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# A Remark on One Iterative Process for Finding the Roots of Equations

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Abstract. In this paper we consider the convergence of one iterative formula for finding the roots of equations.

It is well known that, if the equation

$$x = f(x) \tag{1}$$

has only one root $x = r$ in the interval $[a, b]$, and if the derivative $f'(x)$ of the function $f(x)$ satisfies the condition

$$\max\left|f'(x)\right| = M < 1, \quad \text{for } x \in [a, b], \tag{2}$$

then the iterative method

$$x_{k+1} = f(x_k), \quad k = 0, 1, 2\ldots, \tag{3}$$

converges to the root $x = r$ of the equation (1), where the initial value $x_0$ can be any number from the interval $[a, b]$. The convergence of the process (3) is more rapid if $M$ has a small value.

In this paper we consider the values $f'(a)$ and $f'(b)$ and use it to determine $\max\left|f'(x)\right|$.

Let $f'(x)$ be a negative increasing function. Therefore, we have

$$f'(a) \le f'(x) \le f'(b) < 0, \quad \text{for} \quad x \in [a, b]. \tag{4}$$

From (4), we see that

$$\max\left|f'(x)\right| = \left|f'(a)\right| \tag{5}$$

and

$$1 - f'(a) > 0. \tag{6}$$

In (5), we have either

$$\max\left|f'(x)\right| = \left|f'(a)\right| < 1 \tag{7}$$

or

$$\max\left|f'(x)\right| = \left|f'(a)\right| \ge 1. \tag{8}$$

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In both of these cases, we write the equation (1) in the form

$$x = \frac{1}{1 - f'(a)}\big(f(x) - f'(a)x\big), \tag{9}$$

that is, in the form

$$x = f_1(x), \tag{10}$$

where

$$f_1(x) = \frac{1}{1 - f'(a)}\big(f(x) - f'(a)x\big). \tag{11}$$

From (11) we obtain

$$f_1'(x) = \frac{1}{1 - f'(a)}\big(f'(x) - f'(a)\big). \tag{12}$$

As $f'(x)$ is an increasing function, considering (6), we conclude from (12) that the function $f_1'(x)$ is also increasing.

From (12) we obtain

$$f_1'(a) = 0, \quad f_1'(b) = \frac{f'(b) - f'(a)}{1 - f'(a)} < \frac{-f'(a)}{1 - f'(a)} < 1. \tag{13}$$

From (13) follows that

$$\max\big|f_1'(x)\big| = \frac{f'(b) - f'(a)}{1 - f'(a)} < 1. \tag{14}$$

Let $f'(x)$ be a negative decreasing function. Therefore, we have

$$0 > f'(a) \geq f'(x) \geq f'(b), \quad \text{for} \quad x \in [a, b]. \tag{15}$$

From (15), we see that

$$\max\big|f'(x)\big| = \big|f'(b)\big| \tag{16}$$

and

$$1 - f'(b) > 0. \tag{17}$$

In (16), we have either

$$\max\big|f'(x)\big| = \big|f'(b)\big| < 1 \tag{18}$$

or

$$\max\big|f'(x)\big| = \big|f'(b)\big| \geq 1. \tag{19}$$

In both of these cases, we write the equation (1) in the form

$$x = \frac{1}{1 - f'(b)}\big(f(x) - f'(b)x\big), \tag{20}$$

that is, in the form

$$x = f_2(x), \tag{21}$$

where

(22) $$f_2(x) = \frac{1}{1 - f'(b)}\big(f(x) - f'(b)x\big).$$

From (22) we obtain

(23) $$f_2'(x) = \frac{1}{1 - f'(b)}\big(f'(x) - f'(b)\big).$$

As $f'(x)$ is a decreasing function, considering (17), we conclude from (23) that the function $f_2'(x)$ is also decreasing.

From (23) we obtain

(24) $$f_2'(a) = \frac{f'(a) - f'(b)}{1 - f'(b)} < \frac{-f'(b)}{1 - f'(b)} < 1, \quad f_2'(b) = 0.$$

From (24) follows that

(25) $$\max\left|f_2'(x)\right| = \frac{f'(a) - f'(b)}{1 - f'(b)} < 1.$$

If the function $f'(x)$ satisfies the condition (4), having (9) in mind, we can use the following iterative process for finding the root $x = r$ of the equation (1):

(26) $$x_{k+1} = \frac{1}{1 - f'(a)}\big(f(x_k) - f'(a)x_k\big), \quad k = 0, 1, 2 \ldots$$

when $\max\left|f'(x)\right| < 1$ and when $\max\left|f'(x)\right| \geq 1$.

