

# Common Fixed Point Theorems for Hybrid Pairs of Mappings using Implicit Relations

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ABSTRACT. In this paper, we prove coincidence and fixed point theorems for two hybrid pairs of single valued and multivalued mappings on noncomplete metric spaces. We prove our results without using conditions of compatibility and continuity of mappings. We also give an example to validate our result.

## 1. INTRODUCTION

Jungck [5] generalized the concept of weak commutativity introduced by Sessa [24]. He defined the notion of compatible maps in order to generalize the concept of weak commutativity and showed that the weakly commuting maps are compatible but the converse is not true. The concept of weakly commuting mappings was extended by Kaneko [6] for multivalued set up and extended the result of Jungck [4]. Kaneko and Sessa [7] extended the concept of compatible mappings for multivalued mappings and generalized the result of Kubiak [26].

Nadlar [17] published a paper on multivalued mappings. Since then, the fixed point theory for single valued and multivalued mappings has been studied extensively and applied to diverse problems. This theory provides techniques for solving a variety of applied problems in mathematical science and engineering [19, 23]. A number of generalization of Nadlar's results have appeared.

The study of noncompatible mappings was initiated by Pant [13–16]. He introduced  $R$ -weakly commutativity of mappings and compared  $R$ -weak commutativity and weak compatibility for single valued mappings. He showed that both  $R$ -weak commutativity and weak compatibility for single valued mappings are equivalent at the coincidence points. This idea of  $R$ -weak commutativity was independently extended to the settings of single valued and multivalued mappings by Shahzad and Kamran [12], Singh and Mishra [22].

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Recently, Aamri and Moutawakil [10] defined a property  $(E.A)$  for self maps which contains the class of noncompatible maps. They obtained some fixed point theorems for such mappings using property  $(E.A)$  under strict contractive conditions.

Kamran [25] extended property  $(E.A)$  in the settings of single valued and multivalued mappings and generalized the notion of  $(IT)$ -commutativity for such pairs. He introduced the notion of  $T$ -weakly commuting map and showed that for hybrid pairs of mappings,  $(IT)$ -commuting at coincidence points implies  $T$ -weakly commuting but the converse is not true. He also showed that for single valued mappings  $T$ -weak commutativity at the coincidence points is equivalent to the weak compatibility.

Many authors have proved fixed and coincidence point theorems for hybrid pairs of mappings including Asad and Ahmad [1], Fisher [2], Kubiacyk and Deshpande [8], Sharma and Deshpande [20], Kaneko and Sessa [7], Kamran [25], Shahzad and Kamran [12], Sharma et al. [21].

We have observed that proving fixed point theorems using an implicit relation is a good idea since it covers several contractive conditions rather than one contractive condition (see [11]).

In this paper, we prove coincidence point and fixed point theorems for hybrid pairs of single valued and multivalued mappings on noncomplete metric spaces. The mappings involved in our results are noncompatible and discontinuous. We also give an example to validate our main result. We improve and extend the result of Kamran [25].

## 2. PRELIMINARIES

Let  $(X, d)$  be a metric space and suppose that  $CB(X)$  denotes the set of non-empty, closed and bounded subsets of  $X$ .

For  $A, B \in CB(X)$ , we denote

$$D(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$$

$$D(x, A) = \inf \{d(x, a) : a \in A\}$$

$$H(A, B) = \max\{\sup \{D(a, B) : a \in A\}, \sup \{D(A, b) : b \in B\}\}$$

It is well known that  $(CB(X), H)$  is a metric space with the distance function  $H$ . Moreover,  $(CB(X), H)$  is complete in the event that  $(X, d)$  is complete.

**Definition 1** ([6]). Let  $(X, d)$  be a metric space,  $F : X \rightarrow CB(X)$  and  $T : X \rightarrow X$ . Then the pair  $\{F, T\}$  is said to be weakly commuting if for each  $x \in X$ ,  $TF(x) \in CB(X)$  and  $H(FTx, TFx) \leq D(Tx, Fx)$ .

**Definition 2** ([7]). Let  $(X, d)$  be a metric space,  $F : X \rightarrow CB(X)$  and  $T : X \rightarrow X$ . Then the pair  $\{F, T\}$  is said to be compatible if and only if  $TFx \in CB(X)$  for each  $x \in X$  and  $H(FTx_n, TFx_n) \rightarrow 0$ , whenever  $\{x_n\}$  is a sequence in  $X$  such that  $Fx_n \rightarrow M \in CB(X)$  and  $Tx_n \rightarrow t \in M$ .

