Mathematica Moravica
Vol. **17-1** (2013), 11–24

Fixed Point Theorems in Probabilistic Metric Spaces Using Property (E.A)

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Abstract. In this paper, we prove a common fixed point theorem for even number of self mappings in Menger space by using an implicit relation with property (E.A). We also extend our main result to four finite families of mappings employing the notion of pairwise commuting due to Imdad et al. [Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces, Chaos, Solitons & Fractals 42(5) (2009), 3121–3129]. Our results generalize and extend several well known comparable results existing in literature.

1. Introduction and preliminaries

Karl Menger [14] introduced the notion of a probabilistic metric space (briefly, PM-space) in 1942. The idea of Menger was to utilize distribution functions instead of non-negative real numbers as values of the metric. The notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. The study such spaces received an impetus with the pioneering work of Schweizer and Sklar [21]. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis especially due to its extensive applications in random differential.

In 1986, Jungck [8] introduced the notion of compatible mappings for a pair of self mappings in metric space. Mishra [16] extended this notion of compatibility to PM-space and obtained some interesting common fixed point results in this setting. It can be easily seen that most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the involved mappings. However, the study of common fixed points of non-compatible mappings is also of great interest due to Pant [17]. In 2002, Aamri and

1991 *Mathematics Subject Classification.* Primary: 47H10; Secondary: 54H25.

Key words and phrases. t-norm, probabilistic metric space, property (E.A), weakly compatible mappings, non-compatible mappings, implicit relation.

Moutawakil [1] defined the notion of property (E.A) for a pair of self mappings which contained the class of non-compatible mappings and proved common fixed point theorems in metric spaces under strict contractive condition. Since then, Kubiaczyk and Sharma [10] studied the common fixed points of weakly compatible mappings satisfying the property (E.A) in PM-spaces for the existence of common fixed point. Subsequently, there are a number of results wherein the notion of property (E.A) is used; for instance see [5], [7], [13]. Recently, Ali et al. [2] extended the notion of common property (E.A) in Menger spaces and proved some common fixed point theorems which generalized several known results in Menger space as well as metric spaces (also, see [3]).

In 2005, Miheţ [15] proved some fixed point theorems concerning probabilistic contractions satisfying an implicit relation. Kumar and Pant [12, 13] and Pant and Chauhan [18, 19] proved some fixed point theorems in PM-spaces satisfying implicit relation.

Most recently, Kumar et al. [11] proved a common fixed point theorem for two pairs of non-compatible mappings employing an implicit relation in PM-space. The aim of this paper is to prove a common fixed point theorem for even number of self mappings satisfying an implicit relation. Further, we extend our main result to four finite families of mappings in PM-space.

For the sake of completeness, we recall some definitions and properties of Menger spaces.

Definition 1.1. [21] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

We shall denote by $\Im$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.2. [21] A PM-space is an ordered pair $(X, \mathcal{F})$, where X is a non-empty set of elements and $\mathcal{F}$ is a mapping from $X \times X$ to $\Im$, the collection of all distribution functions. The value of $\mathcal{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$. The function $F_{x,y}$ is assumed to satisfy the following conditions for all $x, y, z \in X$ and $t, s > 0$:

(1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;
(2) $F_{x,y}(0) = 0$;
(3) $F_{x,y}(t) = F_{y,x}(t)$;
(4) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$.

Definition 1.3. [21] A mapping $\triangle : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if the following conditions are satisfied for all $a, b, c, d \in [0, 1]$:

(1) $\triangle(a, 1) = a$ for all $a \in [0, 1]$;
(2) $\triangle(a, b) = \triangle(b, a)$;

(3) $\triangle(a,b) \leq \triangle(c,d)$ for $a \leq c$, $b \leq d$;
(4) $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$;

Examples of continuous t-norms are $\triangle(a,b) = \min\{a,b\}$, $\triangle(a,b) = ab$ and $\triangle(a,b) = \max\{a+b-1,0\}$.

Definition 1.4. [21] A Menger space is a triplet $(X,\mathcal{F},\triangle)$ where $(X,\mathcal{F})$ is a PM-space and $\triangle$ is a t-norm such that the inequality

$$F_{x,z}(t+s) \geq \triangle\left(F_{x,y}(t), F_{y,z}(s)\right),$$

holds for all $x,y,z \in X$ and all $t,s > 0$.

Every metric space (X,d) can be realized as a PM-space by taking $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x,y))$ for all $x,y \in X$.

Definition 1.5. [21] Let $(X,\mathcal{F},\triangle)$ be a Menger space with continuous t-norm $\triangle$.