If the function $f'(x)$ satisfies the condition (15), having (20) in mind, we can use the following iterative process for finding the root $x = r$ of the equation (1):

(27) $$x_{k+1} = \frac{1}{1 - f'(b)}\big(f(x_k) - f'(b)x_k\big), \quad k = 0, 1, 2 \ldots$$

when $\max\left|f'(x)\right| < 1$ and when $\max\left|f'(x)\right| \geq 1$.

In [1, p. 145], the equation

(28) $$F(x) = 0$$

is considered, which has the root $x = r$ in the interval $[a, b]$ in the case

(29) $$0 < m_1 \leq F'(x) \leq M_1, \quad \text{for} \quad x \in [a, b],$$ [1]

where we can take

(30) $$m_1 = F'(a), \; M_1 = F'(b).$$

Now the condition (29) is reduced to

(31) $$F'(a) \leq F'(x) \leq F'(b).$$

[1] If $F'(x) < 0$ instead of equation $F(x) = 0$ we consider the equation $-F(x) = 0$.

In this case the equation (28) can be written in the form

$$x = x - \frac{1}{F'(b)}F(x), \tag{32}$$

that is, in the form

$$x = \varphi(x)$$

where

$$\varphi(x) = x - \frac{1}{F'(b)}F(x), \tag{33}$$

wherefrom

$$\varphi'(x) = 1 - \frac{1}{F'(b)}F'(x). \tag{34}$$

Having (30) and (31) in mind, it follows from (34) that

$$\max\left|\varphi'(x)\right| = 1 - \frac{F'(a)}{F'(b)} = q < 1. \tag{35}$$

**Example 1.** *The equation*

$$F(x) = 8x^3 - 6x - 3 = 0 \tag{$a$}$$

*has a root* $x = r \in [1, 2] = [a, b]$.

*We can write the equation* $(a)$ *in the form*

$$x = \frac{3}{4x} + \frac{3}{8x^2}, \tag{$a_1$}$$

*that is, in the form*

$$x = f(x),$$

*where*

$$f(x) = \frac{3}{4x} + \frac{3}{8x^2}. \tag{$a_2$}$$

*From* $(a_2)$ *we obtain*

$$f'(x) = -\frac{3}{4x^2} - \frac{3}{4x^3}. \tag{$a_3$}$$

*For* $x \in [1, 2] = [a, b]$ *the function* $f'(x)$ *is negative and increasing, and it holds that*

$$f'(1) = f'(a) = -\frac{3}{2},\ f'(2) = f'(b) = -\frac{9}{32}, \tag{$a_4$}$$

*which means that we can apply the formula (26) for finding the root* $x = r \in [1, 2]$ *of the equation* $(a_1)$.

*According to (5) and* $(a_4)$, *we have*

$$\max\left|f'(x)\right| = \left|f'(1)\right| = \left|f'(a)\right| = \frac{3}{2} > 1. \tag{$a_5$}$$

*Having* $(a_4)$ *in mind, we obtain from (14)*

$$(a_6) \qquad \max\left|f_1(x)\right| = \frac{39}{80}.$$

*From* $(a)$ *we obtain*

$$(a_7) \qquad F'(x) = 24x^2 - 6,$$

*wherefrom*

$$(a_8) \qquad F'(1) = F'(a) = 18,\ F'(2) = F'(b) = 90,$$

*which means that the condition (31) is satisfied.*

*For* $x \in [1,2] = [a,b]$ *the function* $F'(x)$ *is increasing, and it holds that*

$$(a_9) \qquad F'(1) = F'(a) = 18,\ F'(2) = F'(b) = 90.$$

*The function* $\varphi'(x)$ *in (34) is decreasing and having* $(a_9)$ *in mind, we obtain from (34)*

$$(a_{10}) \qquad \max\left|\varphi'(x)\right| = \frac{4}{5}.$$

*Now the formula (32) is reduced to*

$$(a_{11}) \qquad x = x - \frac{1}{90}F(x),$$

*from which follows the iterative process*

$$(a_{12}) \qquad x_{k+1} = x_k - \frac{1}{90}(8x_k^3 - 6x_k - 3), \quad k = 0,1,2,\dots .$$

*Having* $(a_2)$ *and* $(a_4)$ *in mind, the iterative process (26) is reduced to*

$$(a_{13}) \qquad x_{k+1} = \frac{3}{5}\left(x_k + \frac{1}{2x_k} + \frac{1}{4x_k^2}\right), \quad k = 0,1,2,\dots,$$

*which converges more rapidly than the process* $(a_{12})$ *for finding the root* $x = r \in [1,2]$ *of the equation* $(a)$*, having* $(a_5)$ *and* $(a_{10})$ *in mind. The initial value* $x_0$ *can be any number from the interval* $[1,2]$.

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