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Fixed Point Theorems for Monotone Mappings on Partial D^*-metric Spaces

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Abstract. In this paper, we introduce the concept of partial D^*-metric on a nonempty set X. In the present paper, we give some fixed point results on these interesting spaces.

1. Introduction

There are a lot of fixed and common fixed point results in different type spaces. For example, metric spaces, fuzzy metric spaces and uniform spaces etc. One of the most interesting is a partial metric space, which is defined by Matthews [9]. In a partial metric space, the distance of a point to it self may not be zero. After the definition of a partial metric space, Matthews proved the partial metric version of Banach fixed point theorem. Then, Valero [21], Oltra and Valero [13] and Altun et al [3] gave some generalizations of the result of Matthews. Again, Romaguera [15] proved the Caristi type fixed point theorem on this space.

On the other hand, there have been a number of generalizations of metric spaces. One of such generalizations is a generalized metric space (or D-metric space) initiated by Dhage [6] in 1992. He proved the existence of unique fixed point of a self-map satisfying a contractive condition in complete and bounded D-metric spaces. Dealing with D-metric space, Ahmad et al. [1], Dhage [6, 7], Dhage et al. [8], Rhoades [14] and Singh and Sharma [20] and others made a significant contribution in fixed point theory of D-metric space. In 2004 Naidu et al. proved that D-metric is not continuous and due to this fact almost all theorems which have been proved are invalid (see [10, 11, 12]. Recently, Sh. Sedghi et al. [16, 17, 18, 19] modified the D-metric space and defined D^*-metric spaces and proved some basic properties and some fixed point and common fixed point theorems in complete D^*-metric spaces. In this paper, using the concept of D^*-metric space, we introduce

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the concept of partial D^*-metric space and prove a common fixed point theorem for three mappings in partial D^*-metric spaces. At first, we recall some concepts and properties of D^*-metric space.

Throughout this paper, denote $\mathbb{N}$ as the set of all natural numbers and $\mathbb{R}^+$ as the set of all positive real numbers.

Definition 1 ([17]). *Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function: $D^* : X^3 \longrightarrow [0, \infty)$ that satisfies the following conditions for each $x, y, z, a \in X$:*

(1) $D^*(x, y, z) \geq 0$,
(2) $D^*(x, y, z) = 0$ *if and only if* $x = y = z$,
(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$, *(symmetry) where p is a permutation function,*
(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a generalized metric (or D^*-metric) space.

Immediate examples of such a function are as follows.

Example 1 ([17]). (a) *Let (X, d) be a metric space then $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ and $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ are D^*-metric on X.*

(b) *If $X = \mathbb{R}^n$, then*

$$D^*(x, y, z) = ||x + y - 2z|| + ||y + z - 2x|| + ||z + x - 2y||$$

for every $x, y, z \in \mathbb{R}^n$ is a D^*-metric on X.

Example 2. *Let $\psi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ be a mapping defined as follows:*

$$\psi(x, y) = 0 \ if \ x = y, \ \ \psi(x, y) = \frac{1}{2} \ \ if \ \ x > y, \ \ \psi(x, y) = \frac{1}{3} \ if \ \ x < y.$$

Then clearly ψ is not a metric, since $\psi(1, 2) \neq \psi(2, 1)$. Define $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ by

$$G(x, y, z) = \max\{\psi(x, y), \psi(y, z), \psi(z, x)\}.$$

Then G is a D^*-metric.

Example 3. *Let $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a mapping defined as follows: $\psi(x, y) = \max\{x, y\}$. Then clearly it is not a metric. Define $G : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by*

$$G(x, y, z) = \max\{x, y\} + \max\{y, z\} + \max\{z, x\} - x - y - z,$$

for every $x, y, z \in \mathbb{R}^+$. Then G is a D^*-metric.

Remark 1 ([17]). *In a D^*-metric space (X, D^*), we have $D^*(x, x, y) = D^*(x, y, y)$.*

For more details of D^*-metric see [16, 18, 19].

2. Partial D^*-metric space

In this section we introduce the concept of a partial D^*-metric space and prove its properties.

Definition 2. *A partial D^*-metric on a nonempty set X is a function $p^* : X \times X \times X \to \mathbb{R}^+$ such that for all $x, y, z, a \in X$:*

(p$_1$) $x = y = z \Longleftrightarrow p^*(x,x,x) = p^*(x,y,z) = p^*(y,y,y) = p^*(z,z,z)$,

(p$_2$) $p^*(x,x,x) \leq p^*(x,y,z)$,

(p$_3$) $p^*(x,y,z) = p^*(p\{x,y,z\})$, *(symmetry) where p is a permutation function,*

(p$_4$) $p^*(x,y,z) \leq p^*(x,y,a) + p^*(a,z,z) - p^*(a,a,a)$.

A partial D^*-metric space is a pair (X, p^*) such that X is a nonempty set and p^* is a partial D^*-metric on X. It is clear that, if $p^*(x,y,z) = 0$, then from (p$_1$) and (p$_2$) $x = y = z$. But if $x = y = z$, $p^*(x,y,z)$ may not be 0. A basic example of a partial D^*-metric space is the pair $(\mathbb{R}^+, p^*)$, where $p^*(x,y,z) = \max\{x,y,z\}$ for all $x,y,z \in \mathbb{R}^+$.