(1) A sequence $\{x_n\}$ in X is said to be converge to a point x in X if and only if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $\mathbb{N}(\epsilon,\lambda)$ such that $F_{x_n,x}(\epsilon) > 1-\lambda$ for all $n \geq \mathbb{N}(\epsilon,\lambda)$.
(2) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $\mathbb{N}(\epsilon,\lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1-\lambda$ for all $n,m \geq \mathbb{N}(\epsilon,\lambda)$.

A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.6. [16] A pair (A,S) of self mappings of a Menger space $(X,\mathcal{F},\triangle)$ is said to be compatible if and only if $F_{ASx_n,SAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Definition 1.7. [5] A pair (A,S) of self mappings of a Menger space $(X,\mathcal{F},\triangle)$ is said to be non-compatible if and only if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$, but for some $t > 0$, $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t)$ is either less than 1 or non-existent.

Definition 1.8. [4, 9] A pair (A,S) of self mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $Az = Sz$ for some $z \in X$, then $ASz = SAz$.

Remark 1.1. Two compatible self mappings are weakly compatible, but the converse is not true in general (see [22, Example 1]). Hence the notion of weak compatibility is more general than compatibility.

Definition 1.9. [10] A pair (A,S) of self mappings of a Menger space $(X,\mathcal{F},\triangle)$ is said to satisfy the property (E.A), if there exists a sequence

$\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Note that weakly compatible and property (E.A) are independent to each other (see [20, Example 2.2]). Also a pair of non-compatible self mappings of $(X, \mathcal{F})$ satisfies the property (E.A) but the converse need not be true (see [5, Example 1]).

2. Implicit Relation

Pant and Chauhan [18] proved a common fixed point theorem concerning probabilistic contractions satisfying the following implicit relation:

Let Φ be the class of all real continuous functions $\varphi : (\mathbb{R}^+)^4 \to \mathbb{R}$, non-decreasing in the first argument and satisfying the following conditions:

(R-1) $u, v \geq 0, \varphi(u, v, u, v) \geq 0$ or $\varphi(u, v, v, u) \geq 0$ implies that $u \geq v$.
(R-2) $\varphi(u, u, 1, 1) \geq 0$ for all $u \geq 1$.

Example 2.1. [18] Define $\varphi(t_1, t_2, t_3, t_4) = at_1 + bt_2 + ct_3 + dt_4$, where $a, b, c, d \in \mathbb{R}$ with $a+b+c+d = 0$, $a > 0$, $a+c > 0$, $a+b > 0$ and $a+d > 0$. Then $\varphi \in \Phi$.

Example 2.2. Define $\varphi(t_1, t_2, t_3, t_4) = 14t_1 - 12t_2 + 6t_3 - 8t_4$. Then $\varphi \in \Phi$.

3. Results

Theorem 3.1. *Let $P_1, P_2, \ldots, P_{2n}, A$ and B be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. Further, let the pairs $(A, P_1P_3 \ldots P_{2n-1})$ and $(B, P_2P_4 \ldots P_{2n})$ be weakly compatible satisfying:*

(1) *$A(X) \subset P_2P_4 \ldots P_{2n}(X)$, $B(X) \subset P_1P_3 \ldots P_{2n-1}(X)$;*
(2) *One of $A(X)$, $B(X)$, $P_1P_3 \ldots P_{2n-1}(X)$ and $P_2P_4 \ldots P_{2n}(X)$ is a complete subspace of X;*
(3) *Suppose that*

$$P_1(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})P_1,$$
$$P_1P_3(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})P_1P_3,$$
$$\vdots$$
$$P_1 \ldots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1 \ldots P_{2n-3},$$
$$A(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})A,$$
$$A(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})A,$$
$$\vdots$$
$$AP_{2n-1} = P_{2n-1}A,$$

similarly,

$$P_2(P_4 \dots P_{2n}) = (P_4 \dots P_{2n})P_2,$$
$$P_2P_4(P_6 \dots P_{2n}) = (P_6 \dots P_{2n})P_2P_4,$$
$$\vdots$$
$$P_2 \dots P_{2n-2}(P_{2n}) = (P_{2n})P_2 \dots P_{2n-2},$$
$$B(P_4 \dots P_{2n}) = (P_4 \dots P_{2n})B,$$
$$B(P_6 \dots P_{2n}) = (P_6 \dots P_{2n})B,$$
$$\vdots$$
$$BP_{2n} = P_{2n}B;$$

(4) $(A, P_1P_3 \dots P_{2n-1})$ *or* $(B, P_2P_4 \dots P_{2n})$ *satisfies the property (E.A);*
(5) *there exist* $k \in (0,1)$ *and* $\varphi \in \Phi$ *such that*

$$\varphi\begin{pmatrix} F_{Ax,By}(kt), F_{P_1P_3 \dots P_{2n-1}x, P_2P_4 \dots P_{2n}y}(t), \\ F_{Ax,P_1P_3 \dots P_{2n-1}x}(kt), F_{By,P_2P_4 \dots P_{2n}y}(t) \end{pmatrix} \geq 0 \tag{1}$$

or,

$$\varphi\begin{pmatrix} F_{Ax,By}(kt), F_{P_1P_3 \dots P_{2n-1}x, P_2P_4 \dots P_{2n}y}(t), \\ F_{Ax,P_1P_3 \dots P_{2n-1}x}(t), F_{By,P_2P_4 \dots P_{2n}y}(kt) \end{pmatrix} \geq 0, \tag{2}$$

for all $x, y \in X$, $t > 0$.

