x)\Big)$$
$$\geq 1-\epsilon\quad [\text{by } (3.30),(3.31)$$

for all $x \in X, t > 0, n \geq m$. Let $c > 0$. Then we can find a positive integer $m' \geq m$ such that

$$N\left(3^{-m'}(g(3^m x) - h(3^m x)), c\right) \geq 1 - \epsilon. \tag{3.34}$$

Now, for all $n \geq m', x \in X$,

$$\begin{aligned}
&N\Big(3^{-n}(g(3^n x) - h(3^n x)), c\Big) \\
&\geq N\Big(3^{-m}(g(3^n x) - h(3^n x)), c\Big) \quad [\because n \geq m] \\
&\geq N\left(3^{-m'}(g(3^m x) - h(3^m x)), \frac{c}{3^{m'-m}(n-m+1)}\right) * \\
&\quad N\left(3^{-m}((g(3^{m+1}x) - h(3^{m+1}x)) - (g(3^m x) - h(3^m x))), \frac{c}{n-m+1}\right) * \\
&\qquad \vdots \qquad\qquad \vdots \\
&\quad N\Bigg(3^{-n+1}((g(3^n x) - h(3^n x)) - (g(3^{n-1}x) - h(3^{n-1}x))), \\
&\qquad\qquad \frac{c}{3^{n-m-1}(n-m+1)}\Bigg) \\
&\geq 1 - \epsilon \quad [\text{by } (3.33), (3.34)].
\end{aligned}$$

Therefore, for all $x \in X$,

$$N - \lim_{n\to\infty} \left(3^{-n}(g(3^n x) - h(3^n x))\right) = 0.$$

It implies that for all $x \in X$,

$$N - \lim_{n\to\infty} 3^{-n} g(3^n x) = N - \lim_{n\to\infty} 3^{-n} h(3^n x) = A(x) [\text{by}(3.17), (3.22)].$$

This completes the proof of the theorem. □

Corollary 3.1. *Let a be a fixed real number with $0 \leq a < 3$ and $\psi : (a, \infty) \to \mathbb{R}^+$ be a function such that for all $t, s > a$*

$$(i)\psi(ts) \leq \psi(t)\psi(s), (ii)\frac{\psi(3)}{3} < 1.$$

Let $f, g, h : X \to Y$ be mappings such that

$$\lim_{t\to\infty} N(f(x+y) - g(x) - h(x), t(\psi(\|x\|) + \psi(\|y\|))) = 1$$

for all x, y with $\|x\|, \|y\| > a$. Then there exists a unique mapping $A : X \to Y$ such that $A(x + y) = A(x) + A(y)$ for all $x, y \in X$ and if for some $\delta > 0, \alpha > 0$

$$N(f(x+y) - g(x) - h(y), \delta(\psi(\|x\|) + \psi(\|y\|))) \geq \alpha$$

for all $x, y \in X$, *with* $\|x\|, \|y\| > a$, *then*

$$N\left(f(x) - A(x) - f(0), \frac{\delta}{3-\psi(3)}\left[12\psi\left(\left\|\frac{x}{2}\right\|\right) + 4\psi\left(\left\|\frac{3x}{2}\right\|\right)\right]\right) \geq \alpha$$

for all $x \in X$ *with* $\|x\| > 2a$.

Proof. Define $\phi(x,y) = \psi(\|x\|) + \psi(\|y\|)$. Then

$$\widetilde{\phi}(x,y) \leq \frac{3}{3-\psi(3)}(\psi(\|x\|) + \psi(\|y\|)).$$

Therefore

$$\frac{1}{3}\left[\widetilde{\phi}\left(\frac{x}{2}, \frac{-x}{2}\right) + \widetilde{\phi}\left(\frac{-x}{2}, \frac{x}{2}\right) + \widetilde{\phi}\left(\frac{x}{2}, \frac{x}{2}\right) + 2\widetilde{\phi}\left(\frac{-x}{2}, \frac{-x}{2}\right) + \right.$$
$$\left.\widetilde{\phi}\left(\frac{-x}{2}, \frac{3x}{2}\right) + \widetilde{\phi}\left(\frac{3x}{2}, \frac{-x}{2}\right) + \widetilde{\phi}\left(\frac{3x}{2}, \frac{3x}{2}\right)\right]$$
$$\leq \frac{1}{3-\psi(3)}\left[12\psi\left(\left\|\frac{x}{2}\right\|\right) + 4\psi\left(\left\|\frac{3x}{2}\right\|\right)\right].$$

