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On Variations of m, n-Totally Projective Abelian p-Groups

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Abstract. We define some new classes of p-torsion Abelian groups which are closely related to the definitions of n-totally projective, strongly n-totally projective and m, n-totally projective groups introduced by P. Keef and P. Danchev in J. Korean Math. Soc. (2013). We also study their critical properties, one of which is the so-named *Nunke's-esque* property.

1. Introduction

All groups examined in the current paper will be p-primary Abelian, where p is an arbitrary fixed prime, and m and n are both non-negative integers which will be used in the sequel as parameters. Most of our notions and notations will be standard being in agreement with [5] and [6]; for the specific ones, we refer the readers to [9], [10] and [11]. About the unstated explicitly terminology, it will be given in all details. We shall say that the group G is Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups. Likewise, in [12] was established that a group G is $p^{\omega+n}$-*projective* precisely when there is $P \leq G[p^n]$ with the property that G/P is Σ-cyclic. Generalizing this concept, in [9] were introduced the following two notions:

- The group G is said to be *n-simply presented* if there exists $P \leq G[p^n]$ with G/P simply presented.
- The group G is said to be *strongly (or nicely) n-simply presented* if there exists a nice subgroup $N \leq G$ with $N \subseteq G[p^n]$ such that G/N is simply presented.

It is self-evident that strongly n-simply presented groups are of necessity n-simply presented; in [9] a concrete example was constructed showing that the converse is false. Furthermore, it was proved again in [9] that G is n-simply presented precisely when it is n-co-simply presented, that is, $G \cong$

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E/F where E is simply presented and $F \subseteq E[p^n]$. So, by analogy, there was stated the following:

The group G is said to be *strongly n-co-simply presented* if $G \cong H/K$ for some simply presented group H and its nice subgroup $K \leq H[p^n]$.

Unfortunately, an explicit construction from [11] demonstrates that there exists a strongly 1-co-simply presented group of length $\omega + 1$ that is not strongly 1-simply presented. However, for groups of length ω these two classes coincide with the class of $p^{\omega+n}$-projectives. Even more, each strongly n-simply presented group of length $\omega + n$, being $p^{\omega+n}$-projective, is strongly n-co-simply presented.

Later on, strengthening the classical notion of total projectivity, in [11] were defined the concepts of n-totally projective groups and strongly n-totally projective groups as follows:

- The group G is said to be *n-totally projective* if, for every (limit) ordinal λ, $G/p^\lambda G$ is $p^{\lambda+n}$-projective.
- The group G is said to be *strongly n-totally projective* if, for each (limit) ordinal λ, $G/p^{\lambda+n}G$ is $p^{\lambda+n}$-projective.

Notice that, when $n = 0$, these groups are just the totally projectives. It is also readily verified that strongly n-totally projective groups are n-totally projective, whereas the converse implication is not true (cf. [11]). However, it was proved in [10] that n-totally projective A-groups are themselves strongly n-totally projective. (For the full definition of *an A-group*, the reader is referred to [7].)

Likewise, note that (strongly) n-simply presented groups are (strongly) n-totally projective, respectively.

- The group G is said to be *weakly n-totally projective* if, for each (limit) ordinal λ, $G/p^\lambda G$ is $p^{\lambda+2n}$-projective.
- The group G is said to be *strong weakly n-totally projective* if, for every (limit) ordinal λ, $G/p^{\lambda+n}G$ is $p^{\lambda+2n}$-projective.

It is apparent that the following inclusions hold:

$$\begin{aligned}\{\text{strongly } n\text{-totally projective}\} &\subseteq \{n\text{-totally projective}\}\\ &\subseteq \{\text{strong weakly } n\text{-totally projective}\}\\ &\subseteq \{\text{weakly } n\text{-totally projective}\}.\end{aligned}$$

Furthermore, in [11] were defined a few more concepts as well. In fact, the above versions of generalizations of simple presentness suggest the following improvements:

- A group G is said to be *m, n-simply presented* if there exists $P \leq G[p^n]$ such that G/P is strongly m-simply presented.

In [4] was showed that G is m, n-simply presented if and only if there is a strongly m-totally projective group A and its p^n-bounded subgroup B such that $G \cong A/B$, that is, G is *m, n-co-simply presented.*

- A group G is said to be *weakly m,n-simply presented* if there exists $N \leq G[p^m]$ such that N is nice in G and G/N is n-simply presented.

A very difficult challenging conjecture says that weakly m,n-simply presented groups are m,n-simply presented, but the most real probability is it to be resolved in the negative. However, for groups of lengths $< \omega^2$ the conjecture holds in the affirmative (see [4]).

- A group G is said to be *m,n-co-weakly simply presented* if there exists an n-simply presented group U and its p^m-bounded nice subgroup V such that $G \cong U/V$.

Again it is interesting what is the relationship between the classes of m,n-co-simply presented groups and m,n-co-weakly simply presented groups.