It is easy to see that every D^*-metric is a partial D^*-metric, but the converse need not be true.

In the following examples a partial D^*-metric fails to satisfy properties of D^*-metric.

Example 4. *Let $p^* : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a mapping defined as follows:*

$$p^*(x,y,z) = |x-y| + |y-z| + |x-z| + \max\{x,y,z\}.$$

Then clearly it is a partial D^*-metric, but it is not a D^*-metric.

Example 5. *Let (X,p) be a partial metric space and $p^* : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a mapping defined as follows:*

$$p^*(x,y,z) = p(x,y) + p(x,z) + p(y,z) - p(x,x) - p(y,y) - p(z,z).$$

Then clearly p^* is a partial D^*-metric, but it is not a D^*-metric.

Remark 2. *Note that $p^*(x,x,y) = p^*(x,y,y)$, because,*

(i) $p^*(x,x,y) \leq p^*(x,x,x) + p^*(x,y,y) - p^*(x,x,x) = p^*(x,y,y)$ *and similarly*

(ii) $p^*(y,y,x) \leq p^*(y,y,y) + p^*(y,x,x) - p^*(y,y,y) = p^*(y,x,x)$.

Hence by (i)and(ii), we get $p^*(x,x,y) = p^*(x,y,y)$.

Lemma 1. *Let (X,p^*) be a partial D^*-metric space. If we define $p(x,y) = p^*(x,y,y)$, then (X,p) is a partial metric space*

Proof. (p$_1$) $x = y \Longleftrightarrow p^*(x,x,x) = p^*(x,y,y) = p(y,y,y) \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(p$_2$) $p^*(x,x,x) \leq p^*(x,y,y)$ implies that $p(x,x) \leq p(x,y)$,

(p$_3$) $p^*(x,y,y) = p^*(y,x,x)$ implies that $p(x,y) = p(y,x)$, (p₄) $p^*(y,y,x) \le p^*(y,y,z) + p^*(z,x,x) - p^*(z,z,z)$ implies that $p(x,y) \le p(y,z) + p(z,x) - p(z,z)$. □

Let (X, p^*) be a partial D^*-metric space. For $r > 0$ define

$$B_{p^*}(x,r) = \{y \in X : p^*(x,y,y) < p^*(x,x,x) + r\}.$$

Definition 3. *Let (X, p^*) be a partial D^*-metric space and $A \subset X$.*

(1) *If for every $x \in A$ there exists $r > 0$ such that $B_{p^*}(x,r) \subset A$, then subset A is called an open subset of X.*

(2) *A sequence $\{x_n\}$ in a partial D^*-metric space (X, p^*) converges to x if and only if $p^*(x,x,x) = \lim_{n\to\infty} p^*(x_n,x_n,x)$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$p^*(x,x,x_n) < p^*(x,x,x) + \varepsilon \ \forall n \ge n_0, \ (1)$$

or equivalently, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$p^*(x,x_n,x_m) < p^*(x,x,x) + \varepsilon \ \forall n,m \ge n_0. \ (2)$$

Indeed, if (1) holds then

$$\begin{aligned} p^*(x,x_n,x_m) &= p^*(x_n,x,x_m) \\ &\le p^*(x_n,x,x) + p^*(x,x_m,x_m) - p^*(x,x,x) \\ &< \varepsilon + \varepsilon + p^*(x,x,x). \end{aligned}$$

Conversely, set $m = n$ in (2) we have $p^*(x_n,x_n,x) < p^*(x,x,x) + \varepsilon$.

(3) *A sequence $\{x_n\}$ in a partial D^*-metric space (X, p^*) is called a Cauchy sequence if $\lim_{n,m\to\infty} p^*(x_n,x_m,x_m)$ exists.*

Let τ_{p^*} be the set of all open subsets X, then τ_{p^*} is a topology on X (induced by the partial D^*-metric p^*).

A partial D^*-metric space (X, p^*) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_{p^*}, to a point $x \in X$.

If a sequence $\{x_n\}$ in a partial D^*-metric space (X, p^*) converges to x then we have

$$\begin{aligned} p^*(x_n,x_n,x_m) &\le p^*(x_n,x_n,x) + p^*(x,x_m,x_m) - p^*(x,x,x) \\ &< \varepsilon + \varepsilon + p^*(x,x,x). \end{aligned}$$

Lemma 2. *Let (X, p^*) be a partial D^*-metric space. If $r > 0$, then ball $B_{p^*}(x,r)$ with center $x \in X$ and radius r is an open ball.*

Proof. Let $y \in B_{p^*}(x,r)$,then $p^*(x,y,y) < p^*(x,x,x) + r$. Let $p^*(x,y,y) - p^*(x,x,x) = \delta$. Let $z \in B_{p^*}(y, r-\delta)$, by triangular inequality we have

$$\begin{aligned} p^*(x,x,z) &\leq p^*(x,x,y) + p^*(y,z,z) - p^*(y,y,y) \\ &= p^*(x,y,y) - p^*(x,x,x) + p^*(z,z,y) - p^*(y,y,y) + p^*(x,x,x) \\ &< \delta + r - \delta + p^*(x,x,x) \\ &= p^*(x,x,x) + r. \end{aligned}$$

Thus $z \in B_{p^*}(x,r)$. Hence $B_{p^*}(y, r-\delta) \subseteq B_{p^*}(x,r)$. Therefore the ball $B_{p^*}(x,r)$ is an open ball. □

Each partial D^*-metric p^* on X generates a topology τ_{p^*} on X which has as a base the family of open p^*-balls $\{B_{p^*}(x,\varepsilon) : x \in X,\, \varepsilon > 0\}$.

