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Fixed Point for Completely Norm Space and Map T_α

S.A.M. Mohsenalhosseini and H. Mazaheri*

Abstract. We consider completely norm space X, and define $\varepsilon-$fixed point for T_α maps, and obtain some sufficient and necessary conditions on that, we also obtain some sufficient and necessary theorems on $\varepsilon-$fixed point for two maps.

1. Introduction

In 1976 Ishikawa obtained a surprising result, a special case of which may be stated as follows: Let K be an arbitrary bounded closed convex subset f a Banach space X, $T : K \to K$ nonexpansive, and $\alpha \in (0, 1)$. Set $T_\alpha = (1-\alpha)I + \alpha T$, then for each $x \in K$, $\|T^n_\alpha(x) - T^{n+1}_\alpha(x)\| \to 0$. In 1978, Edelstein and O'Brien [4] proved that $\{T^n_\alpha(x) - T^{n+1}_\alpha(x)\}$ converges to 0 uniformly for $x \in K$, in 1983 Goebel and Kirk [8] proved that this convergence is even uniform for $T \in \zeta$, where ζ denotes the collection of all nonexpansive self-mappings of K. Also, we obtain some result on T_α and reserch about fixed point and it.

In 1969 M. Furi and A. Vignoli [6, 7] obtained some result on Fixed point theorem in complete metric spaces also, they obtained result of Fixed point for densifying mappings, thereafter the concept of fixed point has been introduced and generalized in different ways by R. Nussbaum [12], S. Park [13], Wee-Tae. Park [14] and Zeqing Liu, Li Wang, Shin Min Kang, and Yong Soo Kim [11]. Also, we obtain some result on its.

2. T_α and $\varepsilon-$Fixed Point

Definition 2.1. Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map. Then $x_0 \in X$ is $\varepsilon-$fixed point for T if $\|Tx_0 - x_0\| < \varepsilon$.

Remark 2.2. In this paper we will denote the set of all $\varepsilon-$fixed points of T, for a given ε, by:

$$AF(T) = \{x \in X | x \text{ is an } \varepsilon \text{ fixed point of } T\}.$$

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*Corresponding author.

Theorem 2.3. *Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map, $x_0 \in X$ and $\varepsilon > 0$. If $\|T^n(x_0) - T^{n+k}(x_0)\| \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\varepsilon-$fixed point.*

Proof. Since $\|T^n(x_0) - T^{n+k}(x_0)\| \to 0$ as $n \to \infty$, $\varepsilon > 0$

$$\exists n_0 > 0 \quad s.t. \quad \forall n \geq n_0 \quad \|T^n(x_0) - T^{n+k}(x_0)\| < \varepsilon.$$

Then

$$\|T^{n_0}(x_0) - T^k(T^{n_0}(x_0)\| < \varepsilon,$$

therefore $T^{n_0}(x_0)$ is an $\varepsilon-$fixed point of T^k. □

Theorem 2.4. *Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map also for all $x, y \in X$,*

$$\|Tx - Ty\| \leq c\|x - y\|, \quad 0 < c < 1$$

then T has an $\varepsilon-$fixed point in completely norm space. Moreover, if $x, y \in X$ are $\varepsilon-$fixed points of T, then $\|x - y\| \leq \dfrac{2\varepsilon}{1-c}$.

Proof. Suppose $x \in X$, then

$$\begin{aligned}
\|(T^n(x) - T^{n+1}(x)\| &= \|T(T^{n-1}(x)) - T(T^n(x))\| \\
&\leq c\|T^{n-1}(x) - T^n(x)\| \\
&\ \ \vdots \\
&\leq c^{n-1}\|T(x) - T^2(x)\| \\
&\leq c^n\|x - Tx\|.
\end{aligned}$$

Therefore $\|(T^n(x) - T^{n+1}(x)\| \to 0$ as $n \to \infty$. From Theorem 2.3 T has an ε-fixed point, and Since

$$\|x - y\| \leq \|x - Tx\| + \|Tx - Ty\| + \|y - Ty\| \leq 2\varepsilon + c\|x - y\|.$$

Then $\|x - y\| \leq \frac{2\varepsilon}{1-c}$. □

Definition 2.5. Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$, and $T_\alpha : X \to X$ be a map as follow:

$$T_\alpha = \alpha I + (1 - \alpha)T, \quad 0 < \alpha < 1.$$

Then $x_0 \in X$ is $\varepsilon-$fixed point for T_α if $\|T_\alpha x_0 - x_0\| < \varepsilon$.

