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Global Behavior of a Rational Difference Equation with Quadratic Term

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Abstract. In this paper, we determine the forbidden set, introduce an explicit formula for the solutions and discuss the global behavior of all solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-1}}{b x_n - c x_{n-2}}, \qquad n = 0, 1, \dots$$

where a, b, c are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

1. Introduction

Li and Zhu [10] discussed the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + a}{x_n + x_{n-1}}, \quad n = 0, 1, \dots$$

where $a \in [0, \infty)$ and x_{-1}, x_0 are positive real numbers.

In [6] H. Sedaghat determined the global behavior of all solutions of the rational difference equations

$$x_{n+1} = \frac{a x_{n-1}}{x_n x_{n-1} + b}, \qquad x_{n+1} = \frac{a x_n x_{n-1}}{x_n + b x_{n-2}}, \qquad n = 0, 1, \dots$$

where $a, b > 0$.

In this paper, we derive the forbidden set, introduce an explicit formula for the solutions and discuss the global behavior of all solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-1}}{b x_n - c x_{n-2}}, \qquad n = 0, 1, \dots \tag{1.1}$$

where a, b, c are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

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2. FORBIDDEN SET AND SOLUTIONS OF QQUATION (1.1)

In this section we derive the forbidden set and introduce an explicit formula for the solutions of the difference equation (1.1).

Suppose that $x_0x_{-1} = 0$. Then we have the following:

If $x_0 = 0$ and $x_{-1} \neq 0$, then x_3 is undefined.
If $x_{-1} = 0$ and $x_0 \neq 0$, then x_2 is undefined.
If $x_{-2} = 0$ and $x_0x_{-1} \neq 0$, then $x_1 = \frac{a}{b}x_{-1} \neq 0$.

Therefore, we can start with the nonzero initial conditions x_{-1}, x_0, x_1, which we shall discuss.

Now suppose that $x_{-i} \neq 0$ for all $i = 0, 1, 2$. Using the substitution $r_n = \frac{x_n}{x_{n-1}}$, equation (1.1) becomes

$$r_{n+1} = \frac{ar_{n-1}}{br_nr_{n-1} - c}, \qquad n = 0, 1, \ldots \tag{2.1}$$

Now using the substitution $l_n = \frac{1}{r_nr_{n-1}}$, we can obtain the linear nonhomogeneous difference equation

$$l_{n+1} = -\frac{c}{a}l_n + \frac{b}{a}, \qquad l_0 = \frac{1}{r_0r_{-1}} = \frac{1}{\alpha}, \qquad n = 0, 1, \ldots \tag{2.2}$$

The solution of equation (2.2) is

$$\begin{aligned} l_n &= \Big(-\frac{c}{a}\Big)^n l_0 + \frac{b}{a}\sum_{i=0}^{n-1}\Big(-\frac{c}{a}\Big)^i \\ &= \frac{(-\frac{c}{a})^n a + b\alpha\sum_{i=0}^{n-1}(-\frac{c}{a})^i}{a\alpha}, \quad n = 0, 1, \ldots \end{aligned} \tag{2.3}$$

But $l_n = \frac{1}{r_nr_{n-1}} = \frac{x_{n-2}}{x_n}$. Therefore,

$$\frac{x_n}{x_{n-2}} = \frac{a\alpha}{(-\frac{c}{a})^n a + b\alpha\sum_{i=0}^{n-1}(-\frac{c}{a})^i}, \qquad n = 0, 1, \ldots \tag{2.4}$$

When $n = n_0$ for some $n_0 \in \mathbb{N}$, if we set $\alpha = \frac{c}{b\sum_{l=0}^{n_0}(-\frac{a}{c})^l}$ in equation (2.4), we obtain

$$\frac{x_n}{x_{n-2}} = \frac{c}{b}.$$

Therefore, y_{n_0+1} is undefined.

