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Invariants of Topological G-Conjugacy on G-Spaces

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Abstract. In this paper we present invariant dynamical properties under G-conjugacy. Moreover we introduce the notions of G-minimal systems, strong G-shadowing property and limit G-shadowing property and we show that these properties are invariant under topological G-conjugacy.

1. Introduction

The dynamical systems is one of the popular areas of researches in Mathematics. Many researchers across the globe are contributing towards the researches on dynamical properties of maps in topological dynamical systems [1]. In order to classify topological dynamical systems, we need a notion of equivalence. For this aim we use the notion of topological conjugacy[8]. Here we are going to study those discrete dynamical system (X, T) which underlying space X is a dynamical space itself. We will see that the basic dynamic makes some changes on the dynamical properties of T effectivelly. More presisely let X be a metric space, G be a topological group, and $\theta : G \times X \to X$ be a map. The triple (X, G, θ) is called a metric G-space if the following three conditions are satisfied:

(1) $\theta(e, x) = x$ for all $x \in X$, where e is the identity of G;
(2) $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $x \in X$ and for all $g, h \in G$;
(3) θ is continuous.

In this paper X is a metric G-space. For each $g \in G$ we define the map $\theta_g : X \to X$ by

$$\theta_g(x) = \theta(g, x), \quad x \in X.$$

If $Y \subseteq X$, then Y is G-invariant if $gY = Y$ for all $g \in G$.

Let $A \subseteq X$ be given. Then the G-orbit of A is defined by

$$G(A) = GA = \{ga \mid g \in G, a \in A\}.$$

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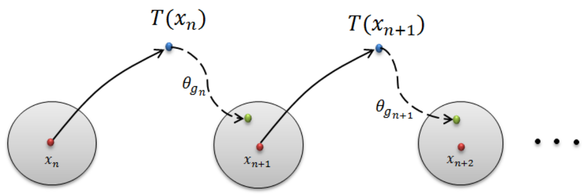
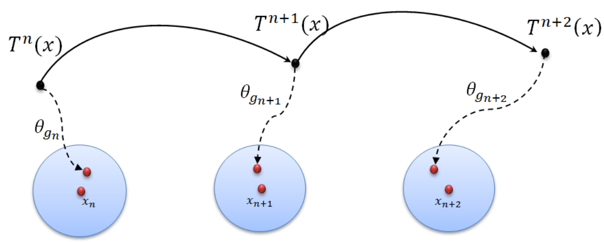
If $x \in X$, then Gx is the G-orbit passing through x. Clearly each G-orbit is a G-invariant subset of X. The orbit space for the action of G on X is the quotient topological space X/G, especially if G is compact then the quotient map $\pi : X \to X/G$ with $\pi(x) = Gx$ is an open, closed and proper(the inverse image of each compact set is compact) map and X/G is a Hausdorff space [4].

Given two metric G-spaces X, Y, a map $T : X \to Y$ is called G-equivariant if for all $x \in X$ and $g \in G$ we have $T(gx) = gT(x)$ and it is called G-pseudoequivariant if for all $x \in X$ we have $T(Gx) = G(Tx)$ [4].

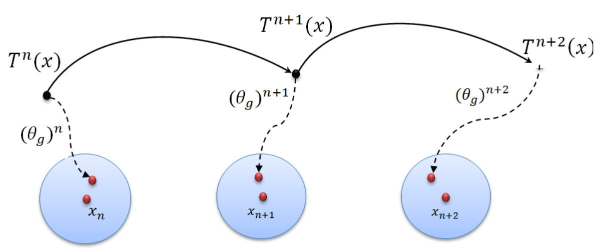
Let T be a continuous map. Then a sequence $\{x_n \in X : n \in \mathbb{Z}\}$ is called a (δ, G)-pseudo orbit for T if for each n , there exists $g_n \in G$ such that $d(g_nT(x_n), x_{n+1}) < \delta$.



A (δ, G)-pseudo orbit $\{x_n\}$ for T is said to be ϵ-shadowed by a point $x \in X$ if for each n , there exists $g_n \in G$ such that $d(g_nT^n(x), x_n) < \epsilon$ [6].