Then $P_1, P_2, \dots, P_{2n}, A$ *and* B *have a unique common fixed point in* X.

Proof. Suppose that the pair $(A, P_1P_3 \dots P_{2n-1})$ enjoys the property (E.A), then there exists a sequence $\{x_n\}$ in X such that $Ax_n \to z$ and $P_1P_3 \dots P_{2n-1}x_n \to z$, for some $z \in X$ as $n \to \infty$. Since $A(X) \subset P_2P_4 \dots P_{2n}(X)$, hence for each $\{x_n\}$ there exists a sequence $\{y_n\}$ in X such that $Ax_n = P_2P_4 \dots P_{2n}y_n$. Therefore, $P_2P_4 \dots P_{2n}y_n \to z$ and $Ax_n \to z$ as $n \to \infty$. Thus in all, we have $Ax_n \to z$, $P_1P_3 \dots P_{2n-1}x_n \to z$ and $P_2P_4 \dots P_{2n}y_n \to z$ as $n \to \infty$. Now we claim that $By_n \to z$ as $n \to \infty$. Suppose $By_n \to w (\neq z) \in X$, then applying inequality (1), we have

$$\varphi\begin{pmatrix} F_{Ax_n,By_n}(kt), F_{P_1P_3 \dots P_{2n-1}x_n, P_2P_4 \dots P_{2n}y_n}(t), \\ F_{Ax_n,P_1P_3 \dots P_{2n-1}x_n}(kt), F_{By_n,P_2P_4 \dots P_{2n}y_n}(t) \end{pmatrix} \geq 0.$$

Taking the limit as $n \to \infty$, we obtain

$$\varphi\left(F_{z,w}(kt), F_{z,z}(t), F_{z,z}(kt), F_{w,z}(t)\right) \geq 0,$$

and so

$$\varphi\left(F_{z,w}(kt), 1, 1, F_{w,z}(t)\right) \geq 0.$$

As φ is non-decreasing in the first argument

$$\varphi\left(F_{z,w}(t), 1, 1, F_{w,z}(t)\right) \geq 0.$$

Using (R-1), we get $F_{z,w}(t) \geq 1$ for all $t > 0$, i.e., $F_{z,w}(t) = 1$, which implies $w = z$. Thus $By_n \to z$ as $n \to \infty$.

Suppose that $P_1P_3\dots P_{2n-1}(X)$ is a complete subspace of X. Then $z=(P_1P_3\dots P_{2n-1})u$ for some $u\in X$. Subsequently, we have $Ax_n\to(P_1P_3\dots P_{2n-1})u$, $By_n\to(P_1P_3\dots P_{2n-1})u$, $P_2P_4\dots P_{2n}y_n\to(P_1P_3\dots P_{2n-1})u$ and $P_1P_3\dots P_{2n-1}x_n\to(P_1P_3\dots P_{2n-1})u$ as $n\to\infty$. Using inequality (1), we have

$$\varphi\left(\begin{array}{c}F_{Au,By_n}(kt),F_{(P_1P_3\dots P_{2n-1})u,P_2P_4\dots P_{2n}y_n}(t),\\ F_{Au,(P_1P_3\dots P_{2n-1})u}(kt),F_{By_n,P_2P_4\dots P_{2n}y_n}(t)\end{array}\right)\geq 0.$$

Taking the limit as $n\to\infty$, we get

$$\varphi\left(F_{Au,z}(kt),F_{z,z}(t),F_{Au,z}(kt),F_{z,z}(t)\right)\geq 0,$$

and so

$$\varphi\left(F_{Au,z}(kt),1,F_{Au,z}(kt),1\right)\geq 0.$$

Using (R-1), we have $F_{Au,z}(kt)\geq 1$ for all $t>0$. Hence $F_{Au,z}(t)=1$, i.e., $Au=z$. Therefore $Au=(P_1P_3\dots P_{2n-1})u=z$ which shows that u is a coincidence point of the pair $(A,P_1P_3\dots P_{2n-1})$.