**Definition 3** ([10]). The maps  $f : X \rightarrow X$  and  $g : X \rightarrow X$  are said to satisfy the property (E.A) if there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t \in X$ .

**Definition 4** ([25]). The maps  $f : X \rightarrow X$  and  $T : X \rightarrow CB(X)$  are said to satisfy the property (E.A) if there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} fx_n = t \in A = \lim_{n \rightarrow \infty} Tx_n$  for some  $t \in X$  and  $A \in CB(X)$ .

**Definition 5** ([18]). The mappings  $T : X \rightarrow X$  and  $F : X \rightarrow CB(X)$  are said to be (IT)–commuting at  $x \in X$  if  $TFx \subseteq FTx$ .

**Definition 6** ([25]). Let  $T : X \rightarrow CB(X)$ , The map  $f : X \rightarrow X$  is said to be  $T$ –weakly commuting at  $x \in X$  if  $ffx \in Tfx$ .

**Example 1.** Let  $X = (0, \infty)$  with the usual metric  $d$ . Define  $f : X \rightarrow X$  and  $F : X \rightarrow CB(X)$  by  $fx = 4x$  and  $Fx = [0, 2 + 4x]$  for all  $x \in X$ . Consider the sequence  $\{x_n\}$  in  $X$  given by  $x_n = 1 + \frac{1}{n}$ ,  $n = 1, 2, 3, \dots$

Then  $\lim_{n \rightarrow \infty} fx_n = 4 \in [0, 6] = \lim_{n \rightarrow \infty} Fx_n$  and  $\lim_{n \rightarrow \infty} H(fFx_n, Ffx_n) \neq 0$ .

Therefore,  $f$  and  $F$  satisfy property (E.A), but they are not compatible.

Also for all  $x \in X$ ,  $fx \in Fx$ ,  $ffx = 16x \in Ffx = [0, 2 + 16x]$ . Therefore  $f$  is  $F$ –weakly commuting.

Further,  $fFx = [0, 8 + 16x] \not\subseteq Ffx = [0, 2 + 16x]$ . Therefore  $f$  and  $F$  are not (IT)–commuting. Also note that  $f$  and  $F$  are not weakly compatible.

**Example 2.** Let  $X = [0, 1)$  with the usual metric  $d$ . Define  $f : X \rightarrow X$  and  $F : X \rightarrow CB(X)$  by  $fx = \frac{x}{2}$  and  $Fx = [0, x]$  for all  $x \in X$ . Consider the sequence  $\{x_n\}$  in  $X$  given by  $x_n = \frac{n-1}{2(n+1)}$ ,  $n = 1, 2, 3, \dots$

Then  $\lim_{n \rightarrow \infty} fx_n = \frac{1}{4} \in \left[0, \frac{1}{2}\right] = \lim_{n \rightarrow \infty} Fx_n$  and  $\lim_{n \rightarrow \infty} H(fFx_n, Ffx_n) = 0$ .

Therefore,  $f$  and  $F$  satisfy property (E.A) and the hybrid pair  $\{f, F\}$  is compatible. Also for all  $x \in X$ ,  $fx \in Fx$ ,  $ffx = \frac{x}{4} \in Ffx = \left[0, \frac{x}{2}\right]$ . Therefore  $f$  is  $F$ –weakly commuting. Also the pair  $\{f, F\}$  is (IT)–commuting because  $fFx \subseteq Ffx$ .

### 3. IMPLICIT RELATIONS

Let  $\Phi$  be the family of all continuous maps  $\varphi : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  such that:

( $\varphi_1$ )  $\varphi$  is non-increasing in variable  $t_6$ .

( $\varphi_2$ ) there exists  $h \in (0, 1)$  such that for all  $u, v \geq 0$  with

( $\varphi_a$ ) :  $\varphi(u, v, v, u, u+v, 0) \leq 0$ , or

( $\varphi_b$ ) :  $\varphi(u, v, u, v, 0, u+v) \leq 0$ ,

then we have  $u \leq hv$ .