□

Corollary 3.2. *Let* $p < 1, 0 \leq a < 3$ *and* $f, g, h : X \to Y$ *be mappings such that*

$$\lim_{t\to\infty} N(f(x+y) - g(x) - h(x), t(\|x\|^p + \|y\|^p)) = 1$$

for all $x, y \in X$ *with* $\|x\|, \|y\| > a$. *Then there exists a unique mapping* $A : X \to Y$ *such that* $A(x+y) = A(x) + A(y)$ *for all* $x, y \in X$ *and if for some* $\delta > 0, \alpha > 0$

$$N(f(x+y) - g(x) - h(y), \delta(\|x\|^p + \|y\|^p)) \geq \alpha$$

for all $x, y \in X$, *with* $\|x\|, \|y\| > a$, *then*

$$N\left(f(x) - A(x) - f(0), \frac{4\delta(3+3^p)}{2^p(3-3^p)}\|x\|^p\right) \geq \alpha$$

for all $x \in X$ *with* $\|x\| > 2a$.

Proof. Define $\psi : (a, \infty) \to \mathbb{R}^+$ by $\psi(t) = t^p$. Then

$$\frac{1}{3-\psi(3)}\left[12\psi\left(\left\|\frac{x}{2}\right\|\right) + 4\psi\left(\left\|\frac{3x}{2}\right\|\right)\right] = \frac{4(3+3^p)}{2^p(3-3^p)}\|x\|^p.$$

□

Theorem 3.2. *Let* $\phi : X^2 \to \mathbb{R}^+$ *be a mapping such that* $\widetilde{\phi}(x,y) := \sum_{i=0}^{\infty} 3^i \phi(3^{-i}x, 3^{-i}y) < \infty$ *for all* $x, y \in X$.

Let $f, g, h : X \to Y$ *be mappings such that* $\lim_{t\to\infty} N(f(x+y) - g(x) - h(x), t\phi(x,y)) = 1$ *uniformly on* X^2. *Then there exists a unique mapping* $A : X \to Y$ *such that* $A(x+y) = A(x) + A(y)$ *for all* $x, y \in X$ *and if for some* $\delta > 0, \alpha > 0$

$$N(f(x+y) - g(x) - h(y), \delta\phi(x,y)) \geq \alpha$$

for all $x, y \in X$, *then*

$$N\left(f(x) - A(x) - f(0), \delta\left[\widetilde{\phi}\left(\frac{x}{6}, \frac{-x}{6}\right) + \widetilde{\phi}\left(\frac{-x}{6}, \frac{x}{6}\right) + \widetilde{\phi}\left(\frac{x}{6}, \frac{x}{6}\right) + 2\widetilde{\phi}\left(\frac{-x}{6}, \frac{-x}{6}\right) + \widetilde{\phi}\left(\frac{-x}{6}, \frac{x}{2}\right) + \widetilde{\phi}\left(\frac{x}{2}, \frac{-x}{6}\right) + \widetilde{\phi}\left(\frac{x}{2}, \frac{x}{2}\right)\right]\right) \geq \alpha$$

for all $x \in X$ *and*

$$N - \lim_{n\to\infty} 3^n(f(3^{-n}x) - f(0)) = A(x)$$

$$N - \lim_{n\to\infty} 3^n(g(3^{-n}x) - g(0)) = N - \lim_{n\to\infty} 3^n(h(3^{-n}x) - h(0)) = A(x).$$