Since $A(X)\subset P_2P_4\dots P_{2n}(X)$, there exists a point v in X such that $Au=(P_2P_4\dots P_{2n})v$. Now, we assert that $(P_2P_4\dots P_{2n})v=Bv$. Using inequality (1), we have

$$\varphi\left(\begin{array}{c}F_{Au,Bv}(kt),F_{(P_1P_3\dots P_{2n-1})u,P_2P_4\dots P_{2n}v}(t),\\ F_{Au,(P_1P_3\dots P_{2n-1})u}(kt),F_{Bv,P_2P_4\dots P_{2n}v}(t)\end{array}\right)\geq 0,$$

and so

$$\varphi\left(F_{z,Bv}(kt),F_{z,z}(t),F_{z,z}(kt),F_{Bv,z}(t)\right)\geq 0,$$

or, equivalently,

$$\varphi\left(F_{z,Bv}(kt),1,1,F_{Bv,z}(t)\right)\geq 0.$$

As φ is non-decreasing in the first argument

$$\varphi\left(F_{z,Bv}(t),1,1,F_{Bv,z}(t)\right)\geq 0.$$

Using (R-1), we have $F_{z,Bv}(t)\geq 1$ for all $t>0$. Hence $F_{z,Bv}(t)=1$. Thus $z=Bv$. Therefore $Bv=(P_2P_4\dots P_{2n})v=z$ which shows that v is a coincidence point of the pair $(B,P_2P_4\dots P_{2n})$.

Since the pairs $(A,P_1P_3\dots P_{2n-1})$ and $(B,P_2P_4\dots P_{2n})$ are weakly compatible, therefore $Az=A(P_1P_3\dots P_{2n-1}u)=(P_1P_3\dots P_{2n-1})Au=(P_1P_3\dots P_{2n-1})z$ and $Bz=B(P_2P_4\dots P_{2n}v)=(P_2P_4\dots P_{2n})Bv=(P_2P_4\dots P_{2n})z$.

Now we prove that $Az=z=P_1P_3\dots P_{2n-1}z$ and $Bz=z=P_2P_4\dots P_{2n}z$, then using inequality (1), we obtain

$$\varphi\left(\begin{array}{c}F_{Az,Bv}(kt),F_{(P_1P_3\dots P_{2n-1})z,P_2P_4\dots P_{2n}v}(t),\\ F_{Az,(P_1P_3\dots P_{2n-1})z}(kt),F_{Bv,P_2P_4\dots P_{2n}v}(t)\end{array}\right)\geq 0,$$

and so

$$\varphi\left(F_{Az,z}(kt),F_{Az,z}(t),F_{Az,Az}(kt),F_{z,z}(t)\right)\geq 0,$$

or, equivalently,

$$\varphi\left(F_{Az,z}(kt),F_{Az,z}(t),1,1\right)\geq 0.$$

As φ is non-decreasing in the first argument, hence

$$\varphi\left(F_{Az,z}(t), F_{Az,z}(t), 1, 1\right) \geq 0.$$

Using (R-2), we have $F_{Az,z}(t) \geq 1$ for all $t > 0$, i.e., $F_{Az,z}(t) = 1$ and so $Az = z$. Therefore $Az = (P_1P_3 \dots P_{2n-1})z = z$.

Similarly, one can prove that $Bz = z = P_2P_4 \dots P_{2n}z$. Therefore $Az = Bz = P_1P_3 \dots P_{2n-1}z = P_2P_4 \dots P_{2n}z = z$. Now we show that z is the common fixed point of all the component mappings. Putting $x = P_3 \dots P_{2n-1}z$, $y = z$, $P_1^{'} = P_1P_3 \dots P_{2n-1}$ and $P_2^{'} = P_2P_4 \dots P_{2n}$ in inequality (1), we get

$$\varphi\left(\begin{array}{c} F_{AP_3\dots P_{2n-1}z,Bz}(kt), F_{P_1^{'}P_3\dots P_{2n-1}z,P_2P_4\dots P_{2n}z}(t), \\ F_{AP_3\dots P_{2n-1}z,P_1^{'}P_3\dots P_{2n-1}z}(kt), F_{Bz,P_2P_4\dots P_{2n}z}(t) \end{array}\right) \geq 0,$$

and so

$$\varphi\left(\begin{array}{c} F_{P_3\dots P_{2n-1}z,z}(kt), F_{P_3\dots P_{2n-1}z,z}(t), \\ F_{P_3\dots P_{2n-1}z,P_3\dots P_{2n-1}z}(kt), F_{z,z}(t) \end{array}\right) \geq 0$$

or, equivalently,

$$\varphi\left(F_{P_3\dots P_{2n-1}z,z}(kt), F_{P_3\dots P_{2n-1}z,z}(t), 1, 1\right) \geq 0.$$