( $\varphi_3$ )  $\varphi(u, u, 0, 0, 0, u) > 0$  for all  $u > 0$ .

**Example 3.**

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max \left\{ \frac{t_2^2 + t_3^2}{t_2 + t_3}, (t_5 + t_6), t_4 \right\}, \quad \alpha \in \left(0, \frac{1}{2}\right).$$

( $\varphi_1$ ) Obviously.

( $\varphi_2$ ) Let  $u, v \geq 0$ , then

$$\begin{aligned} (\varphi_a) \quad \varphi(u, v, v, u, u+v, 0) &= u - \alpha \max \{v, (u+v), u\} \\ &= u - \alpha(u+v) \\ &= (1-\alpha)u - \alpha v \end{aligned}$$

Therefore

$$\varphi(u, v, v, u, u+v, 0) \leq 0 \quad \Rightarrow \quad u \leq \frac{\alpha}{(1-\alpha)}v = hv,$$

where  $h = \frac{\alpha}{(1-\alpha)}$ ,  $0 < h < 1$ .

( $\varphi_3$ )  $\varphi(u, u, 0, 0, 0, u) = u - \alpha \max \{u, u\} = (1-\alpha)u > 0$  for all  $u > 0$ .

**Example 4.**

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \frac{\max \{t_4 - t_2, t_4 - t_3\}}{1 + t_5 t_6} - (1-\alpha)t_5 t_6$$

where  $\alpha \in (0, 1)$ ,  $a, b > 0$ ,  $a + b < 1$ .

( $\varphi_1$ ) Obviously.

( $\varphi_2$ ) Let  $u, v \geq 0$ , then

$$\begin{aligned} (\varphi_a) \quad \varphi(u, v, v, u, u+v, 0) &= u + \alpha(u-v) \\ &= (1+\alpha)u - \alpha v \leq 0 \\ &\Rightarrow u \leq \frac{\alpha}{1+\alpha}v = hv, \end{aligned}$$

where  $h = \frac{\alpha}{1+\alpha}$ ,  $0 < h < 1$ .

( $\varphi_3$ )  $\varphi(u, u, 0, 0, 0, u) = u + \alpha u > 0$  for all  $u > 0$ .

**Example 5.**

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^p - (\alpha t_2^p + \beta(t_3^p - t_4^p))(1 + t_5 t_6)$$

where  $\alpha, \beta \in (0, 1)$ ,  $\alpha + \beta < 1$  and  $p$  is an integer such that  $p \geq 1$ .

( $\varphi_1$ ) Obviously.

( $\varphi_2$ ) Let  $u, v \geq 0$ , then

$$\begin{aligned} (\varphi_a) \quad \varphi(u, v, v, u, u+v, 0) &= u^p - (\alpha v^p + \beta(v^p - u^p)) \\ &= (1+\beta)u^p - (\alpha+\beta)v^p \\ &\leq 0 \\ &\Rightarrow u \leq \left(\frac{\alpha+\beta}{1+\beta}\right)^{1/p} v = hv, \end{aligned}$$

where  $h = \left(\frac{\alpha+\beta}{1+\beta}\right)^{1/p}$ ,  $0 < h < 1$ .

( $\varphi_3$ )  $\varphi(u, u, 0, 0, 0, u) = u^p - \alpha u^p = (1-\alpha)u^p > 0$  for all  $u > 0$ .

**Example 6.**

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = 2t_1 - \frac{\max\{t_2, t_4, t_5\}}{1 + (t_2 - t_3)t_4} + at_5 - t_3t_6,$$

where  $a \in (0, 1)$ .

( $\varphi_1$ ) Obviously.

( $\varphi_2$ ) Let  $u, v \geq 0$ , then

$$\begin{aligned} (\varphi_a) \quad \varphi(u, v, v, u, u + v, 0) &= 2u - (u + v) + a(u + v) \\ &= (1 + a)u - (1 - a)v \\ &\leq 0 \\ &\Rightarrow u \leq \frac{1 - a}{1 + a}v = hv, \end{aligned}$$

where  $h = \frac{1 - a}{1 + a}$ ,  $0 < h < 1$ .

( $\varphi_3$ )  $\varphi(u, u, 0, 0, 0, u) = 2u - u = u > 0$  for all  $u > 0$ .

