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Rhoades Type Fixed Point Theorems for a Family of Hybrid Pairs of Mappings in Metrically Convex Spaces

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Abstract. The present paper establishes some coincidence and fixed point theorems for a sequence of hybrid type nonself mappings defined on a closed subset of a metrically convex metric spaces, which generalize some earlier results due to Rhoades [18], Ahmed and Rhoades [1] and many others. Some related results are also derived.

1. Introduction

The existing literature of fixed point theory contains numerous results for single as well as multi-valued self mappings, but in many applications the mapping under consideration need not always be a self mapping. In an attempt to prove results for nonself mappings in metrically convex complete metric spaces, Rhoades [17] gave sufficient conditions for such mappings to admit a fixed point by proving a fixed point theorem for certain generalized type contractions under suitable boundary conditions on the mapping. The recent literature witnessed various extentions and generalizations of this theorem of Rhoades [17], which includes Rhoades [18], Imdad and Kumar [10] and some others. For the work of this kind one can be referred to Imdad et al. [9], Ahmad and Imdad [2], Ahmad and Khan [3], Rhoades [18] and several others. Recently Ahmed and Rhoades [1] proved a result on coincidence points for two hybrid pairs of compatible continuous mappings which is essentially patterned after Ahmad and Imdad [2]

On the other hand, Huang and Cho [8] and Dhage et al. [5] proved some fixed point theorems for a sequence of set-valued mappings which generalize several results due to Itoh [11], Khan [15], Ahmad and Khan [3] and others. In this paper by combining these two ideas we prove some coincidence and fixed point theorems for a sequence of hybrid type nonself mappings satisfying certain contraction type condition which is essentially patterned after

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Rhoades [18]. Our results either partially or completely generalize earlier results due to Rhoades [18], Imdad and Kumar [10] and several others.

2. PRELIMINARIES

Before proving our results, we collect the relevant definitions and results for our future use.

Let (X, d) be a metric space. Then following Nadler [16], we recall

(i) $CB(X) = \{A : A$ is nonempty closed and bounded subset of $X\}$,
(ii) $C(X) = \{A : A$ is nonempty compact subset of $X\}$.
(iii) For nonempty subsets A, B of X and $x \in X, d(x, A) = \inf\{d(x, a) : a \in A\}$ and

$$H(A, B) = \max[\{\sup\ d(a, B) : a \in A\}, \{\sup d(A, b) : b \in B\}].$$

It is well known (cf. Kuratowski [14]) that $CB(X)$ is a metric space with the distance H which is known as Hausdorff-Pompeiu metric on X.

The following definitions and a lemma will be frequently used in the sequel.

Definition 1 ([6, 7])**.** Let K be a nonempty subset of a metric space (X, d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be weakly commuting (cf.[7]) if for every $x, y \in K$ with $x \in Fy$ and $Ty \in K$, we have

$$d(Tx, FTy) \leq d(Ty, Fy),$$

whereas the pair (F, T) is said to be compatible (cf.[6]) if for every sequence $\{x_n\} \subset K$, from the relation

$$\lim_{n\to\infty} d(Fx_n, Tx_n) = 0$$

and $Tx_n \in K$(for every$n \in N$) it follows that $\lim_{n\to\infty} d(Ty_n, FTx_n) = 0$, for every sequence $\{y_n\} \subset K$ such that $y_n \in Fx_n, n \in N$.

For hybrid pairs of self type mappings these definitions were introduced by Kaneko and Sessa [13].

Definition 2 ([9])**.** Let K be a nonempty subset of a metric space (X, d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be quasi-coincidentally commuting if for all coincidence points 'x' of (T, F), $TFx \subset FTx$ whenever $Fx \subset K$ and $Tx \in K$ for all $x \in K$.

Definition 3 ([9])**.** A mapping $T : K \to X$ is said to be coincidentally idempotent w.r.t mapping $F : K \to CB(X)$, if T is idempotent at the coincidence points of the pair (F, T).

Definition 4 ([4])**.** A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 1 ([4]). *Let K be a nonempty closed subset of a metrically convex metric space (X,d). If $x \in K$ and $y \notin K$ then there exists a point $z \in \delta K$ (the boundary of K) such that $d(x,z)+d(z,y)=d(x,y)$.*

3. Results

Our main result runs as follows.

