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Existence of Coincidence Point for a Pair of Single-Valued and Multivalued Mappings

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Abstract. In this paper we establish some results on the existence of coincidence point for multivalued Kannan maps using the concept of w-distance. Our results generalize and extend some well known results due to Latif and Albar [5] and others.

1. Introduction and Preliminaries

Using the concept of Hausdorff metric, many authers have proved fixed point and coincidence point results in the setting of metric spaces. Nadler [7] has used the concept of Hausdorff metric and obtained a multivalued version of the Banach contraction principle which states that each closed bounded valued contraction map on a complete metric space has a fixed point.On the other hand, Kannan [3] has proved an interesting fixed point result for single-valued maps in the setting of metric spaces which is not an extension of the Banach contraction principle.Latif and Beg [4] have obtained a multivalued version of the Kannan's fixed point result.

In [2] Kada et al. have introduced a notion of w-distance on a metric space and improved several results replacing the involved metric by a generalized distance. While Suzuki [8] generalized Kannan's fixed point result under w-distance. Without using the concept of Hausedorff metric, most recently Feng and Liu [1] introduced a notion of multivalued contractive maps and proved a fixed point result extending Nadler's fixed point result concerning multivalued contractions. The aim of this paper is to obtain some results on the existence of coincidence points for multivalued K_w-maps with weak commutativity condition.

Throughout this paper, X is a metric space with metric d, $Cl(X)$ a collection of all nonempty closed subset of X. Consider a single-valued map $f : X \to X$ and a multivalued map $T : X \to 2^X$.

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(a) An element $x \in X$ is called a coincidence point of f and T if $f(x) \in T(x)$.
(b) f is called Banach contraction if for a fixed constant $h \in (0, 1)$ and for each $x, y \in X$, $d(f(x), f(y)) \leq d(x, y)$.
(c) f is called Kannan contraction if for a fixed constant $r \in [0, \frac{1}{2})$ and for each $x, y \in X$, $d(f(x), f(y)) \leq r[d(x, f(x)) + d(y, f(y))]$. Clearly, Kannan contraction (which may not be continuous) is not a generalized of the Banach contraction principle. Kannan [3] has proved that each Kannan contraction self map on a complete metric space has a unique fixed point.

Definition 1 ([5]). A map $\phi : X \to R$ is called lower semi-continuous if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $\phi(x) \leq \liminf_{n\to\infty} \phi(x)$.

Definition 2 ([5]). A function $w : X \times X \to [0, \infty)$ is called w-distance on X if it satisfies the following conditions;

(w_1) $w(x, z) \leq w(x, y) + w(y, z)$;
(w_2) a map $w(x, .) : X \to [0, \infty)$ is lower semi-continuous;
(w_3) for any ϵ >0, there exists δ >0 such that $w(z, x) \leq \delta$ and $w(z, y) \leq \delta$ imply $w(x, y) \leq \epsilon$ for any $x, y, z \in X$.

Definition 3 ([5]). A multivalued map $T : X \to 2^X$ is K_w-map if there exists a nonnegative number r$\in [0, \frac{1}{2})$ and a w-distance function w such that for any $x \in M$, $u \in T(x)$ there exists $v \in T(y)$ for all $y \in M$ such that

$$w(u, v) \leq r\{w(x, u) + w(y, v)\}.$$

Definition 4 ([6]). Let $T : X \to Cl(X)$ be a multivalued map and $f : X \to X$ a single-valued map such that $T(X) \subset f(X)$. Then f and T are said to be weakly commutative if $f(T(X)) \subset T(f(X))$ for all $x \in X$.

Lemma 1 ([5]). *Let X be a metric space with metric d let w be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

(a) *if $w(x_n, y) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in N$, then $y = z$; in particular, if $w(x, y) = 0$ and $w(x, z) = 0$, then $y = z$;*
(b) *if $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to z;*
(c) *if $w(x_n, x_m) \leq \alpha_n$ for any $n, m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(d) *if $w(y, x_n) \leq \alpha_n$ for any $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

2. Main Results

Theorem 1. *Let $f : X \to X$ be a continuous function and let $T : X \to Cl(X)$ be a multivalued K_w map such that if f and T are weakly commute and*

$$\inf w(x,u) + w(x,T(x)) : x \in X > 0$$

for every $u \in X$ with $u \notin T(u)$. Then f and T has a coincidence point.