A (δ, G)-pseudo orbit $\{x_n\}$ for T is said to be strongly ϵ-shadowed by a point $(g, x) \in G \times X$ if $d(g^nT^n(x), x_n) < \epsilon$ for each $n \in \mathbb{Z}$.



The dynamical system T has the *G-shadowing property* (GSP) on X if for each $\epsilon > 0$ there exists $\delta > 0$ such that for any (δ, G)-pseudo orbit for

T there is a point $x \in X$ such that for each n , there is $g_n \in G$ so that $d(T^n(x), g_n x_n) < \epsilon$[7].

Definition 1.1. The dynamical system T has the *strong G-shadowing property* (SGSP) on X if for each $\epsilon > 0$ there exists $\delta > 0$ such that any (δ, G)-pseudo orbit for T be strongly G-shadowed with a point $(g, x) \in G \times X$.

[[2]] Let (X, d) be a compact metric G-space with G compact. Then the familly $\{\theta_g : \ g \in G\}$ is equicontinuous.

Theorem 1.1. *Let T be a homeomorphism on a compact metric G-space (X, d) with G compact. Then*

(1) *$T \in GSh$ if and only if $T^{-1} \in GSh$,*
(2) *$T \in SGSh$ if and only if $T^{-1} \in SGSh$.*

Proof. Suppose that $T \in$ GSh. Given $\epsilon > 0$ there exists $\delta > 0$ such that any (δ, G)-pseudo orbit for T is (ϵ, G)-shadowed with some point in X. By lemma 1 there exists $\eta > 0$ such that for any $x, y \in X$ the inequality $d(x, y) < \eta$ implies that $d(gx, gy) < \delta$ for each $g \in G$.

Choose $\gamma > 0$ so that for any $x, y \in X$ the inequality $d(x, y) < \gamma$ implies that $d(T(x), T(y)) < \eta$. Now let $\{x_n\}$ be a (δ, G)-pseudo orbit for T^{-1} and put $y_n = x_{-n}$. So for each $n \in \mathbb{Z}$ there is $g_n \in G$ so that $d(g_{-n}T^{-1}(x_{-n}), x_{-n+1}) < \delta$.

Hence for each $n \in \mathbb{Z}$ there exists $h_n \in G$ such that $d(h_{-n}x_{-n}, T(x_{-n+1})) < \eta$. So $d(y_n, h_{-n}^{-1}T(y_{n-1})) < \delta$. Thus there exists a point $x \in X$ and a sequence $\{k_n\}_{n \in \mathbb{Z}} \subset G$ so that $d(k_n T^n(x), y_n) < \epsilon$ for each $n \in \mathbb{Z}$. So $d(l_n T^{-n}(x), x_n) < \epsilon$ where $l_n = k_{-n}$, that is $\{x_n\}$ is (ϵ, G)-shadowed. □

A homeomorphism $T : X \to X$ is said to be G-expansive provided that there exists $\delta > 0$ such that for every $x, y \in X$ with $G(x) \neq G(y)$, there exists $n \in \mathbb{Z}$ such that

$$d(T^n(u), T^n(v)) > \delta \ \text{ for all } u \in G(x), \ v \in G(y).$$

The constant δ is called a G-expansive constant for T [3].

A homeomorphism $T : X \to X$ on a metric G-space X is said to be weak G-expansive provided that there exists $\epsilon > 0$ such that for every $x, y \in X$ with $Gx \neq Gy$ if $u \in Gx$ and $v \in Gy$, then there exists $n = n(u, v) \in \mathbb{Z}$ such that

$$d(T^n(u), T^n(v)) > \delta.$$

The constant δ is called a weak G-expansive constant for T[2]. A finite open cover α of X is called a G-generator for T if for every bisequence $\{A_n\}_{-\infty}^{\infty} \subset \alpha$, the set $\cap_{-\infty}^{\infty} T^{-n}(\overline{A_n})$ contains at most one G-orbit [5].