The following example shows that a convergent sequence $\{x_n\}$ in a partial D^*-metric space (X, p^*) need not be a Cauchy sequence. In particular, it shows that the limit of a convergent sequence is not necessarily unique.

Example 6. *Let $X = [0,\infty)$ and $p^*(x,y,z) = \max\{x,y,z\}$. Then it is clear that (X,p^*) is a complete partial D^*-metric space. Let*

$$x_n = \begin{cases} 1, & n = 2k, \\ 2, & n = 2k+1. \end{cases}$$

Then clearly it is convergent sequence and for every $x \geq 2$ we have $\lim_{n\to\infty} p^*(x_n, x_n, x) = p^*(x,x,x)$, therefore

$$L(x_n) = \{x | x_n \longrightarrow x\} = [2,\infty).$$

But $\lim_{n,m\to\infty} p^*(x_n, x_m, x_m)$ does not exist. Hence $\{x_n\}$ is not a Cauchy sequence.

The following lemma plays an important role in this paper.

Lemma 3. *Let (X,p) be a partial metric space then there exists a partial D^*-metric p^* on X such that*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the partial D^*-metric space (X,p^*),*
(b) *the partial metric space (X,p) is complete if and only if the partial D^*-metric space (X,p^*) is complete. Furthermore, $p^*(x,x,y) = p(x,y)$ for every $x,y \in X$.*

Proof. Define

$$p^*(x,y,z) = \max\{p(x,y), p(x,z), p(y,z)\}\ \forall x,y,z \in X.$$

Then it is easy to see that p^* is a partial D^*-metric and $p^*(x,x,y) = p(x,y)$ for every $x,y \in X$. □

The following Lemma shows that under certain conditions the limit is unique.

Lemma 4. *Let $\{x_n\}$ be a convergent sequence in a partial D^*-metric space (X, p^*) such that $x_n \longrightarrow x$ and $x_n \longrightarrow y$. If*

$$\lim_{n\to\infty} p^*(x_n, x_n, x_n) = p^*(x, x, x) = p^*(y, y, y),$$

then $x = y$.

Proof. As

$$p^*(x, y, y) = p^*(x, x, y) \leq p^*(x, x, x_n) + p^*(x_n, y, y) - p^*(x_n, x_n, x_n),$$

therefore

$$p^*(x_n, x_n, x_n) \leq p^*(x, x, x_n) + p^*(x_n, y, y) - p(x, y, y).$$

By given assumptions, we have

$$\lim_{n\to\infty} p^*(x_n, x_n, x) = p^*(x, x, x),$$
$$\lim_{n\to\infty} p^*(x_n, x_n, y) = p^*(y, y, y),$$
$$\lim_{n\to\infty} p^*(x_n, x_n, x_n) = p^*(x, x, x).$$

Therefore

$$p^*(x, x, x) \leq p^*(x, x, x) + p^*(y, y, y) - p^*(x, y, y),$$

which shows that $p^*(y, y, y) \leq p^*(x, y, y) \leq p^*(y, y, y)$. Also,

$$p^*(x, y, y) = p^*(y, y, x) \leq p^*(y, y, x_n) + p^*(x_n, x, x) - p^*(x_n, x_n, x_n)$$

implies that

$$p^*(x_n, x_n, x_n) \leq p^*(y, y, x_n) + p^*(x_n, x, x) - p^*(x, y, y),$$

which on taking limit as $n \to \infty$ gives

$$p^*(y, y, y) \leq p^*(y, y, y) + p^*(x, x, x) - p^*(x, y, y),$$

which shows that

$$p^*(x, x, x) \leq p^*(x, y, y) \leq p^*(x, x, x).$$

Thus $p^*(x, x, x) = p^*(x, y, y) = p^*(y, y, y)$. Therefore $x = y$. □

Lemma 5. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial D^*-metric space (X, p^*) such that*

$$\lim_{n\to\infty} p^*(x_n, x, x) = \lim_{n\to\infty} p^*(x_n, x_n, x_n) = p^*(x, x, x),$$

and

$$\lim_{n\to\infty} p^*(y_n, y, y) = \lim_{n\to\infty} p^*(y_n, y_n, y_n) = p^*(y, y, y).$$

Then $\lim_{n\to\infty} p^*(x_n, y_n, y_n) = p^*(x, y, y)$. In particular, $\lim_{n\to\infty} p^*(x_n, y_n, z) = p^*(x, y, z)$ for every $z \in X$.