Proof. Let $x_0 \in X$ be an arbitrary element of X and let $y_1 = f(x_1) \in T(x_0)$. Since T is K_w-map, there exists $y_2 = f(x_2) \in T(x_1)$ such that

$$\begin{aligned} w(y_1, y_2) &\le r\big\{w(y_1, x_0) + w(y_1, y_2)\big\} \\ &\le \frac{r}{1-r} w(y_1, x_0); \qquad r \in \big[0, \tfrac{1}{2}\big) . \end{aligned}$$

Thus, we get a sequence $\{y_n\}$ in X such that for every $n \in N$, $y_{n+1} = f(x_{n+1}) \in T(x_n)$ and

$$w(y_n, y_{n+1}) \le \left(\frac{r}{1-r}\right) w(y_{n-1}, y_n)$$

for some fixed r, $0 < r < \frac{1}{2}$. Note that for any $n \in N$, we have

$$w(y_n, y_{n+1}) \le \left(\frac{r}{1-r}\right)^n w(y_1, y_0).$$

Put $\lambda = \frac{r}{1-r}$. Then $0 < r < 1$. For m and n positive integers such that $m > n$, we have

$$\begin{aligned} w(y_n, y_m) &\le w(y_n, y_{n+1}) + w(y_{n+1}, y_{n+2}) + \cdots + w(y_{m-1}, y_m) \\ &\le \lambda^n w(y_1, x_0) + \lambda^{n+1} w(y_1, x_0) + \cdots + \lambda^{m-1} w(y_1, x_0) \\ &\le \frac{\lambda^n}{1-\lambda} w(y_1, x_0), \end{aligned}$$

which implies that $w(y_n, y_m) \to 0$ as $n \to \infty$ and by Lemma 1 $\{y_n\}$ is a Cauchy sequence. From completeness of X, $\{y_n\}$ converges to some $v_0 \in X$. Thus

$$f(x_n) \to v_0.$$

Since f is continuous,

$$f(f(x_n)) \to f(v_0).$$

Note that for each $n \ge 1$

$$f(x_n) \in T(x_{n-1}).$$

By weak commutativity of f and T, we get

$$f(f(x_n)) \in f(T(x_{n-1})) \subseteq T(f(x_{n-1})).$$

Hence

$$f(v_0) \in T(v_0).$$

Let $n \in N$ be fixed. Since $\{y_m\}$ converges to some v_0 and $w(y_n, .)$ is lower semi-continuous, we have

$$w(y_n, v_0) \leq \lim_{m\to\infty} inf w(y_n, y_m) \leq \frac{\lambda^n}{1-\lambda} w(y_1, x_0).$$

Therefore, as $n \to \infty$, we have $w(y_n, v_0)$. Assume that $f(v_0) \in T((v_0)$. Then, by hypothesis, we have

$$\begin{aligned} 0 &< \inf\{w(y, v_0) + w(y, T(y)) : y \in X\} \\ &\leq \inf\{w(y_n, v_0) + w(y_n, T(y_n)) : n \in N\} \\ &\leq \inf\{w(y_n, v_0) + w(y_n, y_{n+1}) : n \in N\} \\ &\leq \inf\{\tfrac{\lambda^n}{1-\lambda} w(y_1, x_0) + \lambda^n w(y_1, x_0) : n \in N\} = 0, \end{aligned}$$

which is impossible and hence $f(v_0) \in T(v_0)$. □

Theorem 2. *Let $f : X \to X$ be a continuous single-valued map and let $\{T_n\}$ be a sequence of multivalued maps from X into $Cl(X)$. Suppose there exists $0 \leq r < \frac{1}{2}$ such that for any two maps T_i, $T_j \in T_n$, $i \neq j$, and for any $x \in X$, $u \in T_i(x)$ there exists $v \in T_j(y)$ for all $y \in X$ with*

$$w(u, v) \leq r\{w(x, u) + w(y, v)\},$$

and for each $n \geq 1$

$$\inf\{w(x, u) + w(x, T_n(x)) : x \in X\} > 0.$$

for any $u \in T_n(u)$, and f is weakly commuting with $\{T_n\}$ for every $n \in N$. Then f and $\{T_n\}_{n\in N}$ has a coincidence point.