Theorem 1. *Let (X,d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S,T : K \to X$ satisfying:*

(iv) $\delta K \subseteq SK \cap TK, F_i(K)\cap K \subseteq SK, F_j(K)\cap K \subseteq TK,$
(v) $Tx \in \delta K \Rightarrow F_i(x) \subseteq K, Sx \in \delta K \Rightarrow F_j(x) \subseteq K,$ *and*

$$\begin{aligned} H(F_i(x),F_j(y)) \le h \max\Big\{ & \frac{1}{a} d(Tx,Sy), d(Tx,F_i(x)), d(Sy,F_j(y)), \\ & \frac{1}{a+h}\big(d(Tx,F_j(y))+d(Sy,F_i(x))\big)\Big\}, \end{aligned} \tag{1}$$

where $i=2n-1$, $j=2n$, $(n\in N)$, $i\neq j$ for all $x,y\in K$ with $x\neq y$, where $0<h<\frac{-1+\sqrt{5}}{2}$, $a \ge 1+\frac{2h^2}{1+h}$,

(vi) (F_i,T) *and* (F_j,S) *are compatible pairs,*
(vii) $\{F_n\}$, S *and* T *are continuous on* K.

Then $\{F_n\}$, S and T have a point of common coincidence.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Assume $\alpha = h(1+h)$. Let $x \in \delta K$. Then (due to $\delta K \subseteq TK$) there exists a point $x_0 \in K$ such that $x = Tx_0$. From the implication $Tx \in \delta K$, implies $F_1(x_0) \subseteq F_1(K)\cap K \subseteq SK$, let $x_1 \in K$ be such that $y_1 = Sx_1 \in F_1(x_0) \subseteq K$. Since $y_1 \in F_1(x_0)$, there exists a point $y_2 \in F_2(x_1)$ such that

$$d(y_1,y_2) \le H(F_1(x_0),F_2(x_1)) + \alpha.$$

Suppose $y_2 \in K$. Then $y_2 \in F_2(K)\cap K \subseteq TK$, implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin K$ then there exists a point $p \in \delta K$ such that

$$d(Sx_1,p)+d(p,y_2) = d(Sx_1,y_2).$$

Since $p \in \delta K \subseteq TK$, there exists a point $x_2 \in K$ with $p = Tx_2$ so that

$$d(Sx_1,Tx_2)+d(Tx_2,y_2) = d(Sx_1,y_2).$$

Let $y_3 \in F_3(x_2)$ be such that $d(y_2,y_3) \le H(F_2(x_1),F_3(x_2)) + \alpha^2$

Thus, repeating the foregoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(viii) $y_{2n} \in F_{2n}(x_{2n-1})$ for all $n \in N$,
$y_{2n+1} \in F_{2n+1}(x_{2n})$ for all $n \in N_0 = N \cup \{0\}$,
(ix) $y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin K \Rightarrow Tx_{2n} \in \delta K$ and
$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n})$,
(x) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \delta K$ and
$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})$,
(xi) $d(y_{2n-1}, y_{2n}) \leq H(F_{2n-1}(x_{2n-2}), F_{2n}(x_{2n-1})) + \alpha^{2n-1}$
$d(y_{2n}, y_{2n+1}) \leq H(F_{2n}(x_{2n-1}), F_{2n+1}(x_{2n})) + \alpha^{2n}$.

We denote

$$\begin{aligned}
&P_\circ = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\}, P_1 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\},\\
&Q_\circ = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\} \quad \text{and}\\
&Q_1 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}.
\end{aligned}$$

One can note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$ and $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now, we distinguish the following three cases.

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_\circ$, then

$$\begin{aligned}
&d(Tx_{2n}, Sx_{2n+1}) \leq H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})) + \alpha^{2n}\\
&\leq h \max\Big\{\frac{1}{a} d(Tx_{2n}, Sx_{2n-1}), d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Sx_{2n-1}, F_{2n}(x_{2n-1})),\\
&\qquad \frac{1}{a+h}\Big(d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))\Big)\Big\} + \alpha^{2n}\\
&= h \max\Big\{\frac{1}{a} d(Tx_{2n}, Sx_{2n-1}), d(Tx_{2n}, Sx_{2n+1}), d(Sx_{2n-1}, Tx_{2n}),\\
&\qquad \frac{1}{a+h}\Big(d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Sx_{2n+1})\Big)\Big\} + \alpha^{2n}\\
&\leq \max\Big\{hd(Tx_{2n}, Sx_{2n-1}) + \alpha^{2n}, \frac{\alpha^{2n}}{1-h},\\
&\qquad \frac{1}{a}(hd(Sx_{2n-1}, Tx_{2n}) + \alpha^{2n}(a+h))\Big\}\\
&\leq hd(Tx_{2n}, Sx_{2n-1}) + \max\Big\{\frac{1}{1-h}, \frac{a+h}{a}\Big\}\alpha^{2n}\\
&\leq hd(Tx_{2n}, Sx_{2n-1}) + \frac{\alpha^{2n}}{1-h}.
\end{aligned}$$