On the other hand, from equation (1.1) we have that

$$\frac{x_{n-1}}{x_{n-3}} = \frac{cx_n/x_{n-2}}{bx_n/x_{n-2} - a}$$

For a fixed $n_0 \in \mathbb{N}$, suppose that we have y_{n_0+1} is undefined. This implies that

$$bx_{n_0} - cx_{n_0-2} = 0.$$

That is

$$\frac{x_{n_0}}{x_{n_0-2}} = \frac{c}{b}.$$

Hence using equation (2.4), we have the following:

$$\frac{x_{n_0-1}}{x_{n_0-3}} = \frac{c^2}{b(c-a)} = \frac{c}{b\sum_{i=0}^{1}(-\frac{a}{c})^i},$$

$$\frac{x_{n_0-2}}{x_{n_0-4}} = \frac{c\frac{c^2}{b(c-a)}}{b\frac{c^2}{b(c-a)} - a} = \frac{c}{b\sum_{i=0}^{2}(-\frac{a}{c})^i}.$$

Now suppose that

$$\frac{x_{n_0-(k-1)}}{x_{n_0-(k+1)}} = \frac{c}{b\sum_{i=0}^{k-1}(-\frac{a}{c})^i}, \quad 0 \le k \le n_0.$$

Then

$$\begin{aligned}\frac{x_{n_0-(k)}}{x_{n_0-(k+2)}} &= \frac{c\frac{c}{b\sum_{i=0}^{k-1}(-\frac{a}{c})^i}}{b\frac{c}{b\sum_{i=0}^{k-1}(-\frac{a}{c})^i} - a} = \frac{c}{b}\frac{c}{c - a\sum_{i=0}^{k-1}(-\frac{a}{c})^i} \\ &= \frac{c}{b}\frac{1}{1+\sum_{i=0}^{k-1}(-\frac{a}{c})^{i+1}} = \frac{c}{b\sum_{i=0}^{k}(-\frac{a}{c})^i}.\end{aligned}$$

Therefore, for $k = n_0$ we have

$$\frac{x_0}{x_{-2}} = \alpha = \frac{c}{b\sum_{i=0}^{n_0}(-\frac{a}{c})^i}.$$

These observations lead us to conclude the following result.

Proposition 2.1. *The forbidden set F of equation (1.1) is*

$$F = \bigcup_{n=0}^{\infty}\Big\{(u_0,u_{-1},u_{-2}) : u_0 = u_{-2}\Big(\frac{c}{b\sum_{i=0}^{n}\big(\frac{-a}{c}\big)^i}\Big)\Big\}\cup$$
$$\Big\{(u_0,u_{-1},u_{-2}) : u_0 = 0\Big\} \cup \Big\{(u_0,u_{-1},u_{-2}) : u_{-1} = 0\Big\}.$$

Using equation (2.4), we obtain the following result.

Theorem 2.2. *Let x_{-2}, x_{-1} and x_0 be real numbers such that $(x_0,x_{-1},x_{-2}) \notin F$. If $a \neq c$, then the solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) is*

$$x_n = \begin{cases} x_{-1}\prod_{j=0}^{\frac{n-1}{2}} \frac{a+c}{\theta(a+c)\left(-\frac{c}{a}\right)^{2j+1}+b}, & n = 1,3,5,\dots \\ x_0\prod_{j=0}^{\frac{n-2}{2}} \frac{a+c}{\theta(a+c)\left(-\frac{c}{a}\right)^{2j+2}+b}, & n = 2,4,6,\dots \end{cases} \tag{2.5}$$

where $\theta = \frac{a+c-b\alpha}{\alpha(a+c)}$.

3. Global Behavior of Equation (1.1)

In this section, we investigate the global behavior of equation (1.1) with $a \neq c$, using the explicit formula of its solution.

We can write the solution of equation (1.1) in the form:

$$x_{2m+i} = x_{-2+i} \prod_{j=0}^{m} \beta_i(j), \quad i = 1,2 \text{ and } m = 0,1,\dots$$

where

$$\beta_i(j) = \frac{a+c}{\theta(a+c)(-\frac{c}{a})^{2j+i} + b}, \quad i = 1,2.$$

Theorem 3.1. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation* (1.1) *such that $(x_0, x_{-1}, x_{-2}) \notin F$. Then the following statements are true.*

(1) *If $a < c$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
(2) *If $a > c$, then we have the following:*
   (a) *If $\frac{a+c}{b} < 1$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
   (b) *If $\frac{a+c}{b} > 1$, then both $\{x_{2n}\}_{n=-1}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ are unbounded.*

Proof.

