Mathematica Moravica
Vol. **17-2** (2013), 15–22

Hybrid Pairs of Maps in Consideration of Common Fixed Point Theorems Using Property (E.A)

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Abstract. In this paper, we prove common fixed point theorems for two hybrid pairs of multivalued and single valued mappings on noncomplete metric spaces using the property (E.A). We improve the results of Damjanović et al [1] and several other authors.

1. Introduction

Jungck [3] defined the notion of compatible maps in order to generalize the concept of weak commutativity introduced by Sessa [16] and showed that the weakly commuting maps are compatible but the converse is not true.

The study of noncompatible mappings was initiated by Pant ([8]-[11]). He introduced $R-$weakly commutativity of mappings and compared $R-$ weak commutativity and weak compatibility for single valued mappings.

Recently, Aamri and Moutawakil [6] defined a property $(E.A)$ for self maps which contains the class of noncompatible maps. They obtained some fixed point theorems for such mappings using property $(E.A)$ under strict contractive conditions.

Nadler [12] published a paper on multivalued mappings. Since then, the fixed point theory for single valued and multivalued mappings has been studied extensively and applied to diverse problems. This theory provides techniques for solving a variety of applied problems in mathematical science and engineering. A number of generalization of Nadler's results have appeared.

Kaneko [4] extended the concept of weakly commuting mappings for multivalued set up and extended the result of Jungck [3]. Kaneko and Sessa [5] extended the concept of compatible mappings for multivalued mappings.

Kamran [17] extended property $(E.A)$ in the settings of single valued and multivalued mappings and generalized the notion of (IT)-commutativity for

2010 *Mathematics Subject Classification.* Primary: 54H25; Secondary: 47H10.
Key words and phrases. Multivalued map, property $(E.A)$, $T-$weakly commuting maps.

such pairs. He introduced the notion of $T-$weakly commuting map and showed that for hybrid pairs of mappings, $(IT)-$commuting at coincidence points implies $T-$weakly commuting but the converse is not true. He also showed that for single valued mappings $T-$weak commutativity at the coincidence points is equivalent to the weak compatibility.

In this paper, we prove common fixed point theorems for two hybrid pairs of single valued and multivalued mappings on noncomplete metric spaces. We improve the results of Damjanović et al [1], Gordji et al [7], Hardy and Rogers [2] and Nadler [12] by dropping the completeness of the whole space and the subspaces and using a weaker condition property (E.A).

2. PRELIMINARIES

Let (X, d) be a metric space and suppose that $CB(X)$ denotes the set of non-empty, closed and bounded subsets of X.

For $A, B \in CB(X)$, we denote

$$D(A, B) = \inf \{d(a, b) : a \in A, b \in B\},$$
$$D(x, A) = \inf \{d(x, a) : a \in A\},$$
$$H(A, B) = \max \{\sup \{D(a, B) : a \in A\}, \sup \{D(A, b) : b \in B\}\}.$$

It is well known that $(CB(X), H)$ is a metric space with the distance function H. Moreover, $(CB(X), H)$ is complete in the event that (X, d) is complete.

Definition 1 ([4])**.** Let (X, d) be a metric space, $F : X \to CB(X)$ and $T : X \to X$. Then the pair $\{F, T\}$ is said to be weakly commuting if for each $x \in X$, $TF(x) \in CB(X)$ and $H(FTx, TFx) \leq D(Tx, Fx)$.

Definition 2 ([5])**.** Let (X, d) be a metric space, $F : X \to CB(X)$ and $T : X \to X$. Then the pair $\{F, T\}$ is said to be compatible if and only if $TFx \in CB(X)$ for each $x \in X$ and $H(FTx_n, TFx_n) \to 0$, whenever $\{x_n\}$ is a sequence in X such that $Fx_n \to M \in CB(X)$ and $Tx_n \to t \in M$.

Definition 3 ([7])**.** Let $T : X \to CB(X)$ be a multivalued map. An element $x \in X$ is said to be a fixed point of T if $x \in Tx$.

Definition 4. [6] The maps $f : X \to X$ and $g : X \to X$ are said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$.

Definition 5 ([17])**.** The maps $f : X \to X$ and $T : X \to CB(X)$ are said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$ for some $t \in X$ and $A \in CB(X)$.