Theorem 1.2. *Let $T : X \to X$ be a homeomorphism on a compact metric G-space. Then T is weak G-expansive if and only if T has a G-generator.*

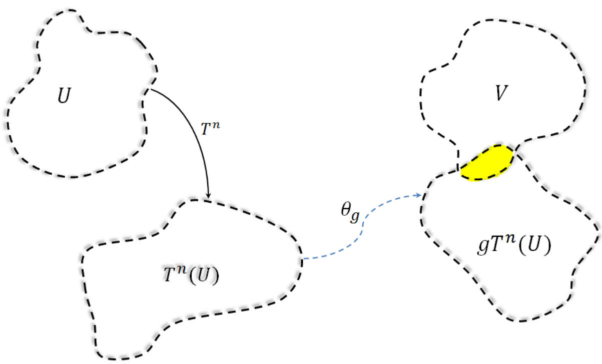
Proof. Suppose that T is weak G-expansive. We show that any finite cover α by open balls of radius $\delta/2$. If $x, y \in \cap_{-\infty}^{\infty} T^{-n}(\overline{A_n})$ then for each $n \in \mathbb{Z}$ we have $d(T^n x, T^n y) \leq \delta$.

Conversely suppose that α is a G-generator for T and δ is the lebesgue number for α. If T is not weak G-expansive then there exist $x, y \in X$ so that $Gx \neq Gy$ and there exist $u \in Gx$ and $v \in Gy$ such that for each $n \in \mathbb{Z}$ we deduce $d(T^n(u), T^n(v)) \leq \delta$. Thus for each n there exists $A_n \in \alpha$ so that $T^n(u), T^n(v) \in A_n$. Hence

$$u, v \in \cap_{-\infty}^{\infty} T^{-n}(A_n) \cap_{-\infty}^{\infty} T^{-n}(\overline{A_n}).$$

which is a contradiction. □

For given points $x, y \in X$ we write $x \overset{\epsilon}{\rightsquigarrow}_G y$ if there exist finite (ϵ, G)-pseudo orbits $x = x_0, x_1, \ldots, x_n = y$ and $y = y_0, y_1, \ldots, y_m = x$ for T . If for every $\epsilon > 0$, $x \overset{\epsilon}{\rightsquigarrow} y$, then x is said to be G-related to y (denoted by $x \sim_G y$). A point x is said to be a G-chain recurrent point of T if $x \sim_G x$. $CR_G(T)$ is denoted by the set of all G-chain recurrent points of T . A homeomorphism $T : X \to X$ is called topologically G-transitive provided that for every nonempty open subsets U and V of X, there exist an integer $n > 0$ and $g \in G$ such that $(gT^n(U)) \cap V \neq \emptyset$.



A homeomorphism $T : X \to X$ is said to be topologically G-mixing provided that for every nonempty open subsets U and V of X, there exists an integer N such that for each $n \geq N$, there is $g_n \in G$ satisfying $(g_n T^n(U)) \cap V \neq \emptyset$ [2].

We say that a homeomorphism $T : X \to X$ is minimal if for each $x \in X$ there exists $g \in G$ such that $\overline{\mathcal{O}(T, gx)} = X$ where $\mathcal{O}(T, x) = \{T^n(x) : n \in \mathbb{Z}\}$ is the orbit of x, that is any G-orbit in X contains a point with dense orbit [2].

We say that a homeomorphism $T : X \to X$ is G-chain transitive if for every $x, y \in X$ we have $x \sim_G y$.

We say that T is (ϵ, G)-chain mixing if there is an $N > 0$ such that for any $x, y \in X$ and any $n \geq N$, there is an (ϵ, G)-pseudo orbit from x to y of length exactly n. The map T is chain mixing if it is ϵ-chain mixing for every $\epsilon > 0$ [2].

In the next section we show that all of these properties are invariant under conjugacy.

2. Topological G-conjugacy

Let $T : X \to X$ and $S : Y \to Y$ be two homeomorphisms of compact G-spaces X and Y. We say that T is topologically G-conjugate to S if there exists a G-equivariant homeomorphism $\phi : X \to Y$ such that $\phi \circ T = S \circ \phi$. The homeomorphism ϕ is called a G-conjugacy [3].

Theorem 2.1. *Let $T : X \to X$ and $S : Y \to Y$ be two homeomorphisms. If T and S are G-conjugate then the following are hold*

1. *T is topologically G-transitive if and only if S is so.*
2. *T is topologically G-mixing if and only if S is so.*
3. *T is weak G-expansive if and only if S is so.*
4. *T has the GSP if and only if so does S.*
5. *T is G-chain transitive if and only if S is so.*
6. *T is G-chain recurrent if and only if S is so.*
7. *T is G-chain mixing if and only if S is so.*
8. *T is minimal if and only if S is so.*

Proof. Suppose that $\phi X \to Y$ is a topological G-conjugacy such that $\phi \circ T = S \circ \phi$.