**Example 7.**

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = \beta t_1^p + \frac{\alpha \max\{(t_2^p - t_3^p), t_4^p\}}{1 + (\alpha t_4^p + \beta t_5^p)(t_2^p - t_3^p)} - \beta(t_3^p + t_6^p),$$

where  $\beta \in (0, 1)$ ,  $\alpha \geq 1$  and  $p$  is an integer such that  $p \geq 1$ .

( $\varphi_1$ ) Obviously.

( $\varphi_2$ ) Let  $u, v \geq 0$ , then

$$\begin{aligned} (\varphi_a) \quad \varphi(u, v, v, u, u + v, 0) &= \beta u^p + \alpha u^p - \beta v^p \\ &= (\beta + \alpha)u^p - \beta v^p \\ &\leq 0 \\ &\Rightarrow u \leq \left(\frac{\beta}{\beta + \alpha}\right)^{1/p} v = hv, \end{aligned}$$

where  $h = \left(\frac{\beta}{\beta + \alpha}\right)^{1/p}$ ,  $0 < h < 1$ .

( $\varphi_3$ )  $\varphi(u, u, 0, 0, 0, u) = \beta u^p + \alpha u^p - \beta u^p = \alpha u^p > 0$  for all  $u > 0$ .

**Example 8.**

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - q \max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\},$$

where  $q \in (0, 1)$ .

( $\varphi_1$ ) Obviously.

( $\varphi_2$ ) Let  $u, v \geq 0$ , then

$$(\varphi_a) \quad \varphi(u, v, v, u, u + v, 0) = u - q \max\left\{v, \frac{v + u}{2}, \frac{u + v}{2}\right\}.$$

If  $\max\left\{v, \frac{v + u}{2}, \frac{u + v}{2}\right\} = v$ , then

$$\varphi(u, v, v, u, u + v, 0) \leq 0 \Rightarrow u \leq qv = hv,$$

where  $h = q$ ,  $0 < h < 1$ .

If  $\max \left\{ v, \frac{v+u}{2}, \frac{u+v}{2} \right\} = \frac{v+u}{2}$ , then

$$\begin{aligned} \varphi(u, v, v, u, u+v, 0) &\leq 0 \\ \Rightarrow u - q \frac{v+u}{2} &\leq 0 \\ \Rightarrow u &\leq \frac{q}{2-q} v = hv, \end{aligned}$$

where  $h = \frac{q}{2-q}$ ,  $0 < h < 1$ .

( $\varphi_3$ )  $\varphi(u, u, 0, 0, 0, u) = u - qu = (1 - q)u \geq 0$  for all  $u > 0$ .

#### 4. MAIN RESULTS

**Theorem 1.** *Let  $(X, d)$  be a metric space. Let  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  be mappings satisfying the following conditions:*

- (1.1)  $F(X) \subseteq f(X)$ ,  $G(X) \subseteq g(X)$ ,
- (1.2)  $\varphi \left\{ \begin{array}{l} H(Fx, Gy), d(fy, gx), D(fy, Fx), \\ D(gx, Gy), D(fy, Gy), D(gx, Fx) \end{array} \right\} \leq 0$  for all  $x, y \in X$ ,  
where  $\varphi \in \Phi$ ,
- (1.3) the pairs  $\{G, f\}$  and  $\{F, g\}$  satisfy property (E.A), then
  - (i) there exists  $u, v \in X$  such that  $fu \in Gu$  and  $gv \in Fv$ .

Further if

- (1.4)  $f$  is  $G$ -weakly commuting and  $g$  is  $F$ -weakly commuting at their coincidence point, then
  - (ii) if  $fu = gv = z \in X$ , then  $fz \in Gz$  and  $gz \in Fz$ ,
  - (iii) if  $fz = gz$  then  $fz = gz \in Fz \cap Gz$ ,
  - (iv) if  $fz = gz = z$ , then  $z$  is a common fixed point of  $f, g, F$  and  $G$ .

*Proof.* Since the pair  $\{F, g\}$  satisfies property (E.A), there is a sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} gx_n = t \in A = \lim_{n \rightarrow \infty} Fx_n$$

for some  $t \in X$  and  $A \in CB(X)$ .

Since  $F(X)$  is closed,  $t \in F(X)$ . Therefore by (1.1) there exists  $u \in X$  such that  $t = fu$ .