Definition 6 ([17])**.** Let $T : X \to CB(X)$. The map $f : X \to X$ is said to be $T-$weakly commuting at $x \in X$ if $ffx \in Tfx$.

Definition 7 ([13])**.** The mappings $T : X \to X$ and $F : X \to CB(X)$ are said to be $(IT)-$commuting at $x \in X$ if $TFx \subseteq FTx$.

Definition 8 ([7])**.** An element $x \in X$ is said to be a coincidence point of $T : X \to CB(X)$ and $f : X \to X$ if $fx \in Tx$. We denote

$$C(f,T) = \{x \in X | fx \in Tx\}$$

the set of coincidence points of T and f.

Definition 9 ([7])**.** An element $x \in X$ is a common fixed point of $T, S : X \to CB(X)$ and $f : X \to X$ if $x = fx \in Tx \cap Sx$.

Example 1. Let $X = (0, \infty)$ with the usual metric d. Define $f : X \to X$ and $F : X \to CB(X)$ by $fx = 4x$ and $Fx = [0, 2 + 4x]$ for all $x \in X$.

Consider the sequence $\{x_n\}$ in X given by $x_n = 1 + \dfrac{1}{n}$, $n = 1, 2, 3, \ldots$

Then $\lim_{n\to\infty} fx_n = 4 \in [0, 6] = \lim_{n\to\infty} Fx_n$ and $\lim_{n\to\infty} H(fFx_n, Ffx_n) \neq 0$.

Therefore, f and F satisfy property $(E.A)$, but they are not compatible.

Also for all $x \in X, fx \in Fx, ffx = 16x \in Ffx = [0, 2 + 16x]$. Therefore f is $F-$weakly commuting.

Further, $fFx = [0, 8 + 16x] \nsubseteq Ffx = [0, 2 + 16x]$. Therefore f and F are not $(IT)-$commuting. Also note that f and F are not weakly compatible.

Example 2. Let $X = [0, 1)$ with the usual metric d. Define $f : X \to X$ and $F : X \to CB(X)$ by $fx = \dfrac{x}{2}$ and $Fx = [0, x]$ for all $x \in X$.

Consider the sequence $\{x_n\}$ in X given by $x_n = \dfrac{n-1}{2(n+1)}$, $n = 1, 2, 3, \ldots$

Then $\lim_{n\to\infty} fx_n = \frac{1}{4} \in \left[0, \frac{1}{2}\right] = \lim_{n\to\infty} Fx_n$ and $\lim_{n\to\infty} H(fFx_n, Ffx_n) = 0$.

Therefore, f and F satisfy property $(E.A)$ and the hybrid pair $\{f, F\}$ is compatible. Also for all $x \in X$, $fx \in Fx$, $ffx = \frac{x}{4} \in Ffx = \left[0, \frac{x}{2}\right]$. Therefore f is $F-$weakly commuting. Also the pair $\{f, F\}$ is $(IT)-$commuting because $fFx \subseteq Ffx$.

The Nadler's [12] fixed-point theorem is the following:

Theorem 1. *Let (X, d) be a complete metric space, and $T : X \to CB(X)$ be a multi valued map satisfying-$H(Tx, Ty) \leq qd(x, y)$ for all $x, y \in X$, where q is a constant such that $q \in [0, 1)$. Then, T has a fixed point.*

Recently, an extension of Theorem 2.1 was obtained by Gordji et al [7]. They proved the following result:

Theorem 2. *Let (X, d) be a complete metric space, and T be a map from X into $CB(X)$ such that*

$$H(Tx, Ty) \leq \alpha d(x, y) + \beta[D(x, Tx) + D(y, Ty)] + \gamma[D(x, Ty) + D(y, Tx)]$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then, T has a fixed point.