As φ is non-decreasing in the first argument, we have

$$\varphi\left(F_{P_3\dots P_{2n-1}z,z}(t), F_{P_3\dots P_{2n-1}z,z}(t), 1, 1\right) \geq 0.$$

Using (R-2), we have $F_{P_3\dots P_{2n-1}z,z}(t) \geq 1$ for all $t > 0$.
Hence $F_{P_3\dots P_{2n-1}z,z}(t) = 1$. Thus $(P_3 \dots P_{2n-1})z = z$.
Thus $P_1z = P_1(P_3 \dots P_{2n-1}z) = z$.
Continuing this procedure, we get $Az = P_1z = P_3z = \dots = P_{2n-1}z = z$. In the same manner, taking $x = z$, $y = P_4 \dots P_{2n}z$, $P_1^{'} = P_1P_3 \dots P_{2n-1}$ and $P_2^{'} = P_2P_4 \dots P_{2n}$ in inequality (1), we get $z = P_4 \dots P_{2n}z$. Hence $P_2z = z$. Continuing this procedure, we get $Bz = P_2z = P_4z = \dots = P_{2n}z = z$. Therefore z is the common fixed point of $P_1, P_2, \dots, P_{2n}, A$ and B. The uniqueness of common fixed point is an easy consequence of inequality (1).

The proof is similar when $P_2P_4 \dots P_{2n}(X)$ is assumed to be a complete subspace of X. The cases wherein $A(X)$ or $B(X)$ is assumed to be a complete subspace of X are similar to the cases in which $P_2P_4 \dots P_{2n}(X)$ or $P_1P_3 \dots P_{2n-1}(X)$ respectively, is complete since $A(X) \subset P_2P_4 \dots P_{2n}(X)$, $B(X) \subset P_1P_3 \dots P_{2n-1}(X)$. We can also find a unique common fixed point of the involved self mappings by using inequality (2) in the same manner. □

Now we prove a common fixed point theorem, which is a slight generalization of Theorem 3.1.

Theorem 3.2. *Let* $\{T_\alpha\}_{\alpha\in J}$ *and* $\{P_i\}_{i=1}^{2n}$ *be two families of self mappings of a Menger space* $(X, \mathcal{F}, \triangle)$, *where* $\triangle$ *is a continuous t-norm. Suppose that there exists a fixed* $\beta \in J$ *such that the pairs* $(T_\alpha, P_1P_3 \dots P_{2n-1})$ *and* $(T_\beta, P_2P_4 \dots P_{2n})$ *be weakly compatible satisfying:*

(1) $T_\alpha(X) \subset P_2P_4\dots P_{2n}(X)$ *for each* $\alpha \in J$ *and* $T_\beta(X) \subset P_1P_3\dots P_{2n-1}(X)$ *for some* $\beta \in J$;
(2) *One of* $T_\alpha(X), T_\beta(X), P_1P_3\dots P_{2n-1}(X)$ *and* $P_2P_4\dots P_{2n}(X)$ *is a complete subspace of* X;
(3) *Suppose that*

$$\begin{aligned}
P_1(P_3\dots P_{2n-1}) &= (P_3\dots P_{2n-1})P_1,\\
P_1P_3(P_5\dots P_{2n-1}) &= (P_5\dots P_{2n-1})P_1P_3,\\
&\vdots\\
P_1\dots P_{2n-3}(P_{2n-1}) &= (P_{2n-1})P_1\dots P_{2n-3},\\
T_\alpha(P_3\dots P_{2n-1}) &= (P_3\dots P_{2n-1})T_\alpha,\\
T_\alpha(P_5\dots P_{2n-1}) &= (P_5\dots P_{2n-1})T_\alpha,\\
&\vdots\\
T_\alpha P_{2n-1} &= P_{2n-1}T_\alpha,
\end{aligned}$$

similarly,

$$\begin{aligned}
P_2(P_4\dots P_{2n}) &= (P_4\dots P_{2n})P_2,\\
P_2P_4(P_6\dots P_{2n}) &= (P_6\dots P_{2n})P_2P_4,\\
&\vdots\\
P_2\dots P_{2n-2}(P_{2n}) &= (P_{2n})P_2\dots P_{2n-2},\\
T_\beta(P_4\dots P_{2n}) &= (P_4\dots P_{2n})T_\beta,\\
T_\beta(P_6\dots P_{2n}) &= (P_6\dots P_{2n})T_\beta,\\
&\vdots\\
T_\beta P_{2n} &= P_{2n}T_\beta.
\end{aligned}$$