(1) If $a < c$, then $\beta_i(j)$ converges to 0 as $j \to \infty$, $i = 1,2$. It follows that, there exists $j_0 \in \mathbb{N}$ such that, $|\beta_i(j)| < \mu$, with some $0 < \mu < 1$ for all $j \geq j_0$. Therefore,

$$\begin{aligned} |x_{2m+i}| &= |x_{-2+i}| \left| \prod_{j=0}^{m} \beta_i(j) \right| \\ &= |x_{-2+i}| \left| \prod_{j=0}^{j_0-1} \beta_i(j) \right| \left| \prod_{j=j_0}^{m} \beta_i(j) \right| \\ &< |x_{-2+i}| \left| \prod_{j=0}^{j_0-1} \beta_i(j) \right| \mu^{m-j_0+1}. \end{aligned}$$

As m tends to infinity, the solution $\{x_n\}_{n=-2}^{\infty}$ converges to zero.

(2) Suppose that $a > c$. Then we have the following:
   (a) If $\frac{a+c}{b} < 1$, then $\beta_i(j)$ converges to $\frac{a+c}{b} < 1$ as $j \to \infty$, $i = 1,2$. Therefore, there exists $j_1 \in \mathbb{N}$ such that, $0 < \beta_i(j) < \mu_1$, with some $0 < \mu_1 < 1$ for all $j \geq j_1$ and the solution $\{x_n\}_{n=-2}^{\infty}$ converges to zero as in (1).
   (b) If $\frac{a+c}{b} > 1$, then $\beta_i(j)$ converges to $\frac{a+c}{b} > 1$ as $j \to \infty$, $i = 1,2$. Then there exists $j_0 \in \mathbb{N}$ such that, $\beta_i(j) > \nu$, for some $\nu > 1$ for all $j \geq j_0$.

Hence

$$\begin{aligned}|x_{2m+i}| &= |x_{-2+i}|\left|\prod_{j=0}^{m}\beta_i(j)\right|\\ &= |x_{-2+i}|\left|\prod_{j=0}^{j_0-1}\beta_i(j)\right|\prod_{j=j_0}^{m}\beta_i(j)\\ &> |x_{-2+i}|\left|\prod_{j=0}^{j_0-1}\beta_i(j)\right|\nu^{m-j_0+1}.\end{aligned}$$

Therefore, both of the subsequences $\{x_{2n}\}_{n=-1}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ are unbounded. □

4. Case $a + c = b$

In order to discuss the case when $a + c = b$ with $a > c$, we need to remember the behavior of equation (2.1) with $a + c = b$ with $a > c$.

Many authors [1–9] discussed the behavior of the solutions of some special cases of the equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + Cx_nx_{n-1}}, \quad n = 0, 1, \ldots$$

where A, B, C are real numbers, but with reducing the numbers of parameters to one or two.

We shall derive only some results concern the behavior of the solutions of equation (2.1), that we shall use.

The following theorem gives the solution of equation (2.1).

Theorem 4.1. *Let r_{-1}, r_0 be real numbers such that $r_{-1}r_0 = \alpha \neq \frac{c}{b\sum_{i=0}^{n}(-\frac{a}{c})^i}$ for any $n \in \mathbb{N}$. Then the solution of equation* (2.1) *is*

$$r_n = \begin{cases} r_{-1}\prod_{j=0}^{\frac{n-1}{2}} \dfrac{\theta(a+c)(-\frac{c}{a})^{2j} + b}{\theta(a+c)(-\frac{c}{a})^{2j+1} + b}, & n = 1, 3, 5, \ldots \\ r_0\prod_{j=0}^{\frac{n-2}{2}} \dfrac{\theta(a+c)(-\frac{c}{a})^{2j+1} + b}{\theta(a+c)(-\frac{c}{a})^{2j+2} + b}, & n = 2, 4, 6, \ldots \end{cases} \tag{4.1}$$

where $\theta = \frac{a+c-b\alpha}{\alpha(a+c)}$.