Damjanović et al [1] proved the following results on a complete metric space:

Theorem 3. *Let (X, d) be a complete metric space. Let $T, S : X \to CB(X)$ be a pair of multi valued maps and $f, g : X \to X$ a pair of single valued maps. Suppose that*

$$H(Sx, Ty) \leq \alpha d(fx, gy) + \beta[D(fx, Sx) + D(gy, Ty)] \\ + \gamma[D(fx, Ty) + D(gy, Sx)]$$

for each $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Suppose also that

(*i*) *$SX \subseteq gX, TX \subseteq fX$,*
(*ii*) *$f(X)$ and $g(X)$ are closed.*

Then, there exist points u and w in X, such that

$$fu \in Su, gw \in Tw, fu = gw \text{ and } Su = Tw.$$

Theorem 4. *Let (X, d) be a complete metric space. Let $T, S : X \to CB(X)$ be multi valued maps and $f : X \to X$ be a single valued map satisfying, for each $x, y \in X$,*

$$H(Sx, Ty) \leq \alpha d(fx, fy) + \beta[D(fx, Sx) + D(fy, Ty)] \\ + \gamma[D(fx, Ty) + D(fy, Sx)]$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. If fX is a closed subset of X and $TX \cup SX \subseteq fX$, then f, T and S have a coincidence in X. Moreover, if f is both $T-$weakly commuting and $S-$weakly commuting at each $z \in C(f, T)$, and $ffz = fz$, then f, T, and S have a common fixed point in X.

We prove the following results:

3. MAIN RESULTS

Theorem 5. *Let (X, d) be a metric space. Let $T, S : X \to CB(X)$ be a pair of multi valued maps and $f, g : X \to X$ a pair of single valued maps. Suppose that:*

(3.1.1) *$SX \subseteq gX, TX \subseteq fX$,*
(3.1.2) *for $\alpha, \beta, \gamma \geq 0$ and $0 < \alpha + 2\beta + 2\gamma < 1$ and for all $x, y \in X$*

$$H(Sx, Ty) \leq \alpha d(fx, gy) + \beta[D(fx, Sx) + D(gy, Ty)] \\ + \gamma[D(fx, Ty) + D(gy, Sx)],$$

(3.1.3) *the pairs $\{S, f\}$ and $\{T, g\}$ satisfy property $(E.A)$.*

Then there exist points u and w in X such that

$$fu \in Su, gw \in Tw, Su = Tw, \text{ and } fu = gw.$$

Proof. Since the pair $\{S, f\}$ satisfies property $(E.A)$, there is a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Sx_n$$

for some $t \in X$ and $A \in CB(X)$.

Since $S(X)$ is closed, $t \in S(X)$. Therefore by (3.1.1) there exists $w \in X$ such that $t = gw$.

By (3.1.2), we have

$$\begin{aligned} H(Sx_n, Tw) \le \alpha d(fx_n, gw) &+ \beta[D(fx_n, Sx_n) + D(gw, Tw)] \\ &+ \gamma[D(fx_n, Tw) + D(gw, Sx_n)]. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} H(A, Tw) \le \alpha d(t, gw) &+ \beta[D(t, A) + D(gw, Tw)] \\ &+ \gamma[D(t, Tw) + D(gw, A)], \end{aligned}$$

which gives

$$\begin{aligned} H(A, Tw) &\le \beta D(t, Tw) + \gamma D(t, Tw) \\ &\le (\beta + \gamma) D(A, Tw) \\ &\le (\beta + \gamma) H(A, Tw) \\ &< H(A, Tw), \end{aligned}$$

as $\beta + \gamma < 1$. This is a contradiction. Therefore $A = Tw$ i.e. $gw \in Tw$.

Again Since the pair $\{T, g\}$ satisfies property $(E.A)$, there is a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} gy_n = q \in M = \lim_{n\to\infty} Ty_n$$

for some $q \in X$ and $M \in CB(X)$.

Since $T(X)$ is closed, $q \in T(X)$. Therefore by (3.1.1) there exists $u \in X$ such that $q = fu$.

By (3.1.2), we have

$$\begin{aligned} H(Su, Ty_n) \le \alpha d(fu, gy_n) &+ \beta[D(fu, Su) + D(gy_n, Ty_n)] \\ &+ \gamma[D(fu, Ty_n) + D(gy_n, Su)]. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} H(Su, M) &\leq \beta D(fu, Su) + \gamma D(fu, Su) \\ &\leq (\beta + \gamma)\, D\,(Su, M) \\ &\leq (\beta + \gamma)\, H(Su, M) \\ &< H(Su, M), \end{aligned}$$

which is a contradiction as $\beta + \gamma < 1$. Therefore $M = Su$ i.e. $fu \in Su$.