Remark 2.6. In this paper we will denote the set of all $\varepsilon-$fixed points of T_α, for a given ε, by:

$$AF(T_\alpha) = \{x \in X \mid x \text{ is an } \varepsilon - \text{fixed point of } T_\alpha\}.$$

Theorem 2.7. *Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map also for all $x, y \in X$,*

$$\|Tx - Ty\| \leq c\ \|x - y\|, \quad 0 < c < 1 \tag{1}$$

If $AF(T)$, the set of Approximate fixed point of T, is nonempty then the mapping

$$T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1.$$

satisfy in (1) and $AF(T) = AF(T_\alpha)$. Moreover $\|T_\alpha^n(x) - T_\alpha^{n+k}(x)\| \to 0$ as $n \to \infty$, for some $k > 0$, $\varepsilon > 0$.

Proof. By the definition of $AF(T)$, $AF(T) = AF(T_\alpha)$. Also, since T satisfy in (1) and I is identify function, it follows that T_α satisfy in (1). Now, we prove $\|T_\alpha^n(x_0) - T_\alpha^{n+k}(x_0)\| \to 0$ as $n \to \infty$. Suppose $x \in X$ now, observe first that $\|T_\alpha x - T_\alpha^2 x\| \leq c\|x - T_\alpha x|$ and, by induction, that $\|T_\alpha^n x - T_\alpha^{n+1} x\| \leq c^n\|x - T_\alpha x\|$. Thus, for any n and any $k > 0$, we have

$$\begin{aligned}
\|T_\alpha^n(x) - T_\alpha^{n+k}(x)\| &\leq \sum_{i=n}^{n+k-1} \|T_\alpha^i(x) - T_\alpha^{i+1}(x)\| \\
&\leq (c^n + \cdots + c^{n+k-1})\|x - T_\alpha(x)\| \\
&\leq \frac{c^n}{1-c}\|x - T_\alpha x\|.
\end{aligned}$$

Since $c < 1$, so that $c^n \to 0$, hence $\|T_\alpha^n(x_0) - T_\alpha^{n+k}(x_0)\| \to 0$ as $n \to \infty$. □

Theorem 2.8. *Let T be a mapping of a completely norm space (X, p) into itself such that $\|Tx - Ty\| \leq \beta(\|x - Tx\| + \|y - Ty\|)$ where $2\beta < 1$.*

If x_0 is an ε-fixed point for T, Then Tx_0 is an ε-fixed point for T^2.

Proof. We have

$$\|(Tx - T^2x\| \leq \beta(\|x - Tx\| + \|Tx - T^2x\|)$$

therefore

$$\|Tx - T^2x\| \leq \frac{\beta}{1-\beta}\|x - Tx\|.$$

since $2\beta < 1$

$$\|Tx - T^2x\| \leq \|x - Tx\|.$$

Since x_0 is an ε-fixed point for T, then $\|Tx_0 - T^2x_0\| \leq \varepsilon$, so Tx_0 is an ε-fixed point for T^2. □

Theorem 2.9. *Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map also for all $x, y \in X$, $\|T_\alpha x - T_\alpha y\| \leq \beta(\|x - T_\alpha x\| + \|y - T_\alpha y\|)$ where $2\beta < 1$.*

If x_0 is an ε-fixed point for T, then $T_\alpha x_0$ is an ε-fixed point for T_α^2.

Proof. We have

$$\|(T_\alpha x - T_\alpha^2 x\| \leq \beta\big(\|x - T_\alpha x\| + \|T_\alpha x - T_\alpha^2 x\|\big),$$

therefore

$$\|T_\alpha x - T_\alpha^2 x\| \leq \frac{\beta}{1-\beta}\|x - T_\alpha x\|.$$

Since $2\beta < 1$, then

$$\|T_\alpha x - T_\alpha^2 x\| \leq \|x - T_\alpha x\|.$$

Since x_0 is an ε-fixed point for T, then by Theorem 2.7 x_0 is an ε-fixed point for T_α, hence $\|T_\alpha x_0 - T_\alpha^2 x_0\| \leq \varepsilon$.