Proof. As $\{x_n\}$ and $\{y_n\}$ converge to a $x \in X$ and $y \in X$ respectively, therefore for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} p^*(x, x, x_n) &< p^*(x, x, x) + \frac{\varepsilon}{2}, \\ p^*(y, y, y_n) &< p^*(y, y, y) + \frac{\varepsilon}{2}, \\ p^*(x, x, x_n) &< p^*(x_n, x_n, x_n) + \frac{\varepsilon}{2}, \end{aligned}$$

and

$$p^*(y, y, y_n) < p^*(y_n, y_n, y_n) + \frac{\varepsilon}{2}$$

for $n \geq n_0$. Now

$$\begin{aligned} p^*(x_n, x_n, y_n) &\leq p^*(x_n, x_n, x) + p^*(x, y_n, y_n) - p^*(x, x, x) \\ &\leq p^*(x_n, x_n, x) + p^*(y, y_n, y_n) + p^*(x, x, y) \\ &\qquad - p^*(y, y, y) - p^*(x, x, x) \\ &< p^*(x, y, y) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= p^*(x, y, y) + \varepsilon, \end{aligned}$$

and so we have

$$p^*(x_n, y_n, y_n) - p^*(x, y, y) < \varepsilon.$$

Also,

$$\begin{aligned} p^*(x, y, y) &\leq p^*(x_n, y, y) + p^*(x, x, x_n) - p^*(x_n, x_n, x_n) \\ &\leq p^*(x, x, x_n) + p^*(x_n, x_n, y_n) + p^*(y_n, y, y) \\ &\qquad - p^*(y_n, y_n, y_n) - p^*(x_n, x_n, x_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + p^*(x_n, x_n, y_n) \\ &= p^*(x_n, x_n, y_n) + \varepsilon. \end{aligned}$$

Thus

$$p^*(x, x, y) - p^*(x_n, x_n, y_n) < \varepsilon.$$

Hence for all $n \geq n_0$, we have $|p^*(x_n, x_n, y_n) - p^*(x, x, y)| < \varepsilon$. Hence the result follows. □

Lemma 6. *If p^* is a partial D^*-metric on X, then the functions p^{*s}, p^{*m} : $X \times X \times X \to \mathbb{R}^+$ given by*

$$\begin{aligned} p^{*s}(x, y, z) = &\, p^*(x, x, y) + p^*(y, y, z) + p^*(z, z, x) \\ &- p^*(x, x, x) - p^*(y, y, y) - p^*(z, z, z) \end{aligned}$$

and

$$p^{*m}(x, y, z) = \max \left\{ \begin{array}{l} 2p^*(x, x, y) - p^*(x, x, x) - p^*(y, y, y), \\ 2p^*(y, y, z) - p^*(y, y, y) - p^*(z, z, z), \\ 2p^*(z, z, x) - p^*(z, z, z) - p^*(x, x, x) \end{array} \right\}$$

for every $x, y, z \in X$, are equivalent D^*-metrics on X.

Proof. It is easy to see that p^{*s} and p^{*m} are D^*-metrics on X. Let $x, y, z \in X$. It is obvious that

$$p^{*m}(x, y, z) \leq 2p^{*s}(x, y, z).$$

On the other hand, since $a + b + c \leq 3 \max\{a, b, c\}$, it provides that

$$\begin{aligned} p^{*s}(x, y, z) &= p^*(x, x, y) + p^*(y, y, z) + p^*(z, z, x) - p^*(x, x, x) \\ &\quad - p^*(y, y, y) - p^*(z, z, z) \\ &= \frac{1}{2}[2p^*(x, x, y) - p^*(x, x, x) - p^*(y, y, y)] \\ &\quad + \frac{1}{2}[2p^*(y, y, z) - p^*(y, y, y) - p^*(z, z, z)] \\ &\quad + \frac{1}{2}[2p^*(z, z, x) - p^*(z, z, z) - p^*(x, x, x)] \\ &\leq \frac{3}{2} \max \left\{ \begin{array}{c} 2p^*(x, x, y) - p^*(x, x, x) - p^*(y, y, y), \\ 2p^*(y, y, z) - p^*(y, y, y) - p^*(z, z, z), \\ 2p^*(z, z, x) - p^*(z, z, z) - p^*(x, x, x) \end{array} \right\} \\ &= \frac{3}{2} p^{*m}(x, y, z). \end{aligned}$$

Thus,we have

$$\frac{1}{2} p^{*m}(x, y, z) \leq p^{*s}(x, y, z) \leq \frac{3}{2} p^{*m}(x, y, z).$$

These inequalities implies that p^{*s} and p^{*m} are equivalent. □

Remark 3. *Note that:*

$$p^{*s}(x, x, y) = 2p^*(x, x, y) - p^*(x, x, x) - p^*(y, y, y) = p^{*m}(x, x, y).$$

A mapping $F : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F(B_{p^*}(x_0, \delta)) \subseteq B_{p^*}(Fx_0, \varepsilon)$.

The following lemma plays an important role to prove fixed point results on a partial D^*-metric space.