By (1.2), we have

$$\varphi \left\{ \begin{array}{l} H(Fx_n, Gu), d(fu, gx_n), D(fu, Fx_n), \\ D(gx_n, Gu), D(fu, Gu), D(gx_n, Fx_n) \end{array} \right\} \leq 0.$$

Letting  $n \rightarrow \infty$ , we get

$$\begin{aligned} & \varphi \{H(A, Gu), 0, D(fu, A), D(fu, Gu), D(fu, Gu), D(fu, A)\} \\ &= \varphi \{H(A, Gu), 0, 0, D(fu, Gu), D(fu, Gu), 0\} \\ &\leq 0. \end{aligned}$$

Since  $fu \in A$ , it follows from the definition of Hausdorff metric that

$$\varphi \{D(fu, Gu), 0, 0, D(fu, Gu), D(fu, Gu), 0\} \leq 0.$$

Using  $(\varphi_a)$ , we get  $D(fu, Gu) \leq 0$ , which is a contradiction. Hence  $fu \in Gu$ .

Similarly, since the pair  $\{G, f\}$  satisfies property  $(E.A)$ , there is a sequence  $\{y_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} fy_n = q \in M = \lim_{n \rightarrow \infty} Gy_n$$

for some  $q \in X$  and  $M \in CB(X)$ .

Since  $G(X)$  is closed,  $q \in G(X)$ . Therefore by (1.1), there exists  $v \in X$  such that  $q = gv$ .

By (1.2), we have

$$\varphi \left\{ \begin{array}{l} H(Fv, Gy_n), d(fy_n, gv), D(fy_n, Fv), \\ D(gv, Gy_n), D(fy_n, Gy_n), D(gv, Fv) \end{array} \right\} \leq 0.$$

Letting  $n \rightarrow \infty$ , we get

$$\begin{aligned} & \varphi \{H(Fv, M), 0, D(gv, Fv), D(gv, M), D(gv, M), D(gv, Fv)\} \\ &= \varphi \{H(Fv, M), 0, D(gv, Fv), 0, 0, D(gv, Fv)\} \\ &\leq 0. \end{aligned}$$

Since  $gv \in M$ , it follows from the definition of Hausdorff metric that

$$\varphi \{D(gv, Fv), 0, D(gv, Fv), 0, 0, D(gv, Fv)\} \leq 0.$$

Using  $(\phi_b)$ , we get  $D(gv, Fv) \leq 0$ , which is a contradiction. Hence  $gv \in Fv$ .

This proves  $(i)$ .

Now since  $f$  is  $G$ -weakly commuting at  $u \in X$  and  $g$  is  $F$ -weakly commuting at  $v \in X$ , we have

$$ffu \in Gfu \text{ and } ggv \in Fgv.$$

If  $fu = gv = z \in X$ , then

$$fz \in Gz \text{ and } gz \in Fz.$$

If  $fz = gz$ , then

$$fz = gz \in Fz \cap Gz.$$

If  $fz = gz = z$ , then  $z$  is the common fixed point of  $f, g, F$  and  $G$ .

This proves the Theorem.  $\square$

Theorem 1 and the Examples 3-8 imply the following:

**Corollary 1.** *Let  $(X, d)$  be a metric space. Let  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  be mappings satisfying the conditions (1.1), (1.3), (1.4) and the following:*

$$(2.1) \quad H(Fx, Gy) \leq \alpha \max \left\{ \begin{array}{l} \frac{d^2(fy, gx) + D^2(fy, Fx)}{d(fy, gx) + D(fy, Fx)}, \\ D(fy, Gy) + D(gx, Fx), D(gx, Gy) \end{array} \right\}$$

for all  $x, y \in X$ , where  $\alpha \in (0, 1/2)$ .

Then  $f, g, F$  and  $G$  have a common fixed point.

**Corollary 2.** *Let  $(X, d)$  be a metric space. Let  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  be mappings satisfying the conditions (1.1), (1.3), (1.4) and the following:*

$$(3.1) \quad \begin{aligned} &H(Fx, Gy) \\ &\leq \alpha \frac{\max \{D(gx, Gy) - d(fy, gx), D(gx, Gy) - D(fy, Fx)\}}{1 + D(fy, Gy) D(gx, Fx)} \\ &\quad + (1 - \alpha) D(fy, Gy) D(gx, Fx) \end{aligned}$$

for all  $x, y \in X$ , where  $\alpha \in (0, 1)$

Then  $f, g, F$  and  $G$  have a common fixed point.