Therefore $T_\alpha x_0$ is an ε-fixed point for T_α^2. □

Theorem 2.10. *Let $(X, \|.\|)$ be a completely norm space, $T : X \to X$ be a mapping and $\varepsilon > 0$. If $\|Tx - Ty\| \leq \alpha\|x - Tx\| + \beta\|y - Ty\|$ and $\alpha + \beta < 1$, then T has ε-fixed point. Moreover, if $x, y \in X$ are $\varepsilon-$fixed points of T, then $\|x - y\| \leq (2 + \alpha + \beta)\varepsilon$.*

Proof. We have

$$\|Tx - T^2 x\| \leq \alpha\|x - Tx\| + \beta\|Tx - T^2 x\|.$$

Therefore

$$\|Tx - T^2 x\| \leq \frac{\alpha}{1-\beta}\|x - Tx\|,$$

also

$$\|T^2 x - T^3 x\| \leq \alpha\|Tx - T^2 x\| + \beta\|T^2 x - T^3 x\|,$$

so

$$\|T^2 x - T^3 x\| \leq (\frac{\alpha}{1-\beta})^2\|Tx - T^2 x\|,$$

and for every $n \geq 1$, we have

$$\|T^n x - T^{n+1} x\| \leq (\frac{\alpha}{1-\beta})^n\|x - Tx\| \text{ and } \frac{\alpha}{1-\beta} < 1.$$

Thus since $\frac{\alpha}{1-\beta} < 1$, $\|T^n x - T^{n+1} x\| \to 0$ as $n \to \infty$. Now by Theorem 2.3 T has an ε-fixed point and since

$$\|x-y\| \leq \|x-Tx\|+\|Tx-Ty\|+\|y-Ty\| \leq (1+\alpha)\|x-Tx\|+(1+\beta)\|y-Ty\|.$$

Then $\|x - y\| \leq (2 + \alpha + \beta)\varepsilon$. □

Theorem 2.11. *Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map and $\varepsilon > 0$, also for all $x, y \in X$, $\|T_\alpha x - T_\alpha y\| \leq \beta\|x - T_\alpha x\| + \gamma\|y - T_\alpha y\|$ and $\beta + \gamma < 1$. If T has an ε-fixed point, then T_α has ε-fixed point. Moreover, if $x, y \in X$ are $\varepsilon-$fixed points of T_α, then $\|x - y\| \leq (2 + \beta + \gamma)\varepsilon$.*

Proof. We have

$$\|T_\alpha x - T_\alpha^2 x\| \leq \alpha\|x - T_\alpha x\| + \beta\|T_\alpha x - T_\alpha^2 x\|.$$

Therefore

$$\|T_\alpha x - T_\alpha^2 x\| \leq \frac{\alpha}{1-\beta}\|x - T_\alpha x\|,$$

also

$$\|T_\alpha^2 x - T_\alpha^3 x\| \leq \alpha\|T_\alpha x - T_\alpha^2 x\| + \beta\|T_\alpha^2 x - T_\alpha^3 x\|,$$

so

$$\|T_\alpha^2 x - T\alpha^3 x\| \leq \left(\frac{\alpha}{1-\beta}\right)^2 \|T_\alpha x - T_\alpha^2 x\|,$$

and for every $n \geq 1$, we have

$$\|T\alpha^n x - T\alpha^{n+1} x\| \leq \left(\frac{\beta}{1-\gamma}\right)^n \|x - T_\alpha x\|.$$

Thus, since $\frac{\beta}{1-\gamma} < 1$, $\|T_\alpha^n x - T_\alpha^{n+1} x\| \to 0$ as $n \to \infty$. Now by Theorem 2.7 and Theorem 2.3, T_α has an ε-fixed point and since

$$\begin{aligned}\|x - y\| &\leq \|x - T_\alpha x\| + \|T_\alpha x - T_\alpha y\| + \|y - T_\alpha y\| \\ &\leq (1+\beta)\|x - T_\alpha x\| + (1+\gamma)\|y - T_\alpha y\|,\end{aligned}$$

then $\|x - y\| \leq (2 + \beta + \gamma)\varepsilon$. □

Corollary 2.12. *Let $(X, \|.\|)$ be a norm, $T : X \to X$ be a mapping and $\varepsilon > 0$. If $\|Tx - Ty\| \leq \beta(\|x - Tx + \|y - Ty)$ and $2\beta < 1$, then T has an ε-fixed point.*

Corollary 2.13. *Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map also for all $x, y \in X$, $\|T_\alpha x - T_\alpha y\| \leq \beta(\|x - T_\alpha x\| + \|y - T_\alpha y\|)$ where $2\beta < 1$, then T_α has an ε-fixed point.*

Definition 2.14. Let $T : X \to X$, be a map and $\varepsilon > 0$. We define diameter $AF(T)$ by

$$\operatorname{diam}(AF(T)) = \sup\{\|x - y\|, \quad x, y \in AF(T)\}.$$

Theorem 2.15. *Let $T : X \to X$, and $\varepsilon > 0$. If there exists a $c \in [0, 1]$ such that for all $x, y \in X$*

$$\|Tx - Ty\| \leq c\|x - y\|.$$

Then

$$\operatorname{diam}(AF(T) \leq \frac{2\varepsilon}{1-c}.$$

Proof. If $x, y \in AF(T)$, then

$$\begin{aligned}\|x - y\| &\leq \|x - Tx\| + \|Tx - Ty\| + \|Ty - y\| \\ &\leq \varepsilon_1 + c\|x - y\| + \varepsilon_2.\end{aligned}$$

put $\varepsilon = Max\{\varepsilon_1, \varepsilon_2\}$, therefore $\|x - y\| \leq \frac{2\varepsilon}{1-c}$.

Hence $\operatorname{diam}(AF(T)) \leq \frac{2\varepsilon}{1-c}$. □

Theorem 2.16. *Let $T : X \to X$, and $\varepsilon > 0$. If there exists a $c \in [0,1]$ such that for all $x, y \in X$ $\|Tx - Ty\| \leq c\|x - y\|$, and $T_\alpha : X \to X$ be a map as follow:*

$$T_\alpha = \alpha I + (1-\alpha)T, \quad 0 < \alpha < 1.$$

Then

$$\operatorname{diam}(AF(T_\alpha) \leq \frac{2\varepsilon}{1-c}.$$

Proof. If $x, y \in AF(T_\alpha)$, then

$$\begin{aligned}\|x - y\| &\leq \|x - T_\alpha x\| + \|T_\alpha x - T_\alpha y\| + \|T_\alpha y - y\| \\ &\leq \varepsilon_1 + c\|x - y\| + \varepsilon_2.\end{aligned}$$

put $\varepsilon = Max\{\varepsilon_1, \varepsilon_2\}$, therefore $\|x - y\| \leq \frac{2\varepsilon}{1-c}$.

Hence $\operatorname{diam}(AF(T_\alpha)) \leq \frac{2\varepsilon}{1-c}$. □

3. $\varepsilon-$FIXED POINT FOR p-SET CONSTRUCTION

Definition 3.1. Let $(X, \|.\|)$ be a completely norm space and D be a subset of X. Define the measure of noncompactness $\beta(D)$ of D by:

$$\beta(D) = inf\{\delta > 0, \quad D \text{ admits a finite covering of subsets of diameter} \leq \delta\}.$$

Definition 3.2. Let $(X, \|.\|)$ be a completely norm space and $T : C \to C$ be a continuous mapping for $C \subset X$, then T is called a $\mathbf{p}-$set contraction if, for all $A \subset C$ with A bounded, $T(A)$ is bounded and $\beta(TA) \leq p\beta(A)$, $0 < p < 1$. If

$$\beta(TA) < \beta(A), \quad \text{for all } \beta(A) > 0,$$

then T is called densifying (or condensing).

Definition 3.3. Let A, B be a closed bounded convex subset of a completely norm space X and $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two maps, such that $T(A) \subseteq B$, $S(B) \subseteq A$. A point (x, y) in $A \times B$ is said to be a pair proximity point for (T, S), if

$$\|Tx - Sy\| = 0.$$

We say (T, S) has the pair proximity property if $P_{(T,S)}(A, B) \neq \varnothing$, where

$$P_{(T,S)}(A, B) = \{(x, y) \in A \times B, \quad \|Tx - Sy\| = 0\}.$$

Theorem 3.4 ([13])**.** *Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a completely continuous compact mapping of X. Then T is a $k-$set contraction.*

Theorem 3.5 ([13])**.** *Let $(X, \|.\|)$ be a completely norm space and $C_1, C_2, \ldots$ a decreasing sequence of nonempty closed subsets of X. Assume that $\beta(C_n) \to 0$ as $N \to \infty$. Then $C_\infty = \cap_{n\geq 1} C_n$ is a nonempty compact set, and approaches C_∞ in the Hausdorff metric.*

Theorem 3.6 ([13]). *Let A be a closed bounded convex subset of a completely norm space X. Let $T : A \to A$ a continuous map and $A_1 = \overline{\text{co}}(T(A))$ and $A_n = \overline{\text{co}}(T(A_{n-1}))$, for $n > 1$. Further, assume $\beta(A_n) \to \infty$ as $n \to \infty$. Then $F(T) \neq \varnothing$ that is, T has at least one fixed point.*

Theorem 3.7. *Let A, B be a closed bounded convex subset of a completely norm space X. Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two continuous maps such that $T(A) \subseteq B$ and $S(B) \subseteq A$.*

Also, we let $A_1 \cup B_1 = \overline{\text{co}}(T(A \cup B))$, and $A_1 \cup B_1 = \overline{\text{co}}(S(A \cup B))$ and $A_n \cup B_n = \overline{\text{co}}(T(A_{n-1} \cup B_{n-1}))$ and $A_n \cup B_n = \overline{\text{co}}(S(A_{n-1} \cup B_{n-1}))$ for $n > 1$. Further, assume $\beta(A_n \cup B_n) \to \infty$ as $n \to \infty$. Then $F(T, S) \neq \varnothing$ that is, (T, S) has a pair proximity property.

Proof. Clearly A_n, B_n is closed bounded convex and nonempty with $A_n \cup B_n \supset A_{n+1} \cup B_{n+1}$ for $n \geq 1$. Then, by the Theorem 3.6, $A_\infty \cup B_\infty = \cap_{n\geq 1} A_n \cup B_n$ is nonempty and compact, also, it is convex. By our construction, $T : A_n \cup B_n \to A_{n+1} \cup B_{n+1}$ and $S : A_n \cup B_n \to A_{n+1} \cup B_{n+1}$ so that $T : A_\infty \cup B_\infty \to A_\infty \cup B_\infty$, $S : A_\infty \cup B_\infty \to A_\infty \cup B_\infty$. Hence, by the Schauder fixed point theorem, (T, S) has a pair proximity property. □

Theorem 3.8. *Let A be a closed bounded convex subset of a completely norm space X, and $T : A \to A$ a $k-$set contraction. Then T has a fixed point.*

Proof. Let $A_n = \overline{\text{co}}(T(A_{n-1}))$, now show that T has a fixed point, it is sufficient to show that $\beta(A_n) \to 0$ as $n \to \infty$,

$$\begin{aligned}
\beta(A_n) = \beta(\overline{\text{co}}(T(A_{n-1})) &= \beta(T(A_{n-1})) \\
&\leq k\beta(A_{n-1}) \\
&\ \vdots \\
&\leq k^{n-1}\beta(A_1) \\
&\leq k^n\beta(A).
\end{aligned}$$

Since $0 < k < 1$, $\beta(A_n) \leq k^n\beta(A) \to 0$ as $n \to \infty$. Thus, T has a fixed point. □

Theorem 3.9. *Let A, B be a closed bounded convex subset of a completely norm space X, and $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two $k-$set contraction such that $T(A) \subseteq B$ and $S(B) \subseteq A$, then $F(T, S) \neq \varnothing$ that is, (T, S) has a pair proximity property.*

Proof. Let $A_n = \overline{\text{co}}(T(A_{n-1}))$, $B_n = \overline{\text{co}}(T(B_{n-1}))$ now show that $F(T,S) \neq \varnothing$, it is sufficient to show that $\beta(A_n \cup B_n) \to 0$ as $n \to \infty$,

$$\begin{aligned}\beta(A_n \cup B_n) = \beta(\overline{\text{co}}(T(A_{n-1} \cup B_{n-1})) &= \beta(T(A_{n-1} \cup B_{n-1}))\\ &\leq k\beta(A_{n-1} \cup B_{n-1})\\ &\ \ \vdots\\ &\leq k^{n-1}\beta(A_1 \cup B_1)\\ &\leq k^n\beta(A \cup B).\end{aligned}$$

Since $0 < k < 1$, $\beta(A_n \cup B_n) \leq k^n\beta(A \cup B) \to 0$ as $n \to \infty$. Hence, $F(T,S) \neq \varnothing$ that is, (T,S) has a pair proximity property. □

Theorem 3.10. *Let A, B be a closed bounded convex subset of a completely norm space X, and $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two $1-$set contraction such that $T(A) \subseteq B$ and $S(B) \subseteq A$, and For every $(x,y) \in A \times B$*

$$\|T(x) - S(y)\| \leq p(\|x - T(x)\| + \|y - S(y)\|)$$

where $p \geq 0$ and $2p < 1$ Then

$$P_{(T,S)}(A,B) \neq \varnothing.$$

Proof. Suppose $x_0, y_0 \in A \cup B$. Define $T_\alpha : A \cup B \to A \cup B$ by

$$T_\alpha(x) = \alpha Tx + (1-\alpha)x_0,\ 0 \leq \alpha < 1,$$

and $S_\alpha : A \cup B \to A \cup B$ by

$$S_\alpha(y) = \alpha Sy + (1-\alpha)y_0,\ 0 \leq \alpha < 1.$$

The maps T_α, S_α are $\alpha-$set contraction for $0 \leq \alpha < 1$. Indeed, if $C \subset A \cup B$ then $T_\alpha(C) = \alpha T(C) + (1-\alpha)x_0$, $S_\alpha(C) = \alpha S(C) + (1-\alpha)x_0 : x_0 \in A \cup B$. Hence,

$$\begin{aligned}\beta(T_\alpha(C)) &= \alpha T(C) + \beta(1-\alpha)x_0\\ &\leq \alpha\beta(T(C)) + (1-\alpha)\beta(x_0)\\ &= \alpha\beta(T(C)),\end{aligned}$$

and

$$\begin{aligned}\beta(S_\alpha(C)) &= \alpha S(C) + \beta(1-\alpha)x_0\\ &\leq \alpha\beta(S(C)) + (1-\alpha)\beta(x_0)\\ &= \alpha\beta(S(C)).\end{aligned}$$

Therefore, by result of Darbo [3], T_α has at least one fixed point $x_\alpha \in A \cup B$ and S_α has at least one fixed point $y_\alpha \in A \cup B$ for any $0 \leq \alpha < 1$. Furthermore, $T_\alpha(x)$ converges to $T(x)$ and $S_\alpha(x)$ converges to $S(x)$ uniformly on $A \cup B$ as $\alpha \to 1$. And but since $\|x_\alpha - T(x_\alpha)\| = \|T_\alpha(x_\alpha) - T(x_\alpha)\|$, $\|y_\alpha - S(x_\alpha)\| = \|S_\alpha(x_\alpha) - S(x_\alpha)\|$. Therefore, $\|x_\alpha - T(x_\alpha)\| \to 0$, $\|y_\alpha - S(y_\alpha)\| \to 0$ as $\alpha \to 1$.

Now, suppose $(x,y) \in A \times B$,

$$\|T(x_\alpha) - S(T(x_\alpha))\| \leq p(\|x_\alpha - T(x_\alpha)\| + \|T(x_\alpha) - S(T(x_\alpha))\|,$$
$$\|T(S(y_\alpha)) - S(y_\alpha)\| \leq p(\|S(y_\alpha) - T(Sy(_\alpha))\| + \|y_\alpha - S(y_\alpha)\|).$$

Therefore,

$$\|T(x_\alpha) - S(T(x_\alpha))\| \leq \frac{p}{1-p}\|x_\alpha - T(x_\alpha)\| \leq \|x_\alpha - T(x_\alpha)\|,$$
$$\|T(S(y_\alpha)) - S(y_\alpha)\| \leq \frac{p}{1-p}\|y_\alpha - S(y_\alpha)\| \leq \|y_\alpha - S(y_\alpha)\|.$$

Since x_α is a fixed point for T_α, or y_α is a fixed point for S_α then

$$P_{(T,S)}(A,B) \neq \varnothing.$$

□

Theorem 3.11. *Let $(X, \|.\|)$ be a strictly convex complete norm space, C be a subset of X, and $T : C \to C$ be a densifying mapping and satisfy in*

$$\|Tx - Ty\| \leq c\ \|x - y\|, \quad 0 < c < 1. \tag{2}$$

Let

$$T_\alpha x = \alpha x + (1-\alpha)Tx,\ 0 < \alpha < 1.$$

Then, for each $x_n \in C$, the sequence $x_{n+1} = \alpha x_n + (1-\alpha)Tx_n$, $n = 0, 1, 2, \ldots$ converges strongly to a approximate fixed point of T in C.

Proof. The set of approximate fixed points of T, $AF(T) \neq \varnothing$, and by Theorem 2.7 $AF(T) = AF(T_\alpha)$. Also, since T is densifying and satisfy in (1), and $0 < c < 1$, $T_\alpha : C \to C$ is also densifying and satisfy in (1).

Since T_α satisfy in (1) and X is strictly convex, we have that

$$\|T_\alpha x_0 - y\| < c\|x_0 - y\|, \quad 0 < c < 1 \text{ for } y \in AF(T) \text{ and } x_0 \in C \setminus AF(T).$$

In order to show that $\{x_{n+1}\}$ convergence strongly to a point in $AF(T)$, it is enough to show that $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$ and that its $\lim = x = \lim x_n$ lies in $AF(T)$.

Now, for each $x_0 \in C$ the sequence $A_0 = \{T_\alpha^n(x_0),\ n = 0, 1, 2, \ldots\}$ is bounded and is transformed into $A_1 = \{T_\alpha^n(x_0),\ n = 0, 1, 2, \ldots\}$. Hence, $\beta(A_0) = \beta(A_1)$, and, therefore, $\beta(A_0) = 0$ since T is densifying map.

Thus, $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$. If we put $z = \lim x_{n_i}$, then it follows that $z \in AF(T)$. □

Theorem 3.12. *Let X be a strictly convex linear space and $T : X \to X$ be a map, Then T has a unique fixed point.*

Proof. Suppose, there exists two fixed point x, y of T, such that $x \neq y$ and $z = \lambda x + (1-\lambda)y$, $0 < \lambda < 1$. Then by strict convexity of X we have that

$$Tx - Ty = Tx - Tz + Tz - Ty < Tx - Ty,$$

which is a contradiction. hence, T has a unique fixed point. □

Theorem 3.13. *Let $T : X \to X$ be a noneexpansive mapping and $\varepsilon > 0$ and*

(3) $$\|T^n(x_0) - T^{n+k}(x_0)\| \to 0 \text{ as } n \to \infty \text{ for some } k > 0.$$

Also let T satisfy the following condition:

(4) *$(I - T)$ maps bounded closed sets into closed sets.*

Then, for $x_0 \in X$ there exists a $\varepsilon-$fixed point that $T^n(x_0)$ converges to it.

Proof. Since $\|T^n(x_0) - T^{n+k}(x_0)\| \to 0$ as $n \to \infty$ for some $k > 0$, $\varepsilon > 0$ then by Theorem 2.3 T^k has an $\varepsilon-$fixed point, now if y be $\varepsilon-$fixed point T^k then

$$\|T^{n+k}(x_0) - y\| \leq \|T^n(x_0) - y\|.$$

So the sequence $\{T^n x_0\}$ is bounded. Let P be the closure of $\{T^n x_0\}$. By condition (4), it follows that $(I - T)P$ is closed. This together with the fact that T satisfy in (2), gives $0 \in (I - T)P$. So there exists a $x_1 \in P$ such that $(I - T)x_1 = 0$, that is $x_1 = Tx_1$.

But this implies that either $x_1 = T^n x_0$ for some n, or there exists a subsequence $\{T^n x_0\}$ converging to x_1. Since x_1 is a approximate fixed point of T, we can conclude that, in either case, the sequence $\{T^n x_0\}$ converges to x_1. □

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S.A.M. Mohsenalhosseini
Faculty of Mathematics
Valiasr Rafsanjan University
Rafsanjan
Iran
E-mail address: amah@vru.ac.ir

H. Mazaheri
Faculty of Mathematics
Yazd University
Yazd
Iran
E-mail address: hmazaheri@yazduni.ac.ir