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On a Statement by I. Aranđelović for Asymptotic Contractions in Appl. Anal. Discrete Math.

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Abstract. We prove that main result of asymptotic contractions by I. Aranđelović [Appl. Anal. Disrete Math. **1** (2007), 211-216, Theorem 1, p. 212] has been for the first time proved 21 years ago in Tasković [Fundamental elements of the fixed point theory, ZUNS-1986, Theorem 4, p. 170]. But, the author (and next other authors) this historical fact is to neglect and to ignore.

1. Main results and facts

Let X be a topological space, $T : X \to X$, and let $A : X \times X \to \mathbb{R}^0_+$. In 1986 Tasković [6] investigated the concept of TCS-convergence in a space X, i.e., a topological space $X := (X, A)$ satisfies the **condition of TCS-convergence** iff $x \in X$ and if $A(T^n x, T^{n+1} x) \to 0$ $(n \to \infty)$ implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ has a convergent subsequence.

The following results, given in the next two theorems are given in 1986 by M. R. Tasković [6] as a natural extension of characterization statements of asymptotically conditions of fixed point theorem given in 1985 by Tasković [7]. These results are according to topological spaces.

Theorem 1 (Tasković [6]). *Let T be a mapping of topological space $X := (X, A)$ into itself, where X satisfies the condition of TCS-convergence. Suppose that there exist a sequence of nonnegative real functions $\{\alpha_n(x, y)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x, y) \to 0$ $(n \to \infty)$ and positive integer $m(x, y)$ such that*

$$(\mathrm{B}) \qquad A\Big(T^n(x), T^n(y)\Big) \le \alpha_n(x, y) \quad \textit{for all} \quad n \ge m(x, y),$$

and for all $x, y \in X$, where $A : X \times X \to \mathbb{R}^0_+$. If $x \mapsto A(x, T(x))$ is a T-orbitally lower semicontinuous function and $A(a, b) = 0$ implies $a = b$,

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then T has a unique fixed point $\xi \in X$ and $T^n(x) \to \xi$ $(n \to \infty)$ for each $x \in X$.

First proof of Theoprem 1 may be found in 1986 year by Tasković [6, p. 170]. Second proof of Theorem 1 may found in Tasković [8, p. 456] and [9, p. 50].

In connection with the preceding, the set $\mathcal{O}(x,\infty) := \{x, Tx, T^2x, \ldots\}$ for $x \in X$ is called the **orbit** of x. A function f mapping X into reals is a **f-orbitally lower semicontinuous** at the point p iff for all sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to p$ $(n \to \infty)$ it follows that $f(p) \le \liminf_{n\to\infty} f(x_n)$. A mapping $T : X \to X$ is said to be orbitally continuous if ξ, $x \in X$ are such that ξ is a cluster point of $\mathcal{O}(x,\infty)$, then $T(\xi)$ is a cluster point of $T(\mathcal{O}(x,\infty))$.

Note that, from the preceding facts of Theorem 1, we can give the following local form of this statement.

Theorem 2 (Localization of (B), Tasković [6]). *Let T be a mapping of topological space $X := (X, A)$ into itself, where X satisfies the condition of TCS-convergence. Suppose that there exist a sequence of nonnegative real functions $\{\alpha_n(x,y)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x,Tx) \to 0$ $(n \to \infty)$ and positive integer $m(x)$ such that*

$$A\Big(T^n(x), T^{n+1}(x)\Big) \le \alpha_n(x,Tx) \text{ for all } n \ge m(x),$$

and for every $x \in X$, where $A : X \times X \to \mathbb{R}^0_+$. If $x \mapsto A(x,Tx)$ is a T-orbitally lower semicontinuous function and $A(a,b) = 0$ implies $a = b$, then T has at least one fixed point in X.

The proof of this statement is an analogous with the former proof of Theorem 1. A brief broof of this statement may be found in Tasković [6]. Also, for other proofs see: Tasković [8, p. 457] and [9, p. 51].

2. QONSEQUNCES AND FURTHER FACTS

In recent years a great number of papers have appeared presenting a various generalizations of the well known Banach-Picard contraction principle (via linear and nonlinear conditions). The following result is a statement with nonlinear conditions given in 2007 by I. Aranđdelović.