- A group G is said to be *strongly m,n-simply presented* if there exists $N \leq G[p^m]$ such that N is nice in G and G/N is strongly n-simply presented.
- A group G is said to be *m,n-co-strongly simply presented* if there exists a strongly n-simply presented group X and its p^m-bounded nice subgroup Y such that $G \cong X/Y$.

A common generalization of both m,n-simply presented groups and weakly m,n-simply presented groups is the following:

- A group G is said to be *widely m,n-simply presented* if there exists $Z \leq G[p^m]$ such that G/Z is n-simply presented.

As in [4] a parallel reformulation of G to be widely m,n-simply presented is that $G \cong J/Q$, where J is n-simply presented and $Q \subseteq J[p^m]$, that is, the group is *widely m,n-co-simply presented.*

The alluded to above versions of extensions of total projectivity propose the next further refinements (cf. [11]):

- A group G is said to be *m,n-totally projective* if, for any ordinal λ, $G/p^{\lambda+m}G$ is $p^{\lambda+m+n}$-projective.

Apparently, if $m = 0$, we get n-totally projective groups, while if $n = 0$, we obtain strongly m-totally projective groups. The combination $m = n = 0$ gives totally projective groups.

Notice also that both m,n-simply presented and weakly m,n-simply presented groups are themselves m,n-totally projective.

Analogously to Propositions 2.1 and 2.2 from [4], and especially similarly to the proof of Proposition 2.1, it follows that even widely m,n-simply presented groups are m,n-totally projective.

Finally, mimicking [3], a group G is termed *nicely m-$p^{\omega+n}$-projective* if there exists a p^m-bounded nice subgroup Y such that G/Y is $p^{\omega+n}$-projective. More generally, a group G is named *strongly m-ω_1-$p^{\omega+n}$-projective* provided that there is a p^m-bounded subgroup T such that G/T is strongly ω_1-$p^{\omega+n}$-projective in the sense of [1], that is, a group A is called *strongly ω_1-$p^{\omega+n}$-projective* if there exists a p^n-bounded nice subgroup B such that

G/B is the direct sum of a countable group and a Σ-cyclic group. Note that $p^{\omega+n}$-projectives are obviously strongly ω_1-$p^{\omega+n}$-projective, by taking the countable summand to be zero. Some other interesting definitions of this kind the reader can see in [2].

Our goal here is to introduce certain non-trivial variations of the given above concepts, needed for applicable purposes. Namely, we state the following definitions.

Definition 1.1. The group G is called *nicely m,n-totally projective* if there is a p^m-bounded nice subgroup N such that G/N is n-totally projective.

Clearly, if $m = 0$, we obtain n-totally projective groups, whereas if $n = 0$, we get strongly m-simply presented groups (see [9]). Besides, choosing $m = n = 0$, we also retrieve totally projective (= simply presented) groups.

On the other hand, it is immediate that weakly m,n-simply presented group are necessarily nicely m,n-totally projective.

Definition 1.2. The group G is called *nicely m,n-strongly totally projective* if there is a p^m-bounded nice subgroup M of G such that G/M is strongly n-totally projective.

Observe that nicely m,n-strongly totally projective groups are obviously nicely m,n-totally projective. Likewise, notice that if $m = 0$, we obtain strongly n-totally projective groups, whereas if $n = 0$, we get strongly m-simply presented groups (cf. [9]). In particular, if both $m = n = 0$, we just retrieve totally projective (= simply presented) groups.

The last definition can be enlarged to the following one:

Definition 1.3. The group G is called *m,n-strongly totally projective* if there is a p^m-bounded subgroup P of G such that G/P is strongly n-totally projective.

Note that n,m-simply presented groups are m,n-strongly totally projective.

Definition 1.4. The group G is called *nicely m,n-weakly totally projective* if there is a p^m-bounded nice subgroup X of G such that G/X is weakly n-totally projective.

Definition 1.5. The group G is called *m,n-weakly totally projective* if there is a p^m-bounded subgroup Y of G such that G/Y is weakly n-totally projective.

Definition 1.6. The group G is called *nicely m,n-strong weakly totally projective* if there is a p^m-bounded nice subgroup K of G such that G/K is strong weakly n-totally projective.

Definition 1.7. The group G is called *m,n-strong weakly totally projective* if there is a p^m-bounded subgroup S of G such that G/S is strong weakly n-totally projective.

Definition 1.8. The group G is called *nicely m,n-co-totally projective* if there is an n-totally projective group T with a nice p^m-bounded subgroup L such that $G \cong T/L$.

Apparently, when $m = 0$, we obtain n-totally projective groups, while if $n = 0$, we get strongly m-co-simply presented groups (see [9]). If both $m = n = 0$, we come to totally projective (= simply presented) groups.

Definition 1.9. The group G is called *nicely m,n-co-strongly totally projective* if there is a strongly n-totally projective group S with a nice p^m-bounded subgroup K such that $G \cong S/K$.

