Mathematica Moravica
Vol. **18-1** (2014), 21–28

Properties of the Quasi-Conformal Curvature Tensor of Kähler-Norden Manifolds

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Abstract. The object of the present paper is to study quasi-conformally flat and parallel quasi-conformal curvature tensor of a Kähler-Norden manifold. Besides this we also study quasi-conformally semisymetric Kähler-Norden manifolds. Finally, we mention an example to verify a Theorem of our paper.

1. Introduction

An anti-Kähler or Kähler-Norden manifold means a triple (M^n, J, g) which consists of a smooth manifold M^n of dimension $n = 2m$, an almost complex structure J and an anti-Hermitian metric g such that $\nabla J = 0$ where ∇ is the Levi-Civita connection of g. The metric g is called anti-Hermitian if it satisfies $g(JX, JY) = -g(X, Y)$ for all vector fields X and Y on M^{2m}. Then the metric g has necessarily a neutral signature (m, m) and M^{2m} is a complex manifold and there exists a holomorphic metric on M^{2m} [1]. This fact gives us some topological obstructions to an anti-Kähler manifold, for instance, all its odd Chern numbers vanish because its holomorphic metric gives us a complex isomorphism between the complex tangent bundle and its dual and a compact simply connected Kähler manifold cannot be anti-Kähler because it does not admit a holomorphic metric.
The conditions of the semisymmetry and pseudosymmetry type for the Riemann, Ricci and Weyl curvature tensors of Kählerian and para-Kählerian manifolds were studied in the papers [9, 10, 11, 12] and many others. In the present paper we extend the result of Sluka [5] in a Kähler-Norden manifold. In [4] Sluka constructed some examples of holomorphically projectively flat as well as semisymmetric and locally symmetric Kähler-Norden manifolds. The present paper is organized as follows:
After preliminaries in section 3, we study quasi-conformally flat Kähler-Norden manifolds. In section 4, we consider parallel quasi-conformal Kähler-Norden manifolds. In section 5, we study quasi-conformally semisymmetric

2010 *Mathematics Subject Classification.* Primary: 53C15; Secondary: 53C56, 53C50.
Key words and phrases. Kähler-Norden manifold, quasi-conformally flat, parallel quasi-conformal curvature tensor and quasi-conformally semisymmetric.

Kähler-Norden manifolds. Finally, we mention an example to verify the Theorem 4.1.

2. Preliminaries

By a Kählerian manifold with Norden metric (Kähler-Norden in short) [2] we mean a triple (M, J, g), where M is a connected differentiable manifold of dimension $n = 2m$, J is a $(1, 1)$-tensor field and g is a pseudo-Riemannian metric on M satisfying the conditions

$$J^2 = -I, \quad g(JX, JY) = -g(X, Y), \quad \nabla J = 0$$

for every X, $Y \in \chi(M)$ is the Lie algebra of vector fields on M and ∇ is the Levi-Civita connection of g.

Let (M, J, g) be a Kähler-Norden manifold. Since in dimension two such a manifold is flat, we assume in the sequel that $dimM \geq 4$. Let $\mathcal{R}(X, Y)$ be the curvature operator $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and let $\mathcal{R}$ be the Riemann-Christoffel curvature tensor, $R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W)$. The Ricci tensor S is defined as $S(X, Y) = trace\{Z \longrightarrow \mathcal{R}(Z, X)Y\}$. These tensors have the following properties [1]

$$\begin{aligned} &\mathcal{R}(JX, JY) = -\mathcal{R}(X, Y), \quad \mathcal{R}(JX, Y) = \mathcal{R}(X, JY), \\ &S(JY, Z) = trace\{X \longrightarrow \mathcal{R}(JX, Y)Z\}, \quad S(JX, Y) = S(JY, X), \\ &S(JX, JY) = -S(X, Y). \end{aligned} \tag{1}$$