1. Let T be topologically G-transitive and let U and V be two open subsets of Y. Then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are open subsets of X. Therefore there exists $g \in G$ and $n > 0$ such that $gT^n(\phi^{-1}(U)) \cap \phi^{-1}(V) \neq \emptyset$. So we have $g\phi^{-1}(S^nU)) \cap \phi^{-1}(V) \neq \emptyset$ and therefore $h(S^nU)) \cap V \neq \emptyset$ for some $h \in G$.
2. Let T be topologically G-mixing and let U and V be two open subsets of Y. Then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are open subsets of X. Therefore there exists $N > 0$ such that for each $n \geq N$ there exists $g_n \in G$ so that $g_nT^n(\phi^{-1}(U)) \cap \phi^{-1}(V) \neq \emptyset$. Hence for each $n \geq N$ we have $g_n\phi^{-1}(S^nU)) \cap \phi^{-1}(V) \neq \emptyset$ and therefore $h_n(S^nU)) \cap V \neq \emptyset$ for some $h_n \in G$.
3. Let ϵ be a weak G-expansive constant of T. Then there exists $\delta > 0$ such that $d'(x, y) < \delta$ implies that $d(\phi^(x), \phi^{-1}(y)) < \epsilon$ for all $x, y \in Y$. Let $Gx \neq Gy$, $u \in G(x)$, $v \in G(y)$, and for each $n \in \mathbb{Z}$, $d'(S^n(u), S^n(u)) \leq \delta$. Thus for each $n \in \mathbb{Z}$, $d'(\phi^{-1}S^n(u), \phi^{-1}S^n(u)) \leq \epsilon$.

   Hence $d'(T^n\phi^{-1}(u), T^n\phi^{-1}(u)) \leq \epsilon$ for each $n \in \mathbb{Z}$. Therefore $G\phi^{-1}(x) = G^{-1}(y)$. So $\phi^{-1}(Gx) = \phi^{-1}(Gy)$ which is a contradiction.

(4) Given $\epsilon > 0$ there exists $\epsilon_1 > 0$ such that $d(x, y) < \epsilon_1$ implies that $d'(\phi(x), \phi(y)) < \epsilon$ for all $x, y \in X$. If T has the GSP, then there is $\delta_1 > 0$ such that each (δ_1, G)-pseudo orbit $\{x_n\}$ for T is ϵ-traced by some point. Hence there exists $\delta > 0$ such that the inequality $d'(x, y) < \delta$ implies to $d(\phi^{-1}(x), \phi^{-1}(y)) < \delta_1$ for all $x, y \in Y$. Let $\{y_n\} \subset Y$ be a (δ, G)-pseudo orbit for S. Thus for each n there exists $g_n \in G$ such that $d'(g_n S(y_n), y_{n+1}) < \delta$. Put $x_n = \phi^{-1}(y_n)$. Then there is a sequence $h_n \in G$ so that

$$d(h_n T(x_n), x_{n+1}) = d(\phi^{-1}(g_n S(y_n)), \phi^{-1} y_{n+1}) < \delta_1.$$

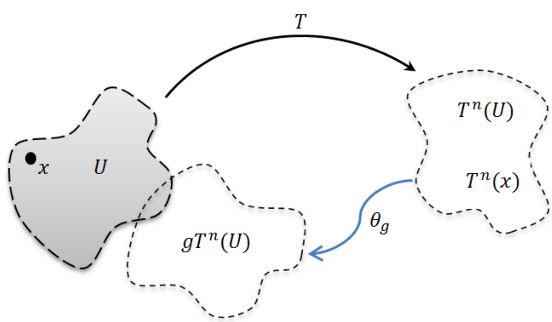
Hence $\{x_n\}$ is a (δ_1, G)-pseudo orbit for T and there exist $x \in X$ and a sequence $k_n \in G$ such that $d'(T^n(x), k_n x_n) < \epsilon_1$. Therefore there exists a sequence $l_n \in G$ so that

$$d'(\phi \circ T^n(x), \phi(k_n x_n)) = d'(S^n(\phi(x)), l_n y_n) < \epsilon.$$

(5), (6) As proof of part 4 one can see that for all $x, y \in X$ if $x \sim_G y$ (with respect to T) then $\phi(x) \sim_G \phi(y)$(with respect to S).