It is observed that nicely m,n-co-strongly totally projective groups are themselves nicely m,n-co-totally projective. Also, note that if $m = 0$, we obtain strongly n-totally projective groups, while if $n = 0$, we get strongly m-co-simply presented groups (cf. [9]). Likewise, the equalities $m = n = 0$ lead to totally projective (= simply presented) groups.

Definition 1.10. The group G is called *m,n-co-strongly totally projective* if there is a strongly n-totally projective group H with a p^m-bounded subgroup V such that $G \cong H/V$.

Definition 1.11. The group G is called *nicely m,n-co-weakly totally projective* if there is a weakly n-totally projective group R with a p^m-bounded nice subgroup C such that $G \cong R/C$.

Definition 1.12. The group G is called *m,n-co-weakly totally projective* if there is a weakly n-totally projective group A with a p^m-bounded subgroup B such that $G \cong A/B$.

Definition 1.13. The group G is called *nicely m,n-co-strong weakly totally projective* if there is a strong weakly n-totally projective group E with a p^m-bounded nice subgroup F such that $G \cong E/F$.

Definition 1.14. The group G is called *m,n-co-strong weakly totally projective* if there is a strong weakly n-totally projective group D with a p^m-bounded subgroup C such that $G \cong D/C$.

In [4] the listed above variations of m,n-simply presented groups were characterized, while the main goal here is to characterize the variations of m,n-totally projectives defined above by comparing them with the previously cited ones from [4], [9] and [11].

2. Basic Results

We begin with the following statement which determines nicely m,n-totally projective groups of length at most $\omega + m$, and which improves Proposition 1.2 from [11].

Theorem 2.1. *Suppose that G is a group with $p^{\omega+m}G = \{0\}$. Then G is nicely m,n-totally projective if and only if G can be embedded in a $p^{\omega+m}$-bounded n-totally projective group.*

Proof. "$\Rightarrow$" Assume that G/N is n-totally projective for some nice subgroup $N \leq G$ with $p^m N = \{0\}$. Hence $G/N/p^\omega(G/N) \cong G/(N+p^\omega G)$ is separable $p^{\omega+n}$-projective. For simpleness we put $N + p^\omega G = P$. Clearly $P \supseteq p^\omega G$ remains nice in G because of separability of the above quotient (or because N is nice in G), as well as $P \leq G[p^m]$.

On the other hand, let B be a totally projective group whose $p^\omega B$ is p^m-bounded and such that there is an isomorphism $\varphi : p^\omega B \to P$. Note that there is an abundance of such groups.

Suppose now that H is the group that is the amalgamated sum of B and G along φ. In other words $H = [B \oplus G]/\{(b, \varphi(b)) \ : \ b \in p^\omega B\}$, i.e., $H = B + G$ where $B \cap G = p^\omega B = P$.

One may see that $p^\omega H = p^\omega B$, so that H will be $p^{\omega+m}$-bounded as well. To that goal, given $x \in p^\omega H = \cap_{i<\omega} p^i H$ hence $x = b_i + g_i = b_j + g_j = \dots$ where $b_i \in p^i B, b_j \in p^j B$ and $g_i \in p^i G, g_j \in p^j G$ for some arbitrary indices i, j with $i < j$. Thus $b_i - b_j = g_j - g_i \in G \cap B = p^\omega B$ whence $b_i \in p^j B$ for every index $j < \omega$, that is, $b_i \in p^\omega B = P$. Similarly, $b_j \in p^\omega B = P$. That is why $g_i \in p^j G + P$ for any $j < \omega$, i.e., $g_i \in \cap_{j<\omega}(p^j G + P) = p^\omega G + P = P$. Finally, $x \in P = p^\omega B$, as required.

Furthermore, one can observe that $H/p^\omega H = (B/p^\omega B) \oplus (G/P)$, and since $B/p^\omega B$ is Σ-cyclic (cf. [5]) while G/P is $p^{\omega+n}$-projective, we deduce that $H/p^\omega H$ is $p^{\omega+n}$-projective. We finally employ Theorem 4.5 from [9] to get appeared that H is n-simply presented. Hence [11] allows us to conclude that G is n-totally projective, as stated.

"$\Leftarrow$". Let $G \subseteq H$ where H is an n-totally projective group of length not exceeding $\omega + m$. Since $G/(p^\omega H \cap G) \cong (G + p^\omega H)/p^\omega H \subseteq H/p^\omega H$ is $p^{\omega+n}$-projective as being a subgroup of the $p^{\omega+n}$-projective group $H/p^{\omega+n}H$, and moreover $p^\omega H \cap G$ is obviously bounded by p^m and is nice in G, we establish the wanted claim. □

We next continue with some relationships between the defined above classes of groups.