Let Q be the Ricci operator. Then we have $S(X, Y) = g(QX, Y)$ and

$$QY = -\sum_i \epsilon_i \mathcal{R}(e_i, Y)e_i,$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis and ϵ_i are the indicators of e_i, $\epsilon_i = g(e_i, e_i) = \pm 1$. The notion of a quasi-conformal curvature tensor was given by Yano and Sawaki [6]. The quasi-conformal curvature tensor $\tilde{C}$ is defined by

$$\begin{aligned} \tilde{C}(X, Y)Z = a\mathcal{R}(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ - g(X, Z)QY] - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2}$$

where a and b are constants and $\mathcal{R}$, Q and r are Riemannain curvature tensor of type $(1, 3)$, the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and the scalar curvature, respectively. If $a = 1$ and $b = -\frac{1}{n-2}$, then (2) takes the form

$$\begin{aligned}\tilde{C}(X,Y)Z &= \mathcal{R}(X,Y)Z - \frac{1}{n-2}\Big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad - g(X,Z)QY\Big] + \frac{r}{(n-1)(n-2)}\big[g(Y,Z)X - g(X,Z)Y\big] \\ &= C(X,Y)Z,\end{aligned}$$

where C is the conformal curvature tensor [8]. Thus the conformal curvature tensor C is the particular case of the tensor $\tilde{C}$. For this reason $\tilde{C}$ is called quasi-conformal curvature tensor. A manifold (M^n, g) $(n > 3)$ shall be called quasi-conformally flat if $\tilde{C} = 0$. It is known [3] that a quasi conformally flat manifold is either conformally flat if $a \neq 0$ or Einstein if $a = 0$ and $b \neq 0$. Since they give no restrictions for manifolds if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$.

Using (1) and (2) we have

$$\begin{aligned}\sum_i \epsilon_i g(\tilde{C}(Je_i, JY)e_i, W) &= b\big[2S(JY, JW) - r^* g(JY, W)\big] \\ &\quad - \frac{r}{n}\Big[\frac{a}{n-1} + 2b\Big]g(JY, JW).\end{aligned} \tag{3}$$

This implies that

$$\begin{aligned}\sum_i \epsilon_i \tilde{C}(Je_i, JY)e_i &= b\big[-2QY - r^* JY\big] \\ &\quad + \frac{r}{n}\Big[\frac{a}{n-1} + 2b\Big]Y,\end{aligned} \tag{4}$$

where r^* is the $*$-scalar curvature, which is defined as the trace of JQ. In the above we have applied the identity $\sum_i \epsilon_i g(Je_i, e_i) = 0$, which is a consequence of the traceless of J.

The holomorphocally projective curvature tensor is defined in the following way [4, 7]

$$P(X,Y) = \mathcal{R}(X,Y) - \frac{1}{n-2}(X \wedge_S Y - JX \wedge_S JY), \tag{5}$$

where the operator $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y, \qquad Z \in \chi(M). \tag{6}$$

We notice, for later use, that this tensor has the following properties

$$\begin{aligned}&P(X,Y,Z,W) = -P(Y,X,Z,W), \quad P(JX,JY,Z,W) = -P(X,Y,Z,W), \\ &\sum_i \epsilon_i P(e_i, Y, Z, Je_i) = 0, \quad \sum_i \epsilon_i P(X, Y, e_i, e_i) = 0,\end{aligned} \tag{7}$$

A Kähler-Norden manifold (M, J, g) is holomorphically projectively flat if and only if its holomorphically projective curvature tensor P vanishes identically.

A Riemannian manifold is said to be quasi-conformally semisymmetric if $\mathcal{R}(X,Y).\tilde{C}=0$, where $\mathcal{R}(X,Y)$ denotes the derivation of the tensor algebra at each point of the manifold for tangent vector fields X, Y.

3. Qusi-conformally flat Kähler-Norden manifolds

In this section we study qusi-conformally flat Kähler-Norden manifolds, that is, $\tilde{C}(X,Y)Z=0$. Therefore from (3) we obtain

$$b[2S(JY,JW)-r^*g(JY,W)]=\frac{r}{n}\left[\frac{a}{n-1}+2b\right]g(JY,JW), \tag{8}$$

Using (1) in (8) yields

$$b[-2S(Y,W)-r^*g(JY,W)]=-\frac{r}{n}\left[\frac{a}{n-1}+2b\right]g(Y,W), \tag{9}$$