(4) $(T_\alpha, P_1P_3\dots P_{2n-1})$ *or* $(T_\beta, P_2P_4\dots P_{2n})$ *satisfies the property (E.A);*
(5) *there exist a constant* $k \in (0,1)$ *and* $\varphi \in \Phi$ *such that*

$$\varphi\left(\begin{array}{c} F_{T_\alpha x, T_\beta y}(kt), F_{P_1P_3\dots P_{2n-1}x, P_2P_4\dots P_{2n}y}(t),\\ F_{T_\alpha x, P_1P_3\dots P_{2n-1}x}(kt), F_{T_\beta y, P_2P_4\dots P_{2n}y}(t)\end{array}\right) \geq 0 \tag{3}$$

or,

$$\varphi\left(\begin{array}{c} F_{T_\alpha x, T_\beta y}(kt), F_{P_1P_3\dots P_{2n-1}x, P_2P_4\dots P_{2n}y}(t),\\ F_{T_\alpha x, P_1P_3\dots P_{2n-1}x}(t), F_{T_\beta y, P_2P_4\dots P_{2n}y}(kt)\end{array}\right) \geq 0 \tag{4}$$

for all $x, y \in X$, $t > 0$.

Then all $\{P_i\}$ *and* $\{T_\alpha\}$ *have a unique common fixed point in* X.

Proof. Let T_{α_0} be a fixed element in $\{T_\alpha\}_{\alpha\in J}$. By Theorem 3.1 with $A = T_{\alpha_0}$ and $B = T_\beta$ it follows that there exists some $z \in X$ such that

$$T_\beta z = T_{\alpha_0} z = P_1P_3\dots P_{2n-1}z = P_2P_4\dots P_{2n}z = z.$$

Let $\alpha \in J$ be arbitrary. Then applying inequality (3), we obtain

$$\varphi\left(\begin{array}{c} F_{T_\alpha z, T_\beta z}(kt), F_{P_1P_3\dots P_{2n-1}z, P_2P_4\dots P_{2n}z}(t), \\ F_{T_\alpha z, P_1P_3\dots P_{2n-1}z}(kt), F_{T_\beta z, P_2P_4\dots P_{2n}z}(t) \end{array}\right) \geq 0,$$

and so

$$\varphi\left(F_{T_\alpha z, z}(kt), F_{z,z}(t), F_{T_\alpha z, z}(kt), F_{z,z}(t)\right) \geq 0,$$

or, equivalently,

$$\varphi\left(F_{T_\alpha z, z}(kt), 1, F_{T_\alpha z, z}(kt), 1\right) \geq 0.$$

Using (R-1), we have $F_{T_\alpha z, z}(kt) \geq 1$ for all $t > 0$. Hence $F_{T_\alpha z, z}(t) = 1$. Thus $T_\alpha z = z$ for each $\alpha \in J$.

Uniqueness of the common fixed point is an easy consequence of inequality (3). □

Corollary 3.1. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. Further, let the pairs (A, S) and (B, T) be weakly compatible satisfying:*

(1) *$A(X) \subset T(X)$, $B(X) \subset S(X)$;*
(2) *One of $A(X), B(X), S(X)$ and $T(X)$ is a complete subspace of X;*
(3) *(A, S) or (B, T) satisfies the property (E.A);*
(4) *there exist a constant $k \in (0, 1)$ and $\varphi \in \Phi$ such that*

(5) $$\varphi\left(F_{Ax,By}(kt), F_{Sx,Ty}(t), F_{Ax,Sx}(kt), F_{By,Ty}(t)\right) \geq 0$$

or,

(6) $$\varphi\left(F_{Ax,By}(kt), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(kt)\right) \geq 0,$$

for all $x, y \in X$, $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

Proof. If we set $P_1P_3\dots P_{2n-1} = S$ and $P_2P_4\dots P_{2n} = T$ in Theorem 3.1, then the result easily follows. □

Remark 3.1. As two non-compatible mappings of a Menger space $(X, \mathcal{F}, \triangle)$ satisfy the property (E.A.), therefore the earlier proved results also valid for non-compatible self mappings.

The following example illustrates Corollary 3.1 in view of Remark 3.1.

Example 3.1. Let $X = [0, 1)$ and d be the usual metric on X and for each $t \in [0, 1]$, define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

for all $x, y \in X$. Clearly, $(X, \mathcal{F}, \triangle)$ is a Menger space, where $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Consider the function $\varphi : (\mathbb{R}^+)^4 \to \mathbb{R}$ as shown in Example 2.2. Let A, B, S and T be self mappings of X defined by

$$A(X) = \begin{cases} 0, & \text{if } x = 0; \\ 0.2, & \text{if } 0 < x < 1. \end{cases}$$

$$B(X) = \begin{cases} 0, & \text{if } x = 0; \\ 0.3, & \text{if } 0 < x < 1. \end{cases}$$

$$S(X) = \begin{cases} 0, & \text{if } x = 0; \\ 0.25, & \text{if } x \in (0, \frac{1}{2}]; \\ x - 0.3, & \text{if } x \in (\frac{1}{2}, 1). \end{cases}$$

$$T(X) = \begin{cases} 0, & \text{if } x = 0; \\ 0.2, & \text{if } x \in (0, \frac{1}{2}]; \\ x - 0.2, & \text{if } x \in (\frac{1}{2}, 1). \end{cases}$$