**Corollary 3.** *Let  $(X, d)$  be a metric space. Let  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  be mappings satisfying the conditions (1.1), (1.3), (1.4) and the following:*

$$(4.1) \quad \begin{aligned} &H^p(Fx, Gy) \\ &\leq (\alpha d^p(fy, gx) + \beta (D^p(fy, Fx) - D^p(gx, Gy))) \\ &\quad (1 + D(fy, Gy) D(gx, Fx)) \end{aligned}$$

for all  $x, y \in X$ , where  $\alpha, \beta \in (0, 1)$ ,  $\alpha + \beta < 1$  and  $p$  is an integer such that  $p \geq 1$ .

Then  $f, g, F$  and  $G$  have a common fixed point.

**Corollary 4.** *Let  $(X, d)$  be a metric space. Let  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  be mappings satisfying the conditions (1.1), (1.3), (1.4) and the following:*

$$(5.1) \quad \begin{aligned} &2H(Fx, Gy) \\ &\leq \frac{\max \{d(fy, gx), D(gx, Gy), D(fy, Gy)\}}{1 + (d(fy, gx) - D(fy, Fx)) D(gx, Gy)} - aD(fy, Gy) \\ &\quad + D(fy, Fx) D(gx, Fx) \end{aligned}$$

for all  $x, y \in X$ , where  $a \in (0, 1)$ .

Then  $f, g, F$  and  $G$  have a common fixed point.

**Corollary 5.** *Let  $(X, d)$  be a metric space. Let  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  be mappings satisfying the conditions (1.1), (1.3), (1.4) and the following:*

$$(6.1) \quad \begin{aligned} &\beta H^p(Fx, Gy) \\ &\leq \beta (d^p(fy, gx) + D^p(gx, Fx)) \end{aligned}$$



$$\frac{\alpha \max \{d^p(fy, gx) - D^p(fy, Fx), D^p(gx, Gy)\}}{1 + (\alpha D^p(gx, Gy) + \beta D^p(fy, Gy))(d^p(fy, gx) - D^p(fy, Fx))}$$

for all  $x, y \in X$ , where  $\beta \in (0, 1)$ ,  $\alpha \geq 1$  and  $p$  is an integer such that  $p \geq 1$ .

Then  $f, g, F$  and  $G$  have a common fixed point.

**Corollary 6.** Let  $(X, d)$  be a metric space. Let  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  be mappings satisfying the conditions (1.1), (1.3), (1.4) and the following:

$$(7.1) \quad H(Fx, Gy) \leq q \max \left\{ d(fy, gx), \frac{D(fy, Fx) + D(gx, Gy)}{D(fy, Gy), D(gx, Fx)} \right\}$$

for all  $x, y \in X$ , where  $q \in (0, 1)$ .

Then  $f, g, H$  and  $G$  have a common fixed point.

**Example 9.** Let  $X = [0, 4)$  with the usual metric  $d$ . Define the mappings  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  by

$$fx = \begin{cases} x, & 0 \leq x \leq 2 \\ \frac{x+3}{4}, & 2 < x < 4 \end{cases}, \quad Fx = \begin{cases} \left[0, \frac{x}{3}\right], & 0 \leq x \leq 2 \\ \left[1, \frac{x+12}{8}\right], & 2 < x < 4 \end{cases}$$

$$gx = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 2 \\ \frac{x+1}{2}, & 2 < x < 4 \end{cases}, \quad Gx = \begin{cases} \left[0, \frac{x}{2}\right], & 0 \leq x \leq 2 \\ \left[1, \frac{x+8}{8}\right], & 2 < x < 4 \end{cases}$$

and the map  $\varphi : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  such that