Again from (3.1.2),

$$\begin{aligned} H(Su, Tw) &\leq \alpha d(fu, gw) + \beta[D(fu, Su) + D(gw, Tw)] \\ &\qquad + \gamma[D(fu, Tw) + D(gw, Su)] \\ &\leq \alpha d(fu, gw) + \gamma[D(fu, Tw) + D(gw, Su)] \\ &\leq \alpha H(Su, Tw) + 2\gamma H(Su, Tw) \\ &= (\alpha + 2\gamma)\, H(Su, Tw) \\ &< H(Su, Tw), \end{aligned}$$

which is a contradiction as $\alpha + 2\gamma < 1$. Therefore $Su = Tw$.

Since for any $h > 1$,

$$d\,(fu, gw) \leq hH\,(Su, Tw)\,,$$

therefore

$$fu = gw.$$

This proves the theorem. □

Remark 1. Theorem 3.1 improves the Theorem 2.3 of Damjanović et al [1] in the sense that the completeness of the whole space the closedness of the subspaces fX and gX are dropped.

Theorem 6. *Let* (X, d) *be a metric space. Let* $T, S : X \to CB(X)$ *be a pair of multi valued maps and* $f, g : X \to X$ *a pair of single valued maps. Suppose that* (3.1.1), (3.1.2) *and* (3.1.3) *hold. Further if,*

(3.2.1) *f is $S-$weakly commuting and g is $T-$weakly commuting at their coincidence point.*

Then

(ii) *if $fu = gw = z \in X$, then $fz \in Tz$ and $gz \in Sz$,*

(iii) *if $fz = gz$ then $fz = gz \in Sz \cap Tz$,*

(iv) *if $fz = gz = z$, then z is a common fixed point of f, g, S and T.*

Proof. It has been established in the Theorem 3.1 that there exist points u and w in X such that

$$fu \in Su, \;\; gw \in Tw, \;\; Su = Tw, \;\; \text{and} \;\; fu = gw.$$

Let $fu = gw = z$. Since f is $S-$weakly commuting and g is $T-$weakly commuting at their coincidence point, we have

$$ffu \in Sfu, ggw \in Tgw.$$

Therefore

$$fz \in Sz, gz \in Tz$$

This proves (ii).

If $fz = gz$, then $fz = gz \in Sz \cap Tz$. This proves (iii).

If $fz = gz = z$, then $z = fz = gz \in Sz \cap Tz$. Therefore z is a common fixed point of f, g, S and T.

This proves the theorem. □

Remark 2. If $f = g$ in Theorem 3.2, we get the following corollary which is an improvement of Theorem 2.4 of Damjanović et al [1].

Corollary 1. *Let (X, d) be a metric space. Let $T, S : X \to CB(X)$ be a pair of multi-valued maps and $f : X \to X$ a single-valued map satisfying:*

(3.3.1) $SX \cup TX \subseteq fX$,

(3.3.2) *for* $\alpha, \beta, \gamma \geq 0$ *and* $0 < \alpha + 2\beta + 2\gamma < 1$ *and for all* $x, y \in X$

$$H(Sx, Ty) \leq \alpha d(fx, fy) + \beta[D(fx, Sx) + D(fy, Ty)] + \gamma[D(fx, Ty) + D(fy, Sx)],$$

(3.3.3) *the pairs* $\{S, f\}$ *and* $\{T, f\}$ *satisfy property* $(E.A)$.

Then there exist points u and w in X such that

$$fu \in Su, \quad fw \in Tw, \quad Su = Tw, \quad \text{and} \quad fu = fw.$$

Further if,

(3.3.4) *f is both $S-$weakly commuting and $T-$weakly commuting at coincidence point.*

Then if

(ii) $fu = fw = z \in X$, *then* $fz \in Sz \cap Tz$.

(iii) $fz = z$, *then z is a common fixed point of f, S and T.*

Remark 3. If $f = g = I_X$ (I_X being the identity map on X) and $S = T$ in Corollary 3.3, then, we obtain an improvement of Theorem 2.2 of Gordji et al [7].

Remark 4. In Theorem 3.1, if:

(i) $\beta = \gamma = 0$ and $S = T$; $f = g = I_X$, then we obtain an improved theorem of Nadler [12].

(ii) $S = T$ and $f = g = I_X$, then we obtain improved results of Reich ([14, 15]).

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