(7) Let T be G-chain mixing. Given $\epsilon > 0$ there exists a $\delta > 0$ such that the inequality $d(x, y) < \delta$ implies to $d'(\phi(x), \phi(y)) < \epsilon$ for all $x, y \in X$. Now if $y, y' \in Y$ then there is $N > 0$ so that for each $n \geq n$ there exists a (δ, G)-chain $y_0 = \phi^{-1}(y), y_1, \ldots, y_n = \phi^{-1}(y')$ from $\phi^{-1}(y')$ to $\phi^{-1}(y)$. Thus $\phi(y_0), \phi(y_1), \ldots, \phi(y_n)$ is an (ϵ, G)-chain from y to y'.

(8) If T is minimal and $y \in Y$ then there exist $g, h \in G$ such that $\overline{\mathcal{O}_T(\phi^{-1}(hy))} = \overline{\mathcal{O}_T(g\phi^{-1}(y))} = X$. We show that $\overline{\mathcal{O}_S(hy)} = Y$. To do this let $\epsilon > 0$ be given. Then there exists $\delta > 0$ so that the inequality $d(x, y) < \delta$ implies to $d'(\phi(x), \phi(y)) < \epsilon$ for all $x, y \in X$. If $z \in Y$ then there exists $n > 0$ such that $d(\phi^{-1} \circ S^n(hy), \phi^{-1}(z)) = d(T^n(\phi^{-1}(hy)), \phi^{-1}(z)) < \delta$. So $d'(S^n(hy), z) < \epsilon$. □



Proposition 2.1. *Let $T : X \to X$ and $S : Y \to Y$ be two homeomorphism on compact metric $G-$ spaces X and Y. If $\phi : X \to Y$ is a topological $G-$conjugacy from T to S, then $\Omega_G(S) = \phi(\Omega_G(T))$.*

Proof. Let $x \in \Omega_G(T)$ and U be an open neighborhood of $\phi(x)$. The continuity of ϕ implies that $\phi^{-1}(U)$ is an open neighborhood of x. So there exist $n > 1$ and $g \in G$ so that

$$(\phi^{-1}(U)) \cap (gT^n(\phi^{-1}(U))) \neq \emptyset.$$

Therefore

$$U \cap (hS^n(U)) = \phi(\phi^{-1}(U) \cap (g\phi^{-1}(S^n(U))) \neq \emptyset$$

for some $h \in G$. Hence $\phi(x) \in \Omega_G(S)$. Thus $\phi(\Omega_G(T)) \subseteq \Omega_G(S)$. A similar argument using the topological G-conjugacy $\phi^{-1} : Y \to X$ shows that $\phi(\Omega_G(T)) \supseteq \Omega_G(S)$. □

Proposition 2.2. *Let $T : X \to X$ and $S : Y \to Y$ be two homeomorphism on compact metric $G-$ spaces X and Y. If $\phi : X \to Y$ is a topological $G-$conjugacy from T to S, then $CR_G(S) = \phi(CR_G(T))$.*

Proof. Let $x \in CR_G(T)$ and $\epsilon > 0$. We shall construct an (ϵ, G)-orbit from $\phi(x)$ to itself. Since ϕ is uniformly continuous on X, there exists $\delta > 0$ such that if $d(z_1, z_2) < \delta$ then $d(\phi(z_1), \phi(z_2)) < \epsilon$. Since $x \in CR_G(T)$ there exists a (δ, G)-chain $x_0 = x, x_1, ..., x_n = x$ from x to itself. For each $i = 0, \ldots, n$ put $y_i = \phi(x_i)$. Then we have

$$d(S(y_i), y_{i+1}) = d(S \circ \phi(x_i), \phi(x_{i+1})) = d(\phi \circ T(x_i), \phi(x_{i+1})) < \epsilon.$$