Proposition 2.1. *Suppose G is a group. If*

(i) G is ω_1-$p^{\omega+m+n}$-projective, then G is widely m,n-simply presented.
(ii) G is strongly ω_1-$p^{\omega+m+n}$-projective, then G is m,n-simply presented.

Proof. (i) In accordance with [8], write G/H is the direct sum of a countable group and a Σ-cyclic group, whence G/H is simply presented, for some $H \leq G$ with $p^{m+n}H = \{0\}$. Observe that $G/H \cong G/p^n H/H/p^n H$. Therefore, $G/p^n H$ is n-simply presented. Since $p^m(p^n H) = \{0\}$, we are finished.

(*ii*) In virtue of [1], one may write G/H as above into the direct sum of a countable group and a Σ-cyclic group, but where H is nice in G and p^{m+n}-bounded. Furthermore, the same idea as that in point (i) works, seeing that H/p^nH remains nice in G/p^nH and hence G/p^nH is strongly n-simply presented. □

Proposition 2.2. *Let G be a nicely m,n-strongly totally projective group such that $p^{\omega+m+n}G = \{0\}$. Then G is nicely m-$p^{\omega+n}$-projective.*

Proof. Assume that G/M is strongly n-totally projective for some nice p^m-bounded subgroup M. Utilizing [11], the quotient $G/M/p^{\omega+n}(G/M) \cong G/(M + p^{\omega+n}G)$ is $p^{\omega+n}$-projective. Since $p^m(M + p^{\omega+n}G) = \{0\}$, and $M + p^{\omega+n}G$ remains nice in G, the result follows. □

With the last statement in hand, one may derive the following:

Theorem 2.2. *Suppose that G is a group with countable $p^{\omega+m+n}G$. Then G is nicely m,n-strongly totally projective if and only if G is strongly m-ω_1-$p^{\omega+n}$-projective.*

Proof. "$\Rightarrow$" Appealing to Proposition 3.1 (*ii*), stated and proved below, the factor-group $G/p^{\omega+m+n}G$ is also nicely m,n-strongly totally projective. Furthermore, Proposition 2.2 is applicable to get that $G/p^{\omega+m+n}G$ is nicely m-$p^{\omega+n}$-projective and hence strongly m-ω_1-$p^{\omega+n}$-projective. Since $p^{\omega+m+n}G$ is countable by assumption, we employ Theorem 3.11 from [3] to deduce the desired implication.
"$\Leftarrow$" It follows immediately because strongly ω_1-$p^{\omega+n}$-projective groups are themselves strongly n-simply presented (see [1]) and so they are strongly n-totally projective. □

Proposition 2.3. *If G is a nicely m,n-totally projective group of length $\lambda < \omega^2$, then G is weakly m,n-simply presented, and vice versa, provided length$(G) < \omega^2$.*

Proof. Suppose that G is a nicely m,n-totally projective group. Thus G/N is n-totally projective for some nice subgroup N of G which is bounded by p^m. Since $p^\lambda(G/N) = (p^\lambda G + N)/N = \{0\}$, we may apply [11] to get that G/N is n-simply presented, as required.

The converse implication is elementary. □

As a consequence, we yield:

Corollary 2.1. *If G is a nicely m,n-totally projective group of length $< \omega^2$, then G is m,n-simply presented (and, in particular, is n,m-strongly totally projective).*

The same can be said adding the word "strongly". Specifically, the following is valid:

Proposition 2.4. *If G is a nicely m,n-strongly totally projective group of length $\lambda < \omega^2$, then G is strongly m,n-simply presented, and visa versa, provided length$(G) < \omega^2$.*

Proof. Utilizing the corresponding definitions, the same idea as that in Proposition 2.3 works. □

Similarly, we derive:

Proposition 2.5. *Suppose that G is a group of length strictly less than ω^2. Then G is nicely m,n-co-totally projective if and only if G is m,n-co-weakly simply presented.*

Proposition 2.6. *If the group G is either*

(*a*) *nicely m,n-totally projective, or*

(*b*) *nicely m,n-co-totally projective,*

then G is m,n-totally projective.

Proof. (a) Assume that there exists a nice p^m-bounded subgroup N of G such that G/N is n-totally projective. Since we have the isomorphism sequence

$$G/N/p^\lambda(G/N) = G/N/(p^\lambda G + N)/N \cong$$
$$G/(p^\lambda G + N) \cong G/p^{\lambda+m}G/(p^\lambda G + N)/p^{\lambda+m}G$$

where $G/N/p^\lambda(G/N)$ is $p^{\lambda+n}$-projective for each limit ordinal λ and $(p^\lambda G + N)/p^{\lambda+m}G$ is p^m-bounded, we apply [11] to infer that $G/p^{\lambda+m}G$ is $p^{\lambda+m+n}$-projective, as required.