The solution of equation (2.1) can be written as:

$$r_{2m+i} = r_{-2+i}\prod_{j=0}^{m}\gamma_i(j), \quad i = 1, 2 \text{ and } m = 0, 1, \ldots$$

where

$$\gamma_i(j) = \frac{\theta(a+c)(-\frac{c}{a})^{2j+i-1} + b}{\theta(a+c)(-\frac{c}{a})^{2j+i} + b}, \quad i = 1, 2.$$

Now assume that $a+c=b$ and $\alpha \neq 0$. Then we have

$$\gamma_i(j) = \frac{\theta(-\frac{c}{a})^{2j+i-1}+1}{\theta(-\frac{c}{a})^{2j+i}+1}, \quad i=1,2.$$

Theorem 4.2. *Assume that $a > c$ and let $\{r_n\}_{n=-1}^{\infty}$ be a solution of equation (2.1) such that $\alpha \neq 1$ and $r_{-1}r_0 = \alpha \neq \frac{c}{b\sum_{i=0}^{n}(-\frac{a}{c})^i}$ for any $n \in N$. Then $\{r_{2m+1}\}_{m=-1}^{\infty}$ and $\{r_{2m}\}_{m=-1}^{\infty}$ converge to finite limits.*

Proof. Let $\{r_n\}_{n=-1}^{\infty}$ is a solution of equation (2.1) such that $r_{-1}r_0 = \alpha \neq \frac{c}{b\sum_{i=0}^{n}(-\frac{a}{c})^i}$ for any $n \in N$.

The condition $\alpha \neq 1$ (where $\frac{a+c}{b} = 1$) ensures that the solution $\{r_n\}_{n=-1}^{\infty}$ is not a period-2 solution.

We claim that, there exists $j_0 \in \mathbb{N}$ such that $\gamma_i(j) > 0$ for all $j \geq j_0$.

For, let $a_i(j) = \theta(-\frac{c}{a})^{2j+i-1}+1$ and $b_i(j) = \theta(-\frac{c}{a})^{2j+i}+1$. Then we can write

$$\gamma_i(j) = \frac{\theta(-\frac{c}{a})^{2j+i-1}+1}{\theta(-\frac{c}{a})^{2j+i}+1} = \frac{a_i(j)}{b_i(j)}, \quad i=1,2.$$

We have the following situations:

- If $\theta < 0$, then, we have the following:
  - If $i=1$, then $b_1(j) > 0$ for all $j \in \mathbb{N}$. But as $a_1(j)$ converges to 1, there exists $j_1 \in \mathbb{N}$ such that $a_1(j) > 0$ for all $j \geq j_1$. Therefore, $\gamma_1(j) = \frac{a_1(j)}{b_1(j)} > 0$ for all $j \geq j_1$.
  - If $i=2$, then $a_2(j) > 0$ for all $j \in \mathbb{N}$. But as $b_2(j)$ converges to 1, there exists $j_2 \in \mathbb{N}$ such that $b_2(j) > 0$ for all $j \geq j_2$. Therefore, $\gamma_2(j) = \frac{a_2(j)}{b_2(j)} > 0$ for all $j \geq j_2$.

  In all cases, there exists a natural number $j_0 = \max\{j_1, j_2\}$ such that $\gamma_i(j) = \frac{a_i(j)}{b_i(j)} > 0$, $i=1,2$ for all $j \geq j_0$.
- If $\theta > 0$, the situation is similar and will be omitted.

Claim is complete.

Now for each $i \in \{1,2\}$, we have for large m

$$\begin{aligned}
r_{2m+i} &= r_{-2+i}\prod_{j=0}^{m}\frac{\theta(-\frac{c}{a})^{2j+i-1}+1}{\theta(-\frac{c}{a})^{2j+i}+1} \\
&= r_{-2+i}\prod_{j=0}^{m}\gamma_i(j) \\
&= r_{-2+i}\prod_{j=0}^{j_0-1}\gamma_i(j)\prod_{j=j_0}^{m}\gamma_i(j) \\
&= r_{-2+i}\prod_{j=0}^{j_0-1}\gamma_i(j)\exp\Big(\sum_{j=j_0}^{m}\ln\gamma_i(j)\Big)
\end{aligned}$$

We shall test the convergence of the series $\sum_{j=j_0}^{\infty} |\ln \gamma_i(j)|$.