Contracting (9) with respect to the pair of arguments Y, W (that is, taking $Y=W=e_i$ into (9), multiplying by ϵ_i and summing up over $i\in\{1,\dots,n\}$), we have

$$-2br=-\frac{r}{n}\left[\frac{a}{n-1}+2b\right]n. \tag{10}$$

This implies

$$-\frac{a}{n-1}r=0. \tag{11}$$

Since $a\neq 0$, then from (11) we obtain

$$r=0. \tag{12}$$

Again using (12) in (9) we obtain

$$S(Y,W)=-\frac{r^*}{2b}g(JY,W). \tag{13}$$

Using (12), (13) in (2) we have

$$\begin{aligned}\mathcal{R}(X,Y)Z=-\frac{r^*}{2a}\big[&-g(JY,Z)X+g(JX,Z)Y\\&-g(Y,Z)JX+g(X,Z)JY\big].\end{aligned} \tag{14}$$

Also holomorphically projectively flatness implies from (5)

$$\begin{aligned}\mathcal{R}(X,Y)Z=\frac{1}{n-2}\big[&S(Y,Z)X-S(X,Z)Y\\&-S(JY,Z)JX+S(JX,Z)JY\big].\end{aligned} \tag{15}$$

Therefore from (13) and (15) it follows that

$$\begin{aligned}\mathcal{R}(X,Y)Z=\frac{r^*}{2b(n-2)}\Big[&-g(JY,Z)X+g(JX,Z)Y-g(Y,Z)JX\\&+\quad g(X,Z)JY\Big].\end{aligned} \tag{16}$$

From equations (14) and (16) we obtain $r^*[a+(n-2)b]=0$. Now, $r^*[a+(n-2)b]=0$ implies either $r^*=0$ or, $a+(n-2)b=0$. If $a+(n-2)b=0$, then putting this into (2), we get $\tilde{C}(X,Y)Z=aC(X,Y)Z$. So the quasi-conformally flatness and conformally flatness are equivalent in this case. Thus in view of the above result we can state the following:

Theorem 3.1. *If a quasi-conformally flat Kähler-Norden manifold is holomorphically projectively flat, then quasi-conformally flatness and conformally flatness are equivalent provided* $r^*\neq 0$.

Corollary 3.1. *The Ricci tensor and curvature tensor of a quasi-conformally flat Kähler-Norden manifold* (M,J,g) *have the shapes* (13) *and* (14)*, respectively.*

4. KÄHLER-NORDEN MANIFOLDS (M,J,g) WITH PARALLEL QUASI-CONFORMAL CURVATURE TENSOR

Assume that the quasi-conformal curvature tensor of a Kähler-Norden manifold is parallel, that is, $\nabla\tilde{C}=0$. From (3) we have

$$\begin{aligned}\sum_i \epsilon_i g(\tilde{C}(Je_i,JY)e_i,W) &= b\big[2S(JY,JW)-r^*g(JY,W)\big]\\ &\quad -\frac{r}{n}\Big[\frac{a}{n-1}+2b\Big]g(JY,JW),\end{aligned}\tag{17}$$

where r^* is the $*$-scalar curvature, which is defined as the trace of JQ. Taking covariant differentiation of (17) and our assumption yields

$$\begin{aligned}0 &= b[-2(\nabla_Z S)(Y,W)-dr^*(Z)g(JY,W)]\\ &\quad +\frac{dr(Z)}{n}\Big[\frac{a}{n-1}+2b\Big]g(Y,W),\end{aligned}\tag{18}$$

since $S(JY,JW)=-S(Y,W)$ and $g(JY,JW)=-g(Y,W)$.