(b) Assume that there exists an n-totally projective group T with a p^m-bounded nice subgroup L such that $G \cong T/L$. Furthermore, we deduce that

$$\begin{aligned} G/p^{\lambda+m}G &\cong T/L/p^{\lambda+m}(T/L) \\ &= T/L/(p^{\lambda+m}T + L)/L \\ &\cong T/(p^{\lambda+m}T + L). \end{aligned}$$

But

$$T/p^\lambda T \cong T/(p^{\lambda+m}T + L)/p^\lambda T/(p^{\lambda+m}T + L)$$

is $p^{\lambda+n}$-projective for every limit ordinal λ and $p^\lambda T/(p^{\lambda+m}T + L)$ is p^m-bounded, so we employ [11] to conclude that $T/(p^{\lambda+m}T + L) \cong G/p^{\lambda+m}G$ is $p^{\lambda+m+n}$-projective, as requested.

Note that the condition $p^m L = \{0\}$ was not utilized. □

Remark 1. For some subclasses of groups of these alluded to above, we refer to [4].

For p^ω-bounded groups, we can say even a little more. Especially the following is true (compare with Theorem 2.5 of [4]):

Theorem 2.3. *Suppose that G is a group with $p^\omega G = \{0\}$. Then the following conditions are equivalent:*

(*i*) *G is m, n-totally projective;*
(*ii*) *G is nicely m, n-totally projective;*
(*iii*) *G is nicely m, n-co-totally projective;*
(*iv*) *G is $p^{\omega+m+n}$-projective.*

Proof. The equivalence $(i) \iff (iv)$ was proved in [11]. What remains to show is that (iv) implies both (iii) and (ii). In fact, since G is $p^{\omega+m+n}$-projective, $G \cong S/Y$ for some Σ-cyclic group S with a p^{m+n}-bounded subgroup Y. Put $X = S[p^n] \cap Y = Y[p^n]$. Thus X is nice in S as the intersection of two closed subgroups (see, for example, [5]). Furthermore, $G \cong S/X/Y/X$, where S/X is obviously $p^{\omega+n}$-projective because $p^n X = \{0\}$, and hence S/X is strongly n-simply presented. But $Y/X = Y/Y[p^n] \cong p^n Y$ is bounded by p^m and is also nice in S/X taking into account that G is separable, so that Y is nice in S (cf. [5]). Now, an appeal to Definition 1.3 gives that G is nicely m, n-co-totally projective.

As for the second implication, since G is $p^{\omega+m+n}$-projective, there is $V \leq G[p^{m+n}]$ such that G/V is Σ-cyclic. Set $U = G[p^m] \cap V = V[p^m]$. Hence U is nice in G as the intersection of two closed subgroups (see, for instance, [5]). Moreover, $G/U/V/U \cong G/V$ is Σ-cyclic with $V/U = V/V[p^m] \cong p^m V$ being bounded by p^n. Consequently, G/U is $p^{\omega+n}$-projective, whence n-totally projective, with $p^m U = \{0\}$. With Definition 1.1 at hand, this guarantees that G is nicely m, n-totally projective, as stated. □

The next example demonstrates that beyond lengths ω, the last result is not longer valid, and also that the concept of m, n-totally projective groups is independent of that of nicely m, n-totally projective groups – the same can be happen for nicely m, n-co-totally projective groups (see [4] too).

Example 2.1. There exists a $p^{\omega+1}$-bounded $1, 1$-totally projective group which is not nicely $1, 1$-totally projective.

Proof. We begin with the following:
CLAIM 1. Let H be a $p^{\omega+1}$-projective group, and let J be a countable subgroup of H. Then $p\overline{J}$ is countable.

To show this, if P is a p-bounded subgroup of H such that H/P is Σ-cyclic, then there is a subgroup L of H containing P and J such that L/P is a countable of H/P. It follows that L is closed in H, so that $\overline{J} \subseteq L$. Since $L = P + X$ for some countable subgroup X, we have $p\overline{J} \subseteq pL = pX$ is countable.
CLAIM 2. Let B be the standard separable free valuated vector space (i.e., all its finite Ulm-Kaplansky invariants equal to 1). Then there is a subspace $V \subseteq \overline{B}$ of uncountable rank, containing B, such that if C is any closed subspace of $\overline{B}$ contained in V, then $C(k) = C \cap \overline{B}(k) = \{0\}$ for some $k < \omega$

(i.e., any closed subspace of $\overline{B}$ - which, in fact, will be a valuated direct summand - contained in V is bounded).

Let b_i for $i < \omega$ be a basis for B. Let C_α for $\alpha < c = 2^{\aleph_0}$ be a list of all the unbounded closed subspaces of $\overline{B}$; note that each C_α has rank c. Construct elements x_α and y_α for $\alpha < c$ such that (1) $y_\alpha \in C_\alpha$, and (2) $\{b_i, x_\alpha, y_\alpha : i < \omega, \alpha < c\}$ is linearly independent. If we let $V = \mathrm{span}\{b_i, x_\alpha : i < \omega, \alpha < c\}$, then for any unbounded closed subspace C_α of $\overline{B}$, we have $y_\alpha \in C_\alpha \setminus V$, which shows that C_α is not contained in V.