Proof. Let x_0 be an arbitrary element of X and let $y_1 = f(x_1) \in T_1(x_0)$. Then there is an element $y_2 = f(x_2) \in T_2(x_1)$ such that

$$w(y_1, y_2) \leq \frac{r}{1-r} w(y_1, x_0).$$

So, there exists a sequence $\{y_n\}$ such that $y_{n+1} = f(x_{n+1}) \in T_{n+1}(x_n)$ for every $n \geq 1$,

$$w(y_n, y_{n+1}) \leq \left(\frac{r}{1-r}\right)^n w(y_1, x_0).$$

Put $\lambda = \frac{r}{1-r}$. Note that $0 < \lambda < 1$ and

$$w(y_n, y_{n+1}) \leq \lambda^n w(y_1, x_0)$$

for all $n \geq 1$. Then as $n \to \infty$, $\{y_n\}$ is a Cauchy sequence in X. By completeness of X, $y_n \to p \in X$ i.e., $f(x_n) \to p$.

By continuity of f

$$f(f(x_n)) \to f(p)$$

and

$$f(x_n) \in T_n(x_{n-1}).$$

By weak commutativity of f and T_n for every n,

$$f(f(x_n)) \in f(T_n(x_{n-1})) \subset T_n(f(x_{n-1})).$$

Since $\{y_n\}$ converges to p and $w(y_n, .)$ is lower semi-continuous, following the proof of Theorem 1 we obtain

$$w(y_n, p) \leq \lim_{m\to\infty} inf w(y_n, y_m) \leq \frac{\lambda^n}{a-\lambda} w(y_1, x_0),$$

which converges to 0 as $n \to \infty$. Now assume that $p \in T_m(p)$. Then by hypothesis, and for $n > m$ and $m \geq 1$ we have

$$\begin{aligned} 0 &< \inf\big\{w(y,p) + w(y, T_m(x)) : x \in X\big\} \\ &\leq \inf\big\{w(y_{m-1}, p) + w(y_{m-1}, T_m(x_{m-1})) : m \in N\big\} \\ &\leq \inf\big\{w(y_{m-1}, p) + w(y_{m-1}, y_m) : m \in N\big\} \\ &\leq \inf\big\{\tfrac{\lambda^{m-1}}{1-\lambda} w(y_1, x_0) + \lambda^{m-1} w(y_1, x_0) : m \in N\big\} = 0, \end{aligned}$$

which is not possible.

Therefore $f(p) \in T_m(p)$. But T_m is arbitrary, hence p is a coincidence point of f and $\{T_n\}_{n\in N}$. □

Example 1. Let $X = [0, 1]$ be a metric space with w-distance function $w : X \times X \to [0, \infty)$ defind by $w(x, y) = \sqrt{(x^2 + y^2)}$ and let

$$T(x) = \begin{cases} [0, \frac{1}{2}], & x \in [0, \frac{1}{2}]; \\ 1 - x, & x \in (\frac{1}{2}, 1]; \end{cases}$$

$$f(x) = \frac{x}{2}, \quad x \in [0, 1].$$

Since f and T are weakly commutative and satisfies all the conditions of Theorem 1. Then for $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \frac{1}{4}$$

and

$$T\left(\frac{1}{2}\right) = \left[0, \frac{1}{2}\right].$$

Therefore $f(\frac{1}{2}) \in T(\frac{1}{2})$. Hence $\frac{1}{2}$ is a coincidence point of f and T in X. Similarly, all points from $[0,\frac{1}{2}]$ are coincidence points of f and T in X.

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