Then A, B, S and T satisfy all the conditions of Corollary 3.1 for some fixed $k \in (0,1)$ and have a unique common fixed point at $x = 0$. It may be noted that, the mappings A and S commute at the coincidence point $0 \in X$ and hence the mappings A and S are weakly compatible. Similarly B and T are weakly compatible mappings. Now we show that the pairs (A, S) and (B, T) are non-compatible, let us consider a decreasing sequence $\{x_n\}$ defined as $x_n \to 0.5$ as $n \to \infty$. Then $Ax_n \to 0.2$, $Sx_n \to 0.2$ (as $n \to \infty$) but $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = \frac{t}{t+|0.2-0.25|} \neq 1$ which shows that the pair (A, S) is non-compatible. Also, $Bx_n \to 0.3$, $Tx_n \to 0.3$ (as $n \to \infty$) but $\lim_{n\to\infty} F_{BTx_n,TBx_n}(t) = \dfrac{t}{t+ \mid 0.3 - 0.2 \mid} \neq 1$. Hence the pair (B, T) is non-compatible. All the mappings involved in this example are discontinuous at the common fixed point $x = 0$.

Remark 3.2. Corollary 3.1 represents the result of Kumar et al. [11, Theorem 12] in absence of inequality (6).

On taking $A = B$ and $S = T$ in Corollary 3.1 then we get the interesting result.

Corollary 3.2. *Let A and S be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. Further, let the pair $\{A, S\}$ is weakly compatible satisfying:*

(1) *$A(X) \subset S(X)$;*
(2) *One of $A(X)$ and $S(X)$ is a complete subspace of X;*
(3) *(A, S) satisfies the property (E.A);*
(4) *there exist a constant $k \in (0,1)$ and $\varphi \in \Phi$ such that*

$$\varphi\left(F_{Ax,Ay}(kt), F_{Sx,Sy}(t), F_{Ax,Sx}(kt), F_{Ay,Sy}(t)\right) \geq 0 \tag{7}$$

or,

$$\varphi\left(F_{Ax,Ay}(kt), F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(kt)\right) \geq 0, \tag{8}$$

for all $x, y \in X$, $t > 0$.

Then A and S have a unique common fixed point in X.

The following definition firstly studied by Imdad et al. [6].

Definition 3.1. [6] Two families of self mappings $\{A_i\}_{i=1}^m$ and $\{B_k\}_{k=1}^n$ are said to be pairwise commuting if

(1) $A_iA_j = A_jA_i$ for all $i, j \in \{1, 2, \ldots, m\}$;
(2) $B_kB_l = B_lB_k$ for all $k, l \in \{1, 2, \ldots, n\}$;
(3) $A_iB_k = B_kA_i$ for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

Now, we utilize Definition 3.1 (which is a natural extension of commutativity condition to two finite families) and prove a common fixed point theorem for four finite families of self mappings in Menger space.

Theorem 3.3. *Let $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm such that $A = A_1A_2 \ldots A_m$, $B = B_1B_2 \ldots B_n$, $S = S_1S_2 \ldots S_p$ and $T = T_1T_2 \ldots T_q$. Also, the pairs of the families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_h\})$ are commuting pairwise satisfying:*

(1) *$A(X) \subset T(X)$, $B(X) \subset S(X)$;*
(2) *One of $A(X), B(X), S(X)$ and $T(X)$ is a complete subspace of X;*
(3) *(A, S) or (B, T) satisfies the property (E.A);*
(4) *there exist a constant $k \in (0, 1)$ and $\varphi \in \Phi$ such that*

$$\varphi\left(F_{Ax,By}(kt), F_{Sx,Ty}(t), F_{Ax,Sx}(kt), F_{By,Ty}(t)\right) \geq 0 \tag{9}$$

or,

$$\varphi\left(F_{Ax,By}(kt), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(kt)\right) \geq 0, \tag{10}$$

for all $x, y \in X$, $t > 0$.

Then $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ have a unique common fixed point in X.