Similarly, if $(Sx_{2n-1}, Tx_{2n}) \in Q_\circ \times P_\circ$, then

$$d(Sx_{2n-1}, Tx_{2n}) \leq hd(Tx_{2n-2}, Sx_{2n-1}) + \frac{\alpha^{2n-1}}{1-h}.$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_1$, then

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}),$$

which in turn yields $d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1})$, and hence $d(Tx_{2n}, Sx_{2n+1}) \leq d(y_{2n}, y_{2n+1}) \leq H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})) + \alpha^{2n}$.

Now, proceeding as in Case 1, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq hd(Tx_{2n}, Sx_{2n-1}) + \frac{\alpha^{2n}}{1-h}.$$

In case $(Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_\circ$ then as earlier, one also obtains

$$d(Sx_{2n-1}, Tx_{2n}) \leq hd(Sx_{2n-1}, Tx_{2n-2}) + \frac{\alpha^{2n-1}}{1-h}.$$

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_\circ$ then $Sx_{2n-1} = y_{2n-1}$. Proceeding as in Case 1, one gets

$$\begin{aligned}
&d(Tx_{2n}, Sx_{2n+1}) = d(Tx_{2n}, y_{2n+1}) \\
&\leq d(Tx_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1}) \\
&\leq d(Sx_{2n-1}, y_{2n}) + H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})) + \alpha^{2n} \\
&\leq d(Sx_{2n-1}, y_{2n}) \\
&\qquad + h \max\Big\{\frac{1}{a} d(Tx_{2n}, Sx_{2n-1}), d(Tx_{2n}, Sx_{2n+1}), d(Sx_{2n-1}, Tx_{2n}), \\
&\qquad\quad \frac{1}{a+h}\big(d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))\big)\Big\} + \alpha^{2n} \\
&\leq d(Sx_{2n-1}, y_{2n}) \\
&\qquad + h \max\Big\{\frac{1}{a} d(Tx_{2n}, Sx_{2n-1}), d(Tx_{2n}, Sx_{2n+1}), d(Sx_{2n-1}, Tx_{2n}), \\
&\qquad\quad \frac{1}{a+h}\big(d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, Sx_{2n+1})\big)\Big\} + \alpha^{2n} \\
&\leq d(Sx_{2n-1}, y_{2n}) \\
&\qquad + h \max\Big\{\frac{1}{a} d(y_{2n}, Sx_{2n-1}), d(Tx_{2n}, Sx_{2n+1}), d(Sx_{2n-1}, y_{2n}), \\
&\qquad\quad \frac{1}{a+h}\big(d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Sx_{2n+1})\big)\Big\} + \alpha^{2n} \\
&\leq \max\Big\{(1+h)d(y_{2n}, Sx_{2n-1}) + \alpha^{2n}, (1+h)d(y_{2n}, Sx_{2n-1}) + \frac{\alpha^{2n}}{1-h}, \\
&\qquad \frac{1}{a}(hd(Sx_{2n-1}, y_{2n}) + (a+h)\alpha^{2n})\Big\} \\
&\leq (1+h)d(y_{2n}, Sx_{2n-1}) + \frac{\alpha^{2n}}{1-h} \\
&\leq h(1+h)d(Tx_{2n-2}, Sx_{2n-1}) + h\frac{\alpha^{2n-1}}{1-h} + \frac{\alpha^{2n}}{1-h}.
\end{aligned}$$

Thus if put $z_{2n} = Tx_{2n}, z_{2n+1} = Sx_{2n+1}$, then one obtains

$$d(z_n, z_{n+1}) \leq \begin{cases} hd(z_{n-1}, z_n) + \frac{\alpha^n}{1-h}, & \text{or} \\ h(1+h)d(z_{n-2}, z_{n-1}) + \frac{h\alpha^{n-1}}{1-h} + \frac{\alpha^n}{1-h}. \end{cases}$$

Now on the lines of Itoh [11] it can be shown that $\{z_n\}$ is Cauchy and there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in $P_\circ$ or $Q_\circ$ respectively. Consequently the subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ for each $k \in N$, converges to z. Using compatibility of (F_j, S), we have

$$\lim_{k\to\infty} d(Sx_{2n_k-1}, F_j(x_{2n_k-1})) = 0 \quad \text{for any even integer } j \in N,$$

which implies that $\lim\limits_{k\to\infty} d(STx_{2n_k}, F_j(Sx_{2n_k-1})) = 0$.