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max \left\{ t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2} \right\},$$

where

$$t_1 = H(Fx, Gy), \quad t_2 = d(fy, gx), \quad t_3 = D(fy, Fx),$$

$$t_4 = D(gx, Gy), \quad t_5 = D(fy, Gy), \quad t_6 = D(gx, Fx).$$

Then

1.  $F(X) \subseteq f(X), G(X) \subseteq g(X)$ .
2. The condition (2) is satisfied.
3. Consider the sequence  $\{x_n\}$  in  $X$  defined by  $x_n = 2 + \frac{1}{n}$ ,  $n = 1, 2, 3, \dots$ . Then

$$\lim_{n \rightarrow \infty} fx_n = \frac{5}{4} \in \lim_{n \rightarrow \infty} Gx_n = \left[1, \frac{5}{2}\right],$$

and

$$\lim_{n \rightarrow \infty} gx_n = \frac{3}{2} \in \lim_{n \rightarrow \infty} Fx_n = \left[1, \frac{7}{4}\right].$$

Therefore the pairs  $\{G, f\}$  and  $\{F, g\}$  satisfy the property (E.A).

4.  $f$  is  $G$ -weakly commuting at the coincidence point 0 and  $g$  is  $F$ -weakly commuting at the coincidence point 0.

Therefore all the conditions of the Theorem 1 are satisfied and 0 is the common fixed point of  $f, g, F$  and  $G$ .

If we put  $g = f$  in the Theorem 1, we get the following:

**Corollary 7.** *Let  $(X, d)$  be a metric space. Let  $f : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  be mappings satisfying the following conditions:*

- (8.1)  $F(X) \subseteq f(X), G(X) \subseteq f(X),$   
 (8.2)  $\varphi \left\{ \begin{array}{l} H(Fx, Gy), d(fy, fx), D(fy, Fx), \\ D(fx, Gy), D(fy, Gy), D(fx, Fx) \end{array} \right\} \leq 0$   
 for all  $x, y \in X$ , where  $\varphi \in \Phi,$   
 (8.3) the pairs  $\{G, f\}$  and  $\{F, f\}$  satisfy property (E.A).

Then

- (i) there exists  $u, v \in X$  such that  $fu \in Gu$  and  $fv \in Fv.$

Further if,

- (8.4)  $f$  is  $G$ -weakly commuting and  $F$ -weakly commuting at their coincidence point, then  
 (ii) if  $fu = fv = z \in X$ , then  $fz \in Fz \cap Gz,$   
 (iii) if  $fz = z$ , then  $z$  is a common fixed point of  $f, F$  and  $G.$

If we put  $F = G$  in Corollary 2, we get the following:

**Corollary 8.** *Let  $(X, d)$  be a metric space. Let  $f : X \rightarrow X$  and  $G : X \rightarrow CB(X)$  be mappings satisfying the following conditions:*

- (9.1)  $G(X) \subseteq f(X),$   
 (9.2)  $\varphi \left\{ \begin{array}{l} H(Gx, Gy), d(fy, gx), D(fy, Gx), \\ D(gx, Gy), D(fy, Gy), D(gx, Gx) \end{array} \right\} \leq 0$   
 for all  $x, y \in X$ , where  $\varphi \in \Phi,$   
 (9.3) the pair  $\{G, f\}$  satisfies property (E.A).

Then

- (i) there exists  $u \in X$  such that  $fu \in Gu.$

Further if,

- (9.4)  $f$  is  $G$ -weakly commuting at their coincidence point, then  
 (ii) if  $fu = z \in X$ , then  $fz \in Gz,$   
 (iii) if  $fz = z$ , then  $z$  is a common fixed point of  $f$  and  $G.$

Since a non-compatible hybrid pair satisfies the property (E.A), we have the following:

**Theorem 2.** *Let  $(X, d)$  be a metric space. Let  $f, g : X \rightarrow X$ , and  $F, G : X \rightarrow CB(X), \varphi \in \Phi$  be mappings satisfying (1.1) and (1.2). If the pairs  $\{G, f\}$  and  $\{F, g\}$  are non-compatible, then (i) of the Theorem 1 holds. Further if (1.4) is satisfied then (ii), (iii) and (iv) of the Theorem 1 hold.*

**Conclusion.** To prove common fixed point theorems for two hybrid pairs of mappings using the property  $(E.A)$ , the condition  $fu = gv = z \in X$  of Theorem 1 is essential. If we drop the property  $(E.A)$  and assume  $(X, d)$  as complete metric space, then the Theorem 1 can be proved without using  $fu = gv = z \in X$ .

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