Lemma 7. *Let (X, p^*) be a partial D^*-metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p^*) if and only if it is a Cauchy sequence in the D^*-metric space (X, p^{*s}).*

(b) *A partial D^*-metric space (X, p^*) is complete if and only if the D^*-metric space (X, p^{*s}) is complete. Furthermore,*

$$\lim_{n\to\infty} p^{*s}(x_n, x_n, x) = 0$$

if and only if

$$p^*(x, x, x) = \lim_{n\to\infty} p^*(x_n, x_n, x) = \lim_{n,m\to\infty} p^*(x_n, x_n, x_m).$$

Proof. First we show that every Cauchy sequence in (X, p^*) is a Cauchy sequence in (X, p^{*s}). To this end let $\{x_n\}$ be a Cauchy sequence in (X, p^*). Then there exists $\alpha \in \mathbb{R}$ such that, for given $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ with $|p^*(x_n, x_n, x_m) - \alpha| < \frac{\varepsilon}{4}$ for all $n, m \geq n_\varepsilon$. Hence

$$\begin{aligned} p^{*s}(x_n, x_n, x_m) &= \Big| 2p^*(x_n, x_n, x_m) - p^*(x_n, x_n, x_n) \\ &\qquad - p^*(x_m, x_m, x_m) + 2\alpha - 2\alpha \Big| \\ &\leq |2p^*(x_n, x_n, x_m) - 2\alpha| + |p^*(x_n, x_n, x_n) - \alpha| \\ &\qquad + |p^*(x_m, x_m, x_m) - \alpha| \\ &< 4\frac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

for all $n, m \geq n_\varepsilon$. Which implies that $\{x_n\}$ is a Cauchy sequence in (X, p^{*s}). Next we prove that completeness of (X, p^{*s}) implies completeness of (X, p^*). Indeed, if $\{x_n\}$ is a Cauchy sequence in (X, p^*) then it is also a Cauchy sequence in (X, p^{*s}). Since the D^*-metric space (X, p^{*s}) is complete we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} p^{*s}(x_n, x_n, y) = 0$. Therefore,

$$\begin{aligned} &\limsup_{n\to\infty} |p^*(x_n, x_n, y) - p^*(y, y, y)| \\ &\qquad \leq \lim_{n\to\infty} |2p^*(x_n, x_n, y) - p^*(x_n, x_n, x_n) - p^*(y, y, y)| = 0. \end{aligned}$$

Hence we follow that $\{x_n\}$ is a convergent sequence in (X, p^*). That is,

$$\lim_{n\to\infty} p^*(x_n, x_n, y) = p^*(y, y, y).$$

Now we prove that every Cauchy sequence $\{x_n\}$ in (X, p^{*s}) is a Cauchy sequence in (X, p^*). Let $\varepsilon = \frac{1}{2}$, then there exists $n_0 \in \mathbb{N}$ such that $p^{*s}(x_n, x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_0$. Since

$$\begin{aligned} p^*(x_n, x_n, x_n) &\leq 4p^*(x_{n_0}, x_{n_0}, x_n) - 3p^*(x_n, x_n, x_n) \\ &\qquad - p^*(x_{n_0}, x_{n_0}, x_{n_0}) + p^*(x_n, x_n, x_n) \\ &\leq 2p^{*s}(x_n, x_n, x_{n_0}) + p^*(x_{n_0}, x_{n_0}, x_{n_0}). \end{aligned}$$

Thus, we have

$$\begin{aligned} p^*(x_n, x_n, x_n) &\leq 2p^{*s}(x_n, x_n, x_{n_0}) + p^*(x_{n_0}, x_{n_0}, x_{n_0}) \\ &\leq 1 + p^*(x_{n_0}, x_{n_0}, x_{n_0}). \end{aligned}$$

Consequently the sequence $\{p^*(x_n, x_n, x_n)\}$ is bounded in $\mathbb{R}$, and so there exists an $a \in \mathbb{R}$ such that a subsequence $\{p^*(x_{n_k}, x_{n_k}, x_{n_k})\}$ is convergent to a, i.e. $\lim_{k\to\infty} p^*(x_{n_k}, x_{n_k}, x_{n_k}) = a$.

It remains to prove that $\{p^*(x_n, x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a Cauchy sequence in (X, p^{*s}), for given $\varepsilon > 0$, there exists n_ε such that $p^{*s}(x_n, x_n, x_m) < \frac{\varepsilon}{2}$ for all $n, m \geq n_\varepsilon$. Thus, for all $n, m \geq n_\varepsilon$,

$$\begin{aligned}|p^*(x_n, x_n, x_n) - p^*(x_m, x_m, x_m)| &\leq 4p^*(x_n, x_n, x_m) \\ &\quad - 3p^*(x_n, x_n, x_n) - p^*(x_m, x_m, x_m) \\ &\quad + p^*(x_n, x_n, x_n) - p^*(x_m, x_m, x_m) \\ &\leq 2p^{*s}(x_n, x_n, x_m) < \varepsilon.\end{aligned}$$

On the other hand,

$$\begin{aligned}|p^*(x_n, x_n, x_n) - a| &\leq |p^*(x_n, x_n, x_n) - p^*(x_{n_k}, x_{n_k}, x_{n_k})| \\ &\quad + |p^*(x_{n_k}, x_{n_k}, x_{n_k}) - a| \\ &< \varepsilon + \varepsilon = 2\varepsilon,\end{aligned}$$

for all $n, n_k \geq n_\varepsilon$. Hence $\lim_{n\to\infty} p^*(x_n, x_n, x_n) = a$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in (X, p^*). We have,

$$\begin{aligned}|2p^*(x_n, x_n, x_m) - 2a| &= \Big|p^{*s}(x_n, x_n, x_m) + p^*(x_n, x_n, x_n) \\ &\qquad - a + p^*(x_m, x_m, x_m) - a\Big| \\ &\leq p^{*s}(x_n, x_n, x_m) + |p^*(x_n, x_n, x_n) - a| \\ &\qquad + |p^*(x_m, x_m, x_m) - a| \\ &< \frac{\varepsilon}{2} + 2\varepsilon + 2\varepsilon = \frac{9}{2}\varepsilon.\end{aligned}$$

That is, $\{x_n\}$ is a Cauchy sequence in (X, p^*).