Consider $V \subseteq \overline{B}$ as in Claim 2. Let Y be a separable group such that $Y[p]$ is isometric to V. Let Y_1 be a group with $Y_1[p] = Y[p]$ and $Y = pY_1 \cong Y_1/Y_1[p]$. If C_1 is the torsion completion of Y_1, then $C = pC_1 \cong C_1/C_1[p]$ is the torsion completion of Y. Let P be the valuated group

$$(C_1/Y_1[p])[p^2] = (Y_1[p^3] + C_1[p^2])/Y_1[p].$$

We can identify $Y[p^2] \cong Y_1[p^3]/Y_1[p]$ with a subgroup of P. In addition,

$$P[p] \cong (Y_1[p^2]/Y_1[p]) \oplus (C_1[p]/Y_1[p]) \cong Y[p] \oplus (C_1[p]/Y_1[p]),$$

$P(\omega) = C_1[p]/Y_1[p]$ and $(P/P(\omega))[p] \cong C_1[p^2]/C_1[p] \cong C[p]$. We will be done if we can show the following:

Claim 3. Suppose G is a group containing P such that the valuation on P agrees with the height function on G, and so that G/P is Σ-cyclic. Then G is $1,1$-simply presented of length $\omega+1$, and hence it is $1,1$-totally projective of the same length, but G is not weakly $1,1$-simply presented; even more, $G \oplus X$ is not weakly $1,1$-simply presented for every Σ-cyclic group X. By virtue of Proposition 2.3, this means that it is not nicely $1,1$-totally projective.

To this aim, suppose M is a nice p-bounded subgroup of G such that G/M is 1-simply presented. Note that $M + p^\omega G$ will also be nice in G and p-bounded, and $G/[M + p^\omega G] \cong G/M/p^\omega(G/M)$ will be $p^{\omega+1}$-projective, and so 1-simply presented. So, we may assume $p^\omega G \subseteq M$.

Since M is nice, $M/p^\omega G$ will be closed in $(G/p^\omega G)[p]$. Consider $M' = (M/p^\omega G) \cap (P/P(\omega))[p]$; so M' is closed in $(P/P(\omega))[p] \cong C[p]$. Observe $M' \subseteq Y[p] = V$, and moreover it follows from Claim 2 that M' is bounded. In other words, for some integer k, we must have $M' \cap V(k) = \{0\}$.

Let Z be a basic subgroup of Y and let $Z = Z'_k \oplus Z_k$ be a decomposition, where Z'_k is a maximal p^k-bounded summand of Z. This determines a decomposition $Y = Z'_k \oplus Y_k$ of Y.

Notice that $Y_k[p^2] \cap M = \{0\}$, so that it embeds isomorphically in G/M. Call this image L and let $J \subseteq L$ be the image of $Z_k[p^2] \subseteq Y_k[p^2] \subseteq G$ in G/M. Note that J is countable, and since $Z_k[p^2]$ is dense in $Y_k[p^2]$, it follows that J is dense in L. However, since $pL \cong pY_k$ is uncountable, we obtain that $p\overline{J}$ is also uncountable. But this contradicts Claim 1, and thus proves our assertion after all. □

The next question arises quite naturally: Does there exist a $p^{\omega+1}$-bounded $1,1$-totally projective group that is not nicely $1,1$-co-totally projective? Even more, in view of Proposition 2.6, is there a nicely $1,1$-totally projective group which is not nicely $1,1$-co-totally projective?

However the converse to that question is true for the "strongly" situation.

Example 2.2. There exists a nicely $1,1$-co-totally projective group of length $\omega+1$ which is not nicely $1,1$-strongly totally projective.

Proof. As already mentioned before, in Example 2.1 from [9] was constructed a $p^{\omega+1}$-bounded strongly 1-co-simply presented group which is not strongly 1-simply presented. We furthermore wish apply Theorem 3.2 of [11] to get the desired claim. □

Recall that it was defined in [8] a group G to be *ω_1-$p^{\omega+n}$-projective*, provided that there exists a countable (nice) subgroup C such that G/C is $p^{\omega+n}$-projective.

In the light of the last constructions, we obtain the following strengthening of Theorem 2.3:

Proposition 2.7. *Suppose that G is a group with countable $p^{\omega+m}G$. Then G is m,n-totally projective if and only if G is ω_1-$p^{\omega+m+n}$-projective.*

Proof. "**Necessity**": Accordingly, $G/p^{\omega+m}G$ is $p^{\omega+m+n}$-projective. We therefore see that the above definition from [8] works to get the assertion.