Theorem 3 (I. Aranđelović [1]). *Let (X,ρ) be a complete metric space, $T : X \to X$ continuous function, and $\{\varphi_n\}_{n\in\mathbb{N}}$ sequence of functions such that $\varphi_n : \mathbb{R}^0_+ \to \mathbb{R}^0_+ := [0,+\infty)$ and*

$$\rho[T^n(x), T^n(y)] \le \varphi_n\big(\rho[x,y]\big) \text{ for all } x, y \in X,$$

and for every $n \in \mathbb{N}$. Assume also that there exists upper semicontinuous function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ such that for any $r > 0$, $\varphi(r) < r$, $\varphi(0) = 0$ and $\varphi_n \to \varphi$ $(n \to \infty)$ uniformly of the range of ρ. If there exists $x \in X$ such that *orbit of* T *at* x *is bounded or* $\liminf_{t\to\infty}(t-\varphi(t)) > 0$ *or* $\limsup_{t\to\infty}\varphi(t)/t < 1$, *then* T *has a unique fixed point* $\xi \in X$ *and all sequences of Picard iterates defined via* T *converges to* ξ.

Main Annotation: *The Theorem 3 is a consequence of Theorem 1.* (In this sense in next we give the following proof of this essential fact).

Proof of Theorem 3. (*Application of Theorem 1*). Suppose that all the conditions of Theorem 3 are satisfied. We prove that all conditions of Theorem 1 are satisfied, too. Since $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ is a continuous function such that $\varphi(t) < t$ for every $t > 0$ and $\varphi(0) = 0$, from Wong's lemma ([10], Lemma 4, p. 201]) it follows that there exists nondecreasing continuous function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ such that $\varphi(t) < \psi(t) < t$ for every $t > 0$ and $\psi(0) = 0$. Let us define $A : X \times X \to \mathbb{R}^0_+$ by $A(a, b) = \psi(\rho[a, b])$, and define a sequence of functions $\{\alpha_n(a, b)\}_{n\in\mathbb{N}}$ by $\alpha_n(a, b) = \rho[T^n(a), T^n(b)]$ for any $a, b \in X$. Since $\psi(t) < t$ we get that

$$A\Big(T^n(x), T^n(y)\Big) = \psi\Big(\rho[T^n(x), T^n(y)]\Big) < \rho[T^n(x), T^n(y)] = \alpha_n(x, y)$$

this is that the condition (B) is satisfied. Since $\psi(t) = 0$ implies $t = 0$, from $A(a, b) = \psi(\rho[a, b]) = 0$ it follows that $\rho[a, b] = 0$, i.e., $a = b$. From the proof given by I. Aranđelović [1] it follows that $\rho[T^n(x), T^n(y)] \to 0$ $(n \to \infty)$ for all $x, y \in X$. Consequently, $\alpha_n(x, y) \to 0$ $(n \to \infty)$. Since T and ψ are continuous mappings the function $x \mapsto A(x, Tx) := \psi(\rho[x, Tx])$ is a T-orbitally lower semicontinouos. Since X is a complete metric space it satisfies the condition of TCS-convergence. Applying Theorem 1 we obtain that T has a unique fixed point $\xi \in X$ and all sequences of Picard iterates converge to ξ. The proof is complete. □

Remark 1. We notice that a form of Wong's lemma may be found in Tasković [8, p. 282].

Further, applying the Theorem 1 we get an asymptotic version of a statement due to Ivanov [3]. This is the following result which is an extension of Kirk's theorem on asymptotic contractions.

Theorem 4. *Let* (X, ρ) *be a complete metric space,* $T : X \to X$ *a continuous function, and* $\varphi_n : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ *for* $n \in \mathbb{N}$ *a sequence such that for all* $n \in \mathbb{N}$ *satisfy*

$$\rho[T^n(x), T^n(y)] \le$$
$$\le \max\{\varphi_n(\rho[x, y]), \varphi_n(\rho[x, Tx]), \varphi_n(\rho[y, Ty]), \varphi_n(\rho[x, Ty]), \varphi_n(\rho[y, Tx])\}$$

for all $x, y \in X$; *and assume also that there exists a function* $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ *such that for any* $t > 0$, $\varphi(t) < t$, $\varphi(0) = 0$ *and* $\varphi_n \to \varphi$ $(n \to \infty)$ *uniformly of the range of* ρ. *If there exists* $x \in X$ *such that orbit of* T *at* x