We shall have established the lemma if we prove that (X, p^{*s}) is complete if so is (X, p^*). Let $\{x_n\}$ be a Cauchy sequence in (X, p^{*s}). Then $\{x_n\}$ is a Cauchy sequence in (X, p^*), and so it is convergent to a point $y \in X$ with,

$$\lim_{n,m\to\infty} p^*(x_n, x_n, x_m) = \lim_{n\to\infty} p^*(y, y, x_n) = p^*(y, y, y).$$

Thus, for given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$p^*(y, y, x_n) - p^*(y, y, y) < \frac{\varepsilon}{2} \; and \;\; |p^*(y, y, y) - p^*(x_n, x_n, x_n)| < \frac{\varepsilon}{2}$$

whenever $n \geq n_\varepsilon$. As a consequence we have

$$\begin{aligned}p^{*s}(y, y, x_n) &= 2p^*(y, y, x_n) - p^*(x_n, x_n, x_n) - p^*(y, y, y) \\ &\leq |p^*(y, y, x_n) - p^*(y, y, y)| + |p^*(y, y, x_n) - p^*(x_n, x_n, x_n)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

whenever $n \geq n_\varepsilon$. Therefore (X, p^{*s}) is complete.

Finally, it is a simple matter to check that $\lim_{n\to\infty} p^{*s}(a, a, x_n) = 0$ if and only if

$$p^*(a, a, a) = \lim_{n\to\infty} p^*(a, a, x_n) = \lim_{n,m\to\infty} p^*(x_n, x_n, x_m). \qquad \square$$

Definition 4. *Let (X, p^*) be a partial D^*-metric space, then p^* is said to be of the first type if for every $x, y \in X$ we have*

$$p^*(x, x, y) \leq p^*(x, y, z),$$

for every $z \in X$.

3. Fixed point Result

We begin this section giving the concept of weakly increasing mappings (see [5]).

Definition 5. *Let $(X, \preceq)$ be a partially ordered set. Two mappings $S, T : X \longrightarrow X$ are said to be S-T weakly increasing if $Sx \preceq TSx$ for all $x \in X$.*

Note that, two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact in [4].

In the sequel, we use the following notations:

(i) $\mathcal{F}$ denote the set of all functions $F : [0, \infty) \longrightarrow [0, \infty)$ such that F is nondecreasing and continuous, $F(0) = 0 < F(t)$ for every $t > 0$ and $F(x + y) \leq F(x) + F(y)$ for all $x, y \in [0, +\infty)$;

(ii) Ψ denote the set of all functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ where ψ is continuous, nondecreasing function such that $\sum_{n=0}^{\infty} \psi^n(t)$ is convergent for each $t > 0$. From the conditions on ψ, it is clear that $\lim_{n\to\infty} \psi^n(t) = 0$ and $\psi(t) < t$ for every $t > 0$.

Our main result is as follows:

Theorem 1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a first type partial D^*-metric p^* on X such that (X, p^*) is a complete partial D^*-metric space.*

Let $S, T, R : X \longrightarrow X$ are three S-T, T-R and R-S weakly increasing mappings such that

$$F(p^*(Sx, Ty, Rz)) \leq \psi(F(\varphi(x, y, z))) \tag{3.1}$$

for all $x, y, z \in X$ with x, y, z are comparable with respect to partially order $\preceq$, where $F \in \mathcal{F}, \psi \in \Psi$ and

$$\varphi(x, y, z) = \max \left\{ \begin{array}{l} p^*(x, y, z), p^*(x, x, Sx), \\ p^*(y, y, Ty), p^*(z, z, Rz) \end{array} \right\}. \tag{3.2}$$

Further assume that if for every increasing sequence $\{x_n\}$ convergent to $x \in X$ we have $x_n \preceq x$.

Then S, T and R have a common fixed point.