"**Sufficiency**": It follows directly from Proposition 2.1 (i) stated and proved above. □

3. Ulm subgroups and Ulm factors

Imitating [5] and/or [6], for any group G and any $n \in \mathbb{N}$, we define $p^nG = \{p^n g \mid g \in G\}$. Set $p^\omega G = \cap_{n<\omega} p^n G$. By induction on an arbitrary ordinal α, one may state $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ whenever α is limit, whereas $p^\alpha G = p(p^{\alpha-1}G)$ provided that α is nonlimit. Clearly $p^\alpha G \leq G$ and these subgroups are called *Ulm subgroups*, while the factor-groups $G/p^\alpha G$ are said to be *Ulm factors*.

We will now study Nunke's type results for the new group classes.

Proposition 3.1. (i) *If G is nicely m,n-totally projective, then so are $p^\alpha G$ and $G/p^\alpha G$ for any ordinal α.*

(ii) *If G is nicely m,n-strongly totally projective, then so are $p^\alpha G$ and $G/p^\alpha G$ for any ordinal α.*

Proof. (i) Let $p^m N = \{0\}$ where N is nice in G such that G/N is n-totally projective. Clearly $N \cap p^\alpha G$ is p^m-bounded and nice in $p^\alpha G$ (see [5]) as well as $p^\alpha G/(p^\alpha G \cap N) \cong (p^\alpha G + N)/N = p^\alpha(G/N)$ is n-totally projective because the same is G/N (cf. [11]), thus proving the first half.

For the other part, $(N+p^\alpha G)/p^\alpha G$ is p^m-bounded and nice in $G/p^\alpha G$ (cf. [5]). Also,

$$G/p^\alpha G/(N+p^\alpha G)/p^\alpha G \cong G/(N+p^\alpha G) \cong G/N/(N+p^\alpha G)/N = G/N/p^\alpha(G/N)$$

is n-totally projective since so is G/N (see [11]), thus showing the second half.
(ii) Follows by similar arguments seeing that $p^\alpha(G/N)$ and $G/N/p^\alpha(G/N)$ are both strongly n-totally projective, provided that G/N is so (cf. [11]). □

Proposition 3.2. *(j) If G is nicely m,n-co-totally projective, then the same are $p^\alpha G$ and $G/p^\alpha G$ for any ordinal α.*

(jj) If G is nicely m,n-co-strongly totally projective, then the same are $p^\alpha G$ and $G/p^\alpha G$ for any ordinal α.

Proof. (j) Let $G \cong T/L$ for some n-totally projective group T with a p^m-bounded nice subgroup L. Hence $p^\alpha G \cong p^\alpha(T/L) = (p^\alpha T + L)/L \cong p^\alpha T/(p^\alpha T \cap L)$, with n-totally projective $p^\alpha T$ (see [11]) and $p^\alpha T \cap L$ being p^m-bounded and nice in $p^\alpha T$ (cf. [5]). This shows that $p^\alpha G$ is nicely m,n-co-totally projective.

Furthermore, concerning the second part-half, $G/p^\alpha G \cong T/L/p^\alpha(T/L) = T/L/(p^\alpha T+L)/L \cong T/(p^\alpha T+L) \cong T/p^\alpha T/(p^\alpha T+L)/p^\alpha T$. The utilization of [11] ensures that $T/p^\alpha T$ is n-totally projective. Moreover, $(p^\alpha T + L)/p^\alpha T \cong L/(p^\alpha T \cap L)$ is p^m-bounded and nice in $T/p^\alpha T$ because $p^\alpha T + L$ is so in T (cf. [5]). This guarantees that $G/p^\alpha G$ is nicely m,n-co-totally projective.
(jj) Follows via identical arguments as above, observing that T being strongly n-totally projective implies the same for both $p^\alpha T$ and $T/p^\alpha T$ (see [11]). □

We now have all the ingredients needed to prove the following assertion. It reduces the study of nicely m,n-strong total projectivity to Ulm subgroups and Ulm factors.

Theorem 3.1. *Suppose that α is an ordinal. Then the group G is nicely m,n-strongly totally projective iff both $p^{\alpha+m+n}G$ and $G/p^{\alpha+m+n}G$ are nicely m,n-strongly totally projective.*

Proof. The necessity follows from Proposition 3.1 (ii), replacing α by $\alpha+m+n$.

Concerning the sufficiency, denote $k=m+n$. With Definition 1.2 at hand, let us assume that $p^{\alpha+k}G/H = p^{\alpha+k}(G/H)$ is strongly n-totally projective for some p^m-bounded nice subgroup H of $p^{\alpha+k}G$. Thus H is nice in G as well (see [5]).

Also, suppose $G/p^{\alpha+k}G/A/p^{\alpha+k}G \cong G/A$ is strongly n-totally projective for some $A \leq G$ such that $A/p^{\alpha+k}G$ is nice in $G/p^{\alpha+k}G$ and $p^m A \subseteq p^{\alpha+k}G$. Therefore, A is nice in G too (cf. [5]).