Since $\lim_{j\to\infty} \left|\frac{\ln \gamma_i(j+1)}{\ln \gamma_i(j)}\right| = \frac{0}{0}$, using L'Hospital's rule we obtain

$$\lim_{j\to\infty} \left|\frac{\ln \gamma_i(j+1)}{\ln \gamma_i(j)}\right| = (\frac{c}{a})^2 < 1.$$

Then from d'Alembert's test that the series $\sum_{j=j_0}^{\infty} |\ln \gamma_i(j)|$ is convergent, it follows that there exist 2 real numbers $\rho_i \in \mathbb{R}$ such that

$$\lim_{m\to\infty} r_{2m+i} = \rho_i, \quad i \in \{0,1\}.$$ □

Now we are ready to introduce the main results in this section.

Theorem 4.3. *Assume that* $\{x_n\}_{n=-2}^{\infty}$ *is a solution of equation* (1.1) *such that* $(x_0, x_{-1}, x_{-2}) \notin F$ *and let* $a + c = b$. *If* $\alpha = 1$, *then* $\{x_n\}_{n=-2}^{\infty}$ *is eventually periodic solution with period* 2.

Proof. Assume that $a + c = b$. If $\alpha = 1$, then $\theta = 0$. Therefore,

$$\begin{aligned} x_{2m+i} &= x_{-2+i} \prod_{j=0}^{m} \frac{a+c}{\theta(a+c)(-\frac{c}{a})^{2j+i} + b} \\ &= x_{-2+i}, \quad i = 1, 2 \text{ and } m = 0, 1, \dots \end{aligned}$$ □

Theorem 4.4. *Assume that* $\{x_n\}_{n=-2}^{\infty}$ *is a solution of equation* (1.1) *such that* $(x_0, x_{-1}, x_{-2}) \notin F$ *and let* $a + c = b$. *If* $\alpha \neq 1$, *then we have the following:*

(1) *If* $a < c$, *then* $\{x_n\}_{n=-2}^{\infty}$ *converges to zero.*
(2) *If* $a > c$, *then* $\{x_n\}_{n=-2}^{\infty}$ *converges to a period-*2 *solution* $\{\mu_0, \mu_1\}$ *such that* $\mu_1 = \mu_0\rho_1$, *where* ρ_1 *is as in Theorem* (4.2).

Proof.

(1) The proof is similar to that in theorem (3.1).
(2) Suppose that $a+c = b$ and $a > c$, then $\beta_i(j) = \frac{1}{\theta(-\frac{c}{a})^{2j+i}+1}$ converges to 1, $i = 1, 2$.

    By an argument similar to that in theorem (4.2), there exists $j_0 \in \mathbb{N}$ such that, $\beta_i(j) > 0$, for all $j \geq j_0$.

    Hence

$$\begin{aligned} x_{2m+i} &= x_{-2+i} \prod_{j=0}^{m} \beta_i(j) = x_{-2+i} \prod_{j=0}^{j_0-1} \beta_i(j) \prod_{j=j_0}^{m} \beta_i(j) \\ &= x_{-2+i} \prod_{j=0}^{j_0-1} \beta_i(j) \exp\Big(\sum_{j=j_0}^{m} \ln \beta_i(j)\Big). \end{aligned}$$

    We shall test the convergence of the series $\sum_{j=j_0}^{\infty} |\ln \beta_i(j)|$.

Since $\lim_{j\to\infty}\left|\frac{\ln\beta_i(j+1)}{\ln\beta_i(j)}\right| = \frac{0}{0}$, using L'Hospital's rule we obtain

$$\lim_{j\to\infty}\left|\frac{\ln\beta_i(j+1)}{\ln\beta_i(j)}\right| = \left(\frac{c}{a}\right)^2 < 1.$$

It follows from D' Alemberts' test that the series $\sum_{j=j_0}^{\infty}|\ln\beta_i(j)|$ is convergent.

This ensures that there are two real numbers μ_0, μ_1 such that

$$\lim_{m\to\infty} x_{2m+i} = \mu_i, \quad i \in \{0,1\}.$$

Moreover, as $x_{2m+1} = x_{2m}r_{2m+1}$, then $\mu_1 = \mu_0\rho_1$ where

$$\rho_1 = r_{-1}\prod_{j=0}^{\infty}\frac{\theta(-\frac{c}{a})^{2j}+1}{\theta(-\frac{c}{a})^{2j+1}+1} \quad \text{and} \quad \mu_0 = x_0\prod_{j=1}^{\infty}\frac{1}{\theta(-\frac{c}{a})^{2j}+1}. \qquad \square$$

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