Proof. Let x_0 be an arbitrary point of X. We can define a sequence in X as follows:

$$x_{3n+1} = Sx_{3n}\ ,\ x_{3n+2} = Tx_{3n+1} \quad \text{and} \quad x_{3n+3} = Rx_{3n+2} \quad \text{for} \quad n = 0, 1, \ldots.$$

Since S, T, R are three $S-T$, $T-R$ and $R-S$ weakly increasing mappings, we have

$$x_1 = Sx_0 \preceq TSx_0 = x_2 = Tx_1 \preceq RTx_1 = x_3 = Rx_2 \preceq SRx_2 = x_4$$

and continuing this process we have

$$x_1 \preceq x_2 \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

Case: Suppose there exists $n_0 \in \mathbb{N}$ such that $p^*(x_{3n_0}, x_{3n_0+1}, x_{3n_0+2}) = 0$. Now we show that $p^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3}) = 0$. Otherwise, from (3.1), we get

$$\begin{aligned} F(p^*(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3})) &\leq F(p^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3})) \\ &= F(p^*(Sx_{3n_0}, Tx_{3n_0+1}, Rx_{3n_0+2})) \\ &\leq \psi(F(\varphi(x_{3n_0}, x_{3n_0+1}, x_{3n_0+2}))) \\ &= \psi(F(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3})) \\ &< F(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3})), \end{aligned}$$

which is a contradiction. Hence $p^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3}) = 0$. Therefore, $x_{3n_0} = x_{3n_0+1} = x_{3n_0+2} = x_{3n_0+3}$. Thus $Sx_{3n_0} = Tx_{3n_0} = Rx_{3n_0} = x_{3n_0}$. That is x_{3n_0} is a common fixed point of S, T and R.

Case: Assume that $p^*(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$ for every $n \in \mathbb{N}$.
Now we prove that

$$F(p^*(x_{n-1}, x_n, x_{n+1})) \leq \psi(F(p^*(x_{n-2}, x_{n-1}, x_n))). \tag{3.3}$$

Setting $x = x_{3n}$, $y = x_{3n+1}$ and $z = x_{3n+2}$ in (3.2), we have

$$\varphi(x_{3n}, x_{3n+1}, x_{3n+2}) = \max \left\{ \begin{array}{l} p^*(x_{3n}, x_{3n+1}, x_{3n+2}), \\ p^*(x_{3n}, x_{3n}, x_{3n+1}), \\ p^*(x_{3n+1}, x_{3n+1}, x_{3n+2}), \\ p^*(x_{3n+2}, x_{3n+2}, x_{3n+3}) \end{array} \right\}.$$

Since, p^* is of the first type, we get

$$\varphi(x_{3n}, x_{3n+1}, x_{3n+2}) \leq \max\{p^*(x_{3n}, x_{3n+1}, x_{3n+2}), p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})\}.$$

If $p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})$ is maximum in the R.H.S. of the above inequality, we have from (3.1)that

$$\begin{aligned} F(p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})) &= F(p^*(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2})) \\ &< \psi(F(\varphi(x_{3n}, x_{3n+1}, x_{3n+2}))) \\ &\leq \psi\Big(F(\max\{p^*(x_{3n}, x_{3n+1}, x_{3n+2}), \\ &\qquad p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})\})\Big) \\ &= \psi\Big(F(p^*(x_{3n+1}, x_{3n+2}, x_{3n+3}))\Big) \\ &< F(p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})), \end{aligned}$$

which is a contradiction. Thus,

$$F(p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})) \leq \psi(F(p^*(x_{3n}, x_{3n+1}, x_{3n+2})).$$

Similarly, we have

$$F(p^*(x_{3n+2}, x_{3n+3}, x_{3n+4})) \leq \psi(F(p^*(x_{3n+1}, x_{3n+2}, x_{3n+3}))),$$

and

$$F(p^*(x_{3n}, x_{3n+1}, x_{3n+2})) \leq \psi(F(p^*(x_{3n-1}, x_{3n}, x_{3n+1}))).$$

Therefore, for every $n \in \mathbb{N}$ we have

$$F(p^*(x_n, x_{n+1}, x_{n+2})) \leq \psi(F(p^*(x_{n-1}, x_n, x_{n+1}))).$$

Now, we have

$$F(p^*(x_n, x_{n+1}, x_{n+2})) \leq \psi(F(p^*(x_{n-1}, x_n, x_{n+1}))) \leq \cdots \leq \psi^n(F(p^*(x_0, x_1, x_2))).$$

Hence

$$\lim_{n\to\infty} F(p^*(x_n, x_{n+1}, x_{n+2})) = 0,$$

so that

$$\lim_{n\to\infty} p^*(x_n, x_{n+1}, x_{n+2}) = 0. \tag{3.4}$$

Since $p^*$is of the first type and F is nondecreasing, we have

$$F(p^*(x_n, x_n, x_{n+1}) \leq F(p^*(x_n, x_{n+1}, x_{n+2})) \leq \psi^n(F(p^*(x_0, x_1, x_2))).$$

Since $F(x+y) \leq F(x) + F(y)$ and $p^{*s}(x_n, x_n, x_{n+1}) \leq 2p^*(x_n, x_n, x_{n+1})$ we have

$$F(p^{*s}(x_n, x_n, x_{n+1}) \leq 2F(p^*(x_n, x_n, x_{n+1})) \leq 2\psi^n(F(p^*(x_0, x_1, x_2))).$$