Using the continuity of S and F_j, one obtains $Sz \in F_j(z)$, for any even integer $j \in N$. Similarly the continuity of T and F_i implies $Tz \in F_i(z)$, for any odd integer $i \in N$. Now

$$\begin{aligned} d(Tz, Sz) &\leq H(F_i(z), F_j(z)) \\ &\leq h \max\Big\{\frac{1}{a} d(Tz, Sz), d(Tz, F_i(z)), d(Sz, F_j(z)), \\ &\qquad\qquad \frac{1}{a+h}\big(d(Tz, F_j(z)) + d(Sz, F_i(z))\big)\Big\} \\ &\leq h \max\Big\{\frac{1}{a} d(Tz, Sz), 0, 0, \frac{2}{a+h} d(Tz, Sz)\Big\} \\ &\leq \max\Big\{\frac{h}{a}, \frac{2h}{a+h}\Big\} d(Tz, Sz), \end{aligned}$$

yielding thereby $Tz = Sz$, which shows that z is a common coincidence point of $\{F_n\}, S$ and T. □

Remark 1. By setting $F_i = F$ (for any odd integer $i \in N$) and $F_j = G$ (for any even integer $j \in N$) in Theorem 1, one deduces a result due to Ahmed and Rhoades [1].

In the next theorem we utilize the closedness of TK and SK so as to relax the continuity requirements besides limiting the commutativity to points of coincidence.

Theorem 2. *Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S, T : K \to X$ satisfying* (1), *(iv) and (v). Suppose that*

(xii) *TK and SK are closed subspaces of X. Then*
  (a) *(F_i, T) has a point of coincidence,*
  (b) *(F_j, S) has a point of coincidence.*

Moreover, (F_i, T) has a common fixed point if T is quasi-coincidentally commuting and coincidentally idempotent w.r.t F_i, whereas (F_j, S) has a common fixed point provided S is quasi-coincidentally commuting and coincidentally idempotent w.r.t F_j.

Proof. On the lines of the proof of the Theorem 1, one assumes that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ and TK as well as SK are closed subspaces of X. Since $\{Tx_{2n_k}\}$ is Cauchy in TK, it converges to a point $u \in TK$. Let $v \in T^{-1}u$, then $Tv = u$. Since $\{Sx_{2n_k+1}\}$ is a subsequence of Cauchy sequence, $\{Sx_{2n_k+1}\}$ converges to u as well. Using (1), one can write

$$\begin{aligned}
&d(F_i(v), Tx_{2n_k}) \leq H(F_i(v), F_j(x_{2n_k-1})) \\
&\leq h \max\Big\{\frac{1}{a} d(Tv, Sx_{2n_k-1}), d(Sx_{2n_k-1}, F_j(x_{2n_k-1})), d(Tv, F_i(v)), \\
&\qquad \frac{1}{a+h}\big(d(Tv, F_j(x_{2n_k-1})) + d(Sx_{2n_k-1}, F_i(v))\big)\Big\}
\end{aligned}$$

which on letting $k \to \infty$, reduces to

$$\begin{aligned}
d(F_i(v), u) &\leq h \max\Big\{0, 0, d(u, F_i(v)), \frac{1}{a+h}(0 + d(F_i(v), u))\Big\} \\
&\leq \max\Big\{h, \frac{h}{a+h}\Big\} d(u, F_i(v))
\end{aligned}$$

yielding thereby $u \in F_i(v)$, which implies that $u = Tv \in F_i(v)$ as $F_i(v)$ is closed.

Since Cauchy sequence $\{Tx_{2n}\}$ converges to $u \in K$ and $u \in F_i(v)$, $u \in F_i(K) \cap K \subseteq SK$, there exists $w \in K$ such that $Sw = u$. Again using (1), one gets

$$\begin{aligned}
d(Sw, F_j(w)) = d(Tv, F_j(w)) &\leq H(F_i(v), F_j(w)) \\
&\leq h \max\Big\{\frac{1}{a} d(Tv, Sw), d(Tv, F_i(v)), d(Sw, F_j(w)), \\
&\qquad \frac{1}{a+h}\big(d(Tv, F_j(w)) + d(Sw, F_i(v))\big)\Big\} \\
&\leq \max\Big\{h, \frac{h}{a+h}\Big\} d(Sw, F_j(w))
\end{aligned}$$

implying thereby $Sw \in F_j(w)$, that is w is a coincidence point of (S, F_j).