We will now use a trick used in [4], [9] and [11], respectively. Let V be a maximal p^m-bounded summand of $p^{\alpha+n}G$; so there exists a decomposition $p^{\alpha+n}G = U \oplus V$ for some $U \leq p^{\alpha+n}G$. Besides, let K be a $p^{\alpha+k}$-high subgroup of G containing V. Now, it follows that (see, for instance, [9] and [11])

$$(G/p^{\alpha+k}G)[p^m] = (U \oplus K[p^m])/p^{\alpha+k}G,$$

whence $A \subseteq U \oplus K[p^m]$. Therefore, $U + A \subseteq U \oplus K[p^m]$ and hence the modular law from [5] yields $U + A = (U \oplus K[p^m]) \cap (U + A) = U + (U + A) \cap K[p^m]$. Letting $(U + A) \cap K[p^m] = B$, we deduce that $U + A = U + B$ with $p^m B = \{0\}$. Since $U \subseteq p^{\alpha+n}G \subseteq p^{\alpha}G$, we have that $p^{\alpha+n}G + A = p^{\alpha+n}G + B$.

Next put $Z = B + H$. By what we have already established above, it follows that $p^m Z = \{0\}$ and that $p^{\alpha+n}G + Z = p^{\alpha+n}G + B = p^{\alpha+n}G + A$. Furthermore, A being nice in G elementary insures that $p^{\alpha+n}G + Z = p^{\alpha+n}G + A$ is nice in G as well. Moreover, the modular law ensures that $p^{\alpha+k}G \cap Z = p^{\alpha+k}G \cap (B + H) = p^{\alpha+k}G \cap B + H = p^{\alpha+k}G \cap K[p^m] \cap (U + A) + H = H$ is nice in $p^{\alpha+k}G$. Applying Lemma 2.9 from [4], we conclude that $p^{\alpha+n}G \cap Z$ is nice in $p^{\alpha+n}G$, and hence in G (cf. [5]), because $k \geq n$. Finally, we again employ [5] to get that after all Z is, in fact, nice in G.

On the other hand, using the niceness of Z in G, we derive that $p^{\alpha+k}(G/Z) = (p^{\alpha+k}G + Z)/Z \cong p^{\alpha+k}G/(p^{\alpha+k}G \cap Z) = p^{\alpha+k}G/H$ is strongly n-totally projective. So, [11] applies to infer that $p^{\alpha+n}(G/Z)$ is strongly n-totally projective since $k \geq n$. In virtue again of ([11], Theorem 2.5), $G/Z/p^{\alpha+n}(G/Z) = G/Z/(p^{\alpha+n}G+Z)/Z \cong G/(p^{\alpha+n}G+Z) = G/(p^{\alpha+n}G+A) \cong G/A/(p^{\alpha+n}G+A)/A = G/A/p^{\alpha+n}(G/A)$ is strongly n-totally projective, too. We once again employ ([11], Corollary 2.8) to detect that G/Z is strongly n-totally projective, as wanted. □

Remark 2. It seems that $k = m + n$ cannot be minimized to m or n as it was done in [4].

4. Left-open Problems

In closing we pose the following list of still unsettled questions and conjectures.

Question 3.1. Suppose G is a group such that $G/p^{\lambda}G$ is totally projective for some ordinal λ. Is then G nicely m,n-totally projective if and only if $p^{\lambda}G$ is?

Question 3.2. Suppose G is a group such that $G/p^{\lambda}G$ is totally projective for some ordinal λ. Is then G nicely m,n-strongly totally projective if and only if $p^{\lambda}G$ is?

These questions will have a positive solution provided the following implication holds: If A is a group such that $p^{\lambda}A$ is n-totally projective and $A/p^{\lambda}A$ is totally projective, then A is n-totally projective.

In regard to Corollary 2.1, one can state the following:

Question 3.3. If G is a nicely m,n-totally projective group, is then G an n,m-strongly totally projective group?

Conjecture 3.1. Every n-simply presented group is a summand of a strongly n-simply presented group; in particular, for any n, there is an n-simply presented group which is not strongly n-simply presented.

Same for the co-case.

Conjecture 3.2. For any $n \geq 0$, there exists a strongly n-simply presented group of length $\omega + n + 1$ that is not strongly n-co-simply presented.

As noted above, the definition of an A-group is stated in [7].

Conjecture 3.3. Let G be an A-group. Then G is n-simply presented if and only if G is strongly n-simply presented.

Same for the co-case.

Since as aforementioned G is n-simply presented exactly when it is n-co-simply presented, if the last conjecture is true one may derive that G is strongly n-simply presented uniquely when it is strongly n-co-simply presented, provided G is an A-group.

Conjecture 3.4. Suppose G is an A-group. Then G is weakly n-totally projective if and only if G is strong weakly n-totally projective.

Thus, since it was demonstrated in [10] that there exists a weakly n-totally projective A-group which is not n-totally projective, if this conjecture holds in the affirmative, we will have an example of a strong weakly n-totally projective A-group that is not n-totally projective.

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