Now from $p^{*s}(x_{n+k}, x_n, x_n) \leq p^{*s}(x_{n+k}, x_{n+k-1}, x_{n+k-1}) + \cdots + p^{*s}(x_{n+1}, x_n, x_n)$, we have

$$\begin{aligned} F(p^{*s}(x_{n+k}, x_n, x_n)) &\leq F(p^{*s}(x_{n+k}, x_{n+k-1}, x_{n+k-1})) + \cdots + F(p^{*s}(x_{n+1}, x_n, x_n)) \\ &\leq 2\psi^{n+k-1}(p^*(x_0, x_1, x_2)) + \cdots + 2\psi^n(p^*(x_0, x_1, x_2)) \\ &\leq 2\sum_{i=n}^{\infty} \psi^i(p^*(x_0, x_1, x_2)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \psi^n(t)$ is convergent for each $t > 0$ it follows that $\{x_n\}$ is a Cauchy sequence in the D^*-metric space (X, p^{*s}). Since (X, p^*) is complete, then from Lemma 7 follows that the sequence $\{x_n\}$ converges to some x in the D^*-metric space (X, p^{*s}). Hence $\lim_{n\to\infty} p^{*s}(x_n, x, x) = 0$. Again, from Lemma 7, we have

$$p^*(x, x, x) = \lim_{n\to\infty} p^*(x_n, x, x) = \lim_{n,m\to\infty} p^*(x_n, x_m, x_m). \tag{3.5}$$

Since $\{x_n\}$ is a Cauchy sequence in the D^*-metric space (X, p^{*s}) and

$$p^{*s}(x_n, x_m, x_m) = 2p^*(x_n, x_m, x_m) - p^*(x_n, x_n, x_n) - p^*(x_m, x_m, x_m),$$

we have

$$\lim_{n,m\to\infty} p^{*s}(x_n, x_m, x_m) = 0$$

and by (3.4) we have

$$\lim_{n\to\infty} p^*(x_n, x_n, x_n) = 0,$$

thus by definition p^{*s} we have

$$\lim_{n,m\to\infty} p^*(x_n, x_m, x_m) = 0.$$

Therefore by (3.5), we have

$$\begin{aligned} p^*(x,x,x) &= \lim_{n\to\infty} p^*(x_n, x, x) \\ &= \lim_{n,m\to\infty} p^*(x_n, x_m, x_m) = 0. \end{aligned}$$

Now by the inequality (3.1) for $x = x$, $y = x_{3n+1}$ and $z = x_{3n+2}$, then we have

$$F(p^*(Sx, x_{3n+2}, x_{3n+3})) \le \psi(F(\varphi(x, x_{3n+1}, x_{3n+2})),$$

and by letting $n \to \infty$ and using Lemma 5, we obtain

$$F(p^*(Sx, x, x)) \le \psi(F(p^*(Sx, x, x)) < F(p^*(Sx, x, x)),$$

which is a contradiction. Hence, $p^*(Sx, x, x) = 0$. Thus $Sx = x$. Similarly, by using the inequality (3.1) for $y = x$, $x = x_{3n}$ and $z = x_{3n+2}$, then we have

$$F(p^*(x_{3n}, Tx, x_{3n+3})) \le \psi(F(\varphi(x_{3n}, x, x_{3n+2})),$$

and letting $n \to \infty$ and using Lemma 5, we obtain

$$F(p^*(x, Tx, x)) \le \psi(F(p^*(x, Tx, x)) < F(p^*(x, Tx, x)),$$

which is a contradiction.

Hence, $p^*(x, Tx, x) = 0$. Thus $Tx = x$. Similarly, by using the inequality (3.1) for $z = x$, $x = x_{3n}$ and $y = x_{3n+1}$, we can show that $Rx = x$. □

Corollary 1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a first type partial D^*-metric p^* on X such that (X, p^*) is a complete partial D^*-metric space.*

Let $S : X \longrightarrow X$ be a mapping such that $Sx \preceq S^2x$ and

$$F(p^*(Sx, Sy, Sz)) \le \psi(F(\varphi(x, y, z))) \tag{3.6}$$

for all $x, y, z \in X$ with x, y, z are comparable with respect to partially order $\preceq$, where $F \in \mathcal{F}, \psi \in \Psi$ and

$$\varphi(x, y, z) = \max \left\{ \begin{array}{l} p^*(x, y, z), p^*(x, x, Sx), \\ p^*(y, y, Sy), p^*(z, z, Sz). \end{array} \right\} \tag{3.7}$$

Further assume that if for every increasing sequence $\{x_n\}$ convergent to $x \in X$ we have $x_n \preceq x$.
Then S has a fixed point.

Example 7. *Let* $X = [0, \infty)$ *and* $p^*(x, y, z) = \max\{x, y, z\}$, *then* (X, p^*) *is a partial* D^**-metric space.*

Define self-map S *on* X *as* $Sx = \frac{x}{2}$, *and the relation* $\preceq$ *on* X *as follows:*

$$x \preceq y \iff x \geq y,$$

for any $x, y \in X$. *Then* $\preceq$ *is a (partial) order on* X *induced by* $\leq$. *Since* $Sx \geq S^2x$ *it follows that* $Sx \preceq S^2x$. *If define* $F(t) = t$ *and* $\psi(t) = kt$ *for* $0 < k < 1$ *then it is easy to see that*

$$p^*(Sx, Sy, Sz) \leq k\varphi(x, y, z),$$

for every x *in* X *and* $\frac{1}{2} \leq k < 1$. *Thus all conditions of Corollary 1 are satisfied and* $x = 0$ *is the unique fixed point of* S.

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