If one assumes that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in $Q_\circ$ with TK as well as SK are closed subspaces of X, then noting that $\{Sx_{2n_k+1}\}$ is Cauchy in SK, the foregoing arguments establish that $Tv \in F_i(v)$ and $Sw \in F_j(w)$.

Since v is a coincidence point of (F_i, T) therefore using quasi-coincidentally commuting as well as coincidentally idempotent property of T w.r.t F_i, one can have

$$Tv \in F_i(v) \text{and} u = Tv \Rightarrow Tu = TTv = Tv = u,$$

therefore $u = Tu = TTv \in TF_i(v) \subset F_i(Tv) = F_i(u)$, which shows that u is the common fixed point of (F_i, T). Similarly using the quasi-coincidentally commuting as well as coincidentally idempotent property of S w.r.t F_j, one can show that (F_j, S) has a common fixed point as well. □

Remark 2. By setting $F_n = F$ (for $n \in N$) and $S = T = I_K$ in Theorem 2, one deduces a result due to Rhoades [18].

Remark 3. A fixed point theorem similar to Theorem 3.2 can also be outlined in respect of Theorem 2.

Finally, we prove a theorem when "closedness of K" is replaced by "compactness of K".

Theorem 3. *Let (X, d) be a complete metrically convex metric space and K a nonempty compact subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $T : K \to X$ satisfying:*

(xiii) $\delta K \subseteq TK, (F_i(K) \cup F_j(K)) \cap K \subseteq TK,$
(xiv) $Tx \in \delta K \Rightarrow F_i(x) \cup F_j(x) \subseteq K$ *with* $H(F_i(x), F_j(y)) < M(x, y)$ *when* $M(x, y) > 0$, *for all* $x, y \in K$ *where*

$$M(x,y) = h \max\Big\{\frac{1}{a} d(Tx,Ty), d(Tx,F_i(x)), d(Ty,F_j(y)), \frac{1}{a+h}\big(d(Tx,F_j(y)) + d(Ty,F_i(x))\big)\Big\} \tag{2}$$

where $i = 2n-1$, $j = 2n$, $(n \in N)$, $i \neq j$ *for all* $x, y \in K$ *with* $x \neq y$, *where* $0 \leq h \leq \frac{-1+\sqrt{5}}{2}$, $a \geq 1 + \frac{2h^2}{1+h}$.

If T is compatible with $\{F_n\}$ $(n \in N)$ then $\{F_n\}$ and T have a common point of coincidence, provided all involves maps are continuous.

Proof. We assert that $M(x, y) = 0$ for some $x, y \in K$. Otherwise $M(x, y) \neq 0$, for any $x, y \in K$ implies that

$$f(x,y) = \frac{H(F_i(x), F_j(y))}{M(x,y)}$$

is continuous and satisfies $f(x, y) < 1$ for all $(x, y) \in K \times K$. Since $K \times K$ is compact, there exists $(u, v) \in K \times K$ such that $f(x, y) \leq f(u, v) = c < 1$ for $x, y \in K$, which in turn yields $H(F_i(x), F_j(y)) \leq cM(x, y)$ for $x, y \in K$ and $0 < c < 1$. Therefore using (2), one obtains

$$\max\left\{\frac{1}{1-ch}, \frac{a+h}{a+h(1-c)}\right\} < 1.$$

Now, by Theorem 1 (with restriction $S = T$, we get $Tz \in F_i(z) \cap F_j(z)$ for some $z \in K$ and one concludes $M(z, z) = 0$ contradicting the facts that $M(x, y) > 0$. Therefore $M(x, y) = 0$ for some $x, y \in K$ which implies $Tx \in F_i(x)$ for any odd integer$i \in N$ and $Tx = Ty \in F_j(y)$ for any even integer $j \in N$. If $M(x, x) = 0$ then $Tx \in F_j(x)$ for any even integer $j \in N$ and if $M(x, x) \neq 0$ then using (2), one infers that $d(Tx, F_j(x)) \leq 0$ yielding thereby $Tx \in F_j(x)$ for any even integer $j \in N$. Similarly in either of the cases $M(y, y) = 0$ or $M(y, y) > 0$, one concludes that $Ty \in F_i(y)$ for any odd integer $i \in N$. Thus we have shown that $\{F_n\}$ and T have a common point of coincidence. This completes the proof. □

While proving Theorem 3 the following question remains unresolved: Does Theorem 3.3 hold for $\{F_n\}$, S and T instead of $\{F_n\}$ and T?

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