Corollary 3.3. *Let* a *be a fixed real number with* $a > 3$ *and* $\psi : (0, a) \to \mathbb{R}^+$ *be a function such that for all* $0 < t, s < a$

$$(i)\psi(ts) \geq \psi(t)\psi(s), (ii)\frac{\psi(3)}{3} > 1.$$

Let $f, g, h : X \to Y$ *be mappings such that*

$$\lim_{t\to\infty} N(f(x+y) - g(x) - h(x), t(\psi(\|x\|) + \psi(\|y\|))) = 1$$

for all x, y *with* $\|x\|, \|y\| < a$. *Then there exists a unique mapping* $A : X \to Y$ *such that* $A(x+y) = A(x) + A(y)$ *for all* $x, y \in X$ *and if for some* $\delta > 0, \alpha > 0$

$$N(f(x+y) - g(x) - h(y), \delta(\psi(\|x\|) + \psi(\|y\|))) \geq \alpha$$

for all $x, y \in X$, *with* $\|x\|, \|y\| < a$, *then*

$$N\left(f(x) - A(x) - f(0), \frac{\delta\psi(3)}{\psi(3) - 3}\left[12\psi\left(\left\|\frac{x}{6}\right\|\right) + 4\psi\left(\left\|\frac{x}{2}\right\|\right)\right]\right) \geq \alpha$$

for all $x \in X$ *with* $\|x\| < a$.

Proof. Define $\phi(x, y) = \psi(\|x\|) + \psi(\|y\|)$. Then

$$\widetilde{\phi}(x, y) \leq \frac{\psi(3)}{\psi(3) - 3}(\psi(\|x\|) + \psi(\|y\|)).$$

Therefore

$$\widetilde{\phi}\left(\frac{x}{6}, \frac{-x}{6}\right) + \widetilde{\phi}\left(\frac{-x}{6}, \frac{x}{6}\right) + \widetilde{\phi}\left(\frac{x}{6}, \frac{x}{6}\right) + 2\widetilde{\phi}\left(\frac{-x}{6}, \frac{-x}{6}\right) + \widetilde{\phi}\left(\frac{-x}{6}, \frac{x}{2}\right) + \widetilde{\phi}\left(\frac{x}{2}, \frac{-x}{6}\right) + \widetilde{\phi}\left(\frac{x}{2}, \frac{x}{2}\right)$$

$$\leq \frac{\psi(3)}{\psi(3) - 3}\left[12\psi\left(\left\|\frac{x}{6}\right\|\right) + 4\psi\left(\left\|\frac{x}{2}\right\|\right)\right].$$

□

Corollary 3.4. *Let $p > 1, a > 3$ and $f, g, h : X \to Y$ be mappings such that*

$$\lim_{t\to\infty} N(f(x+y) - g(x) - h(x), t(\|x\|^p + \|y\|^p)) = 1$$

for all $x, y \in X$ with $0 \le \|x\|, \|y\| < a$. Then there exists a unique mapping $A : X \to Y$ such that $A(x+y) = A(x) + A(y)$ for all $x, y \in X$ and if for some $\delta > 0, \alpha > 0$

$$N(f(x+y) - g(x) - h(y), \delta(\|x\|^p + \|y\|^p)) \ge \alpha$$

for all $x, y \in X$, with $0 \le \|x\|, \|y\| < a$, then

$$N\left(f(x) - A(x) - f(0), \frac{4\delta(3^p+3)}{2^p(3^p-3)}\|x\|^p\right) \ge \alpha$$

for all $x \in X$ with $0 < \|x\| < a$.

Proof. Define $\psi : (0, a) \to \mathbb{R}^+$ by $\psi(t) = t^p$. Then

$$\frac{\psi(3)}{\psi(3)-3}\left[12\psi\left(\left\|\frac{x}{6}\right\|\right) + 4\psi\left(\left\|\frac{x}{2}\right\|\right)\right] = \frac{4(3^p+3)}{2^p(3^p-3)}\|x\|^p. \qquad \square$$

4. CONCLUSION

In this paper the generalized Hyers-Ulam-Rassias stability of the functional equation $f(x+y) = g(x) + h(y)$ has been studied in fuzzy Banach spaces. What could be the general solution of such functional equation and what are the properties of the general solution of this equation, it should be studied in future. Instead of crisp functional equation, if we consider fuzzy functional equation, how can we study the corresponding Hyers-Ulam stability property.

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