Therefore $\phi(CR_G(T)) \subset CR_G(S)$. A similar argument using the topological G-conjugacy $\phi^{-1} : Y \to X$ shows that $\phi(CR_G(T)) \supset CR_G(S)$. □

3. EXAMPLES

Example 3.1. Consider the compact space $X = \cup_{i=1}^n C_i$ with the usual metric where each C_i is the circle in $\mathbb{R}^2$ with center in origin and radius i. Let

$$G = \mathcal{SO}(2) = \{A \in \mathcal{M}_{2\times 2}(\mathbb{R}) : \ \det(A) = \pm 1\}$$

and define $\theta : G \times X \to X$ by the usual rotation on X. Then one can see that X is a G-space and the identity map on X has the generator $\alpha = \{C_i : \ i = 1, \ldots, n\}$. Hence the identity map is weak G-expansive.

The following example shows that G-shadowing doesn't implies strong G-shadowing.

Example 3.2. Consider the compact space $X = \{1/n, 1 - 1/n \ : \ n \in \mathbb{Z}\}$ with the usual metric and let the topological group $G = \{1, -1\}$ act on X with the action defined by $\theta(1, x) = x$ and $\theta(-1, x) = 1 - x$. Define a homeomorphism $T : X \to X$ by

$$T(x) = \begin{cases} x, & x = 0, 1; \\ \text{next to the right of } x, & x \in X \backslash \{0, 1\}. \end{cases}$$

We show that T has the GSP. Given $\epsilon > 0$, there exists $m \in \mathbb{N}$ so that $1/m < \epsilon$. Put $\delta = 1/[2m(m+1)]$ and let $\{x_n\}$ be an (δ, G)-pseudo orbit. Then we have two cases:

<u>Case 1:</u> If there exists $n \in \mathbb{Z}$ such that $x_n \in (1/m, 1-1/m)$, then $x_{n+1} = T(x_n)$ or $x_{n+1} = 1-T(x_n)$. In both cases we deduce $x_{n+1} \in (1/m, 1-1/m)$. From $|T(x_{n-1}) - x_n| < \delta$ we deduce $x_{n-1} \in (1/m, 1-1/m)$. Hence for each $n \in \mathbb{Z}$ we have $x_n \in (1/m, 1-1/m)$. Now for each n we define

$$g_n = \begin{cases} 1, & x_{n+1} = T(x_n); \\ -1, & x_{n+1} = 1 - T(x_n). \end{cases}$$

Thus $|g_n T^n(x_0) - x_n| = 0 < \epsilon$.

<u>Case 2:</u> If $x_n \in [0, 1/m] \cup [1-1/m, 1]$ for each n, then we define

$$g_n = \begin{cases} 1, & x_n \in [0, 1/m]; \\ -1, & x_n \in [1-1/m, 1]. \end{cases}$$

Therefore $|g_n T^n(0) - x_n| \leq 1/m < \epsilon$.

Now we show that T does not have the SGSP. Suppose that $\epsilon = 1/3$ and $\delta > 0$. Choose $m \geq 6$ such that $1/m < \delta$ and define a (δ, G) pseudo orbit x_n as follow

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 1/m, \quad x_4 = 1/(m-1),$$
$$x_n = 1 \text{ for } n \leq 0,$$
$$x_{n+1} = T(x_n) \text{ for } n \geq 5.$$

Let (g, x) strongly ϵ-shadows x_n. Then we have two cases:

<u>Case 1:</u> If $g = 1$, then we have $|T^n(x) - x_n| < \epsilon$ for each $n \in \mathbb{Z}$. Hence

$$|T(x) - 1| < 1/3 \quad \Rightarrow \quad T^n(x) \in (2/3, 1] \forall n \in \mathbb{Z}.$$

But

$$|T^2(x) - x_2| < 1/3 \quad \Rightarrow \quad T^2(x) < 1/3$$

which is a contradiction.

<u>Case 2:</u> If $g = -1$ then we have

$$|1 - T^3(x) - 1/m| < 1/3 \quad \Rightarrow \quad T^3(x) > 1/2 \quad \Rightarrow \quad T^n(x) > 1/2 \forall n \geq 3.$$

But

$$|T^4(x) - 1/(m-1)| < 1/3 \quad \Rightarrow \quad T^4(x) < 1/2,$$

which is a contradiction.

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