Proof. From the notion of pair-wise commuting, one can prove that $AS = SA$ as

$$\begin{aligned}
AS &= (A_1A_2 \ldots A_m)(S_1S_2 \ldots S_p) \\
&= (A_1A_2 \ldots A_{m-1})(A_mS_1S_2 \ldots S_p) \\
&= (A_1A_2 \ldots A_{m-1})(S_1S_2 \ldots S_pA_m) \\
&= (A_1A_2 \ldots A_{m-2})(A_{m-1}S_1S_2 \ldots S_pA_m) \\
&= (A_1A_2 \ldots A_{m-2})(S_1S_2 \ldots S_pA_{m-1}A_m) \\
&\ \ \vdots \\
&= A_1(S_1S_2 \ldots S_pA_2 \ldots A_{m-1}A_m) \\
&= (S_1S_2 \ldots S_p)(A_1A_2 \ldots A_m) \\
&= SA.
\end{aligned}$$

One can also prove that $BT = TB$. Hence the pairs (A, S) and (B, T) are weakly compatible. In view of Corollary 3.1, we conclude that A, B, S and T have a unique common fixed point z in X.

Now, we assert that w remains the fixed point of all the component mappings.

$$\begin{aligned} A(A_iw) &= ((A_1A_2\dots A_m)A_i)\,w \\ &= (A_1A_2\dots A_{m-1})(A_mA_i)w \\ &= (A_1A_2\dots A_{m-1})(A_iA_m)w \\ &= (A_1A_2\dots A_{m-2})(A_{m-1}A_iA_m)w \\ &= (A_1A_2\dots A_{m-2})(A_iA_{m-1}A_m)w \\ &\vdots \\ &= A_1(A_iA_2\dots A_m)w \\ &= (A_1A_i)(A_2\dots A_m)w \\ &= (A_iA_1)(A_2\dots A_m)w \\ &= A_i(A_1A_2\dots A_m)w \\ &= A_iw. \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} A(S_kw) &= S_k(Aw) = S_kw, \\ S(S_kw) &= S_k(Sw) = S_kw, \\ S(A_iw) &= A_i(Sw) = A_iw, \\ B(B_rw) &= B_r(Bw) = B_rw, \\ B(T_hw) &= T_h(Bw) = T_hw, \\ T(T_hw) &= T_h(Tw) = T_hw, \\ T(B_rw) &= B_r(Tw) = B_rw, \end{aligned}$$

which shows that (for all i, r, k and h) A_iw and S_kw are other fixed point of the pair (A, S) whereas B_rw and T_hw are other fixed points of the pair (B, T).

Now appealing to the uniqueness of common fixed points of mappings A, B, S and T, we get

$$w = A_iw = S_kw = B_rw = T_hw,$$

for all $i \in \{1, 2, \dots, m\}$, $k \in \{1, 2, \dots, p\}$, $r \in \{1, 2, \dots, n\}$ and $h \in \{1, 2, \dots, q\}$ which shows that w is the unique common fixed point of $\{A_i\}_{i=1}^{m}$, $\{B_r\}_{r=1}^{n}$, $\{S_k\}_{k=1}^{p}$ and $\{T_h\}_{t=1}^{q}$. □

By setting $A_1 = A_2 = \dots = A_m = A$, $B_1 = B_2 = \dots = B_n = B$, $S_1 = S_2 = \dots = S_p = S$ and $T_1 = T_2 = \dots = T_h = T$ in Theorem 3.3, we deduce the following:

Corollary 3.3. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm satisfying:*

(1) $A^m(X) \subset T^q(X), B^n(X) \subset S^p(X)$;
(2) *One of* $A^m(X), B^n(X), S^p(X)$ *and* $T^q(X)$ *is a complete subspace of* X;
(3) (A^m, S^p) *or* (B^n, T^q) *satisfies the property (E.A);*
(4) *there exist a constant* $k \in (0,1)$ *and* $\varphi \in \Phi$ *such that*

$$\varphi\left(F_{A^m x, B^n y}(kt), F_{S^p x, T^q y}(t), F_{A^m x, S^p x}(kt), F_{B^n y, T^q y}(t)\right) \geq 0 \tag{11}$$

or,

$$\varphi\left(F_{A^m x, B^n y}(kt), F_{S^p x, T^q y}(t), F_{A^m x, S^q x}(t), F_{B^n y, T^q y}(kt)\right) \geq 0, \tag{12}$$

for all $x, y \in X$, $t > 0$ *and and* m, n, p *and* q *are fixed positive integers.*

Then A, B, S *and* T *have a unique common fixed point in* X *provided that* $AS = SA$ *and* $BT = TB$.

Remark 3.3. Corollary 3.3 is a slight but partial generalization of Corollary 3.1 as the commutativity requirements (i.e., $AS = SA$ and $BT = TB$) in this corollary are stronger as compared to weak compatibility in Corollary 3.1.

Acknowledgements

The authors are thankful to the Editor and anonymous referees for their fruitful comments on this paper.

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