Contracting (18) with respect to the pair of arguments Y, W (that is, taking $Y=W=e_i$ into (18), multiplying by ϵ_i and summing up over $i\in\{1,\dots,n\}$), we have

$$-2bdr(Z)+\frac{dr(Z)}{n}\left[\frac{a}{n-1}+2b\right]n=0.\tag{19}$$

Since $a\neq 0$, then (19) implies

$$dr(Z)=0.\tag{20}$$

Using (20) in (18) we have

$$(\nabla_Z S)(Y,W)=-\frac{1}{2}dr^*(Z)g(JY,W).\tag{21}$$

Putting $Y=JY$ in (21) we obtain

$$(\nabla_Z S)(JY,W)=\frac{1}{2}dr^*(Z)g(Y,W).\tag{22}$$

Contracting (22) with respect to the pair of arguments Y, W (that is, taking $Y = W = e_i$ into (22), multiplying by ϵ_i and summing up over $i \in \{1, \dots, n\}$), we have

$$dr^*(Z) = 0. \tag{23}$$

Again using (20) and (23) in (18) yields

$$(\nabla_Z S)(Y, W) = 0. \tag{24}$$

In view of (2), the covariant derivative $\nabla\tilde{C}$ can be expressed in the following form

$$\begin{aligned}(\nabla_W\tilde{C})(X,Y)Z) = a(\nabla_W\mathcal{R})(X,Y)Z + b\big[&(\nabla_W S)(Y,Z)X \\ &- (\nabla_W S)(X,Z)Y + g(Y,Z)(\nabla_W Q)X \\ &- g(X,Z)(\nabla_W Q)Y\big].\end{aligned} \tag{25}$$

Using (24) in (25) we obtain

$$(\nabla_W\tilde{C})(X,Y)Z) = a(\nabla_W\mathcal{R})(X,Y)Z. \tag{26}$$

Since $a \neq 0$, then in view of the above result we can state the following:

Theorem 4.1. *A Kähler-Norden manifold (M, J, g) is quasi-conformally symmetric if and only if it is locally symmetric.*

5. QUASI-CONFORMALLY SEMISYMMETRIC KÄHLER-NORDEN MANIFOLDS

In this section we study Quasi-conformally semisymmetric Kähler-Norden manifolds. Assume that $\mathcal{R}.\tilde{C} = 0$. From (4) we have

$$\begin{aligned}\sum_i \epsilon_i\tilde{C}(Je_i, JY)e_i = b[-2QY - r^*JY] \\ + \frac{r}{n}\Big[\frac{a}{n-1} + 2b\Big]Y,\end{aligned} \tag{27}$$

where r^* is the $*$-scalar curvature, which is defined as the trace of JQ.

Since $\mathcal{R}.\tilde{C} = 0$, then from (27) we have $\mathcal{R}.Q = 0$ and hence $\mathcal{R}.S = 0$. Again

$$\begin{aligned}\tilde{C}(X,Y)Z = a\mathcal{R}(X,Y)Z + b\big[&S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\big] \\ &- \frac{r}{n}\Big[\frac{a}{n-1} + 2b\Big]\big[g(Y,Z)X - g(X,Z)Y\big],\end{aligned} \tag{28}$$

where a and b are constants and $\mathcal{R}$, Q and r are Riemannain curvature tensor of type $(1,3)$, the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and the scalar curvature, respectively.

By the $\mathcal{R}.\tilde{C} = 0$ and $\mathcal{R}.S = 0$ from (28) we have $\mathcal{R}.\mathcal{R} = 0$.

Convers by,

$$\mathcal{R}.\mathcal{R} = 0 \;\Rightarrow\; \mathcal{R}.S = 0 \;\Rightarrow\; \mathcal{R}.Q = 0 \;\Rightarrow\; \mathcal{R}.\tilde{C} = 0. \tag{29}$$

From the above results we can state the following:

Theorem 5.1. *A Kähler-Norden manifold* (M, J, g) *is quasi-conformally semisymmetric if and only if it is semisymmetric.*

In [4], Sluka proved that

Theorem 5.2. [4] *A Kähler-Norden manifold* (M, J, g) *is holomorphically projectively semisymmetric if and only if it is semisymmetric.*

In view of Theorems 5.1 and 5.2, we can state the following:

Theorem 5.3. *A Kähler-Norden manifold* (M, J, g) *is quasi-conformally semisymmetric if and only if it is holomorphically projectively semisymmetric.*

6. Example

In [4] Sluka cited an example of a Kähler-Norden manifold which is locally symmetric. This example verifies our Theorem 4.1.

Acknowledgement: The authors are thankful to the referee for his/her comments and valuable suggestions towards the improvement of this paper.

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