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A Question of Priority Regarding a Fixed Point Theorem in a Cartesian Product of Metric Spaces

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Abstract. We prove that a result of Ćirić and Prešić [Acta Math. Univ. Comenianae, **76** (2007), 143-147, Theorem 2, p. 144] has been for the first time proved before 31 years in Tasković [Publ. Inst. Math., **34** (1976), 231-242, Theorem 3, p. 238]. But the authors neglected and ignored this historical fact.

1. Main results and facts

We say that the mapping $f : (\mathbb{R}^0_+)^k \to \mathbb{R}^0_+ := [0, +\infty)$ (for a fixed $k \in \mathbb{N}$) has the M**-property** iff f is increasing (i.e., $u_i \leq v_i$ for $i = 1, \dots, k$ implies that $f(u_1, \dots, u_k) \leq f(v_1, \dots, v_k)$), semihomogeneous (that is to say, $f(\delta x_1, \dots, \delta x_k) \leq \delta f(x_1, \dots, x_k)$ for every $\delta \geq 0$), and $g(x) := f(\alpha_1 x, \dots, \alpha_k x^k)$ be continuous at the point $x = 1$, where α_i ($i = 1, \dots, k$) are nonnegative real constants.

In 1976 in Tasković [3] we have proved the following localization theorem on a Cartesian product of metric spaces as a solution of Kuratowski's problem in 1932, see: Brown [1].

Theorem 1 (Tasković [3, p. 238]). *Let $X := (X, \rho)$ be a complete metric space and let T be a mapping of X^k (for a given fixed $k \in \mathbb{N}$) into X satisfying the following condition:*

$$\text{(A)}\quad \rho\Big[T(u_1, \dots, u_k), T(u_2, \dots, u_{k+1})\Big] \leq f\Big(\alpha_1 \rho[u_1, u_2], \dots, \alpha_k \rho[u_k, u_{k+1}]\Big)$$

for all $u_1, \dots, u_k, u_{k+1} \in X$, where the mapping $f : (\mathbb{R}^0_+)^k \to \mathbb{R}^0_+$ has the M-property and $f(\alpha_1, \dots, \alpha_k) \in [0, 1]$. Then:

(a) *There exists a fixed point $\zeta \in X$ of the mapping $\mathfrak{F}(x) := T(x, \dots, x)$ and it is unique when $f(\alpha_1, 0, \dots, 0) + \dots + f(0, \dots, 0, \alpha_k) < 1$;*

(b) *The point $\zeta \in X$ is the limit of the sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfying*

$$x_{n+k} = T(x_n, \dots, x_{n+k-1}), \quad \text{for } n \in \mathbb{N}, \tag{1}$$

independently of initial values $x_1, \dots, x_k \in X$.

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(c) *The rapidity of convergence of the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *to the point* $\zeta \in X$ *is evaluated for* $n \in \mathbb{N}$ *by*

$$\rho[x_{n+k}, \zeta] \leq \frac{\theta^n}{1-\theta} \max_{i=1,\dots,k} \left(\frac{\rho[x_i, x_{i+1}]}{\theta^i} \right) \quad \textit{for } \theta \in (0,1).$$

First proof of Theorem 1 may be found in 1976 by Tasković [3, p. 238-239]. Other proofs may be found by Tasković [4], [5], and [6]. Also see: [7].

Recently, in 2007 Ćirić and Prešić have proved the following statement (see: [2, Theorem 2, p. 144]).

Theorem 2. *Let* (X, ρ) *be a complete metric space and* $T : X^k \to X$ $(k \in \mathbb{N}$ *is a fixed number) satisfying the following contractive type condition*

$$(2) \quad \rho\Big[T(u_1, \dots, u_k), T(u_2, \dots, u_{k+1})\Big] \leq \lambda \max\Big\{\rho[u_1, u_2], \dots, \rho[u_k, u_{k+1}]\Big\}$$

for all $u_1, \dots, u_k, u_{k+1} \in X$*, where the constant* $\lambda \in (0,1)$*. Then there exists a point* $\zeta \in X$ *such that* $T(\zeta, \dots, \zeta) = \zeta$*. Moreover, if* $x_1, \dots x_k \in X$ *are arbitrary point in* X *and* $n \in \mathbb{N}$*, the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by (1) is convergent.*

If in addition we suppose that $\rho[T(u, \dots, u), T(v, \dots, v)] < \rho[u, v]$ *for all* $u, v \in X$ $(u \neq v)$*, then* ζ *is the unique point in* X *such that* $T(\zeta, \dots, \zeta) = \zeta$*.*

However, Theorem 2 is a simple consequence of Theorem 1 which we proved first time 31 years ago in: Tasković [3].

Indeed, if in Theorem 1 we let $f(t_1, \dots, t_5) = \max\{t_1, \dots, t_5\}$ for $\max\{\alpha_1, \dots, \alpha_5\} := \lambda \in [0,1)$, then the condition (A) and other conditions are satisfied.

Hence, we obtain Theorem 2, as a directly consequence of my Theorem 1.

Remark. We notice that Theorem 2 is an example (Problem **72** on 77 page) in the book by M. R. Tasković/ D. Aranđelović: *Functional Analysis and Functions Theory* – Theorems, tasks and problems, NIRO "Književne novine", Beograd 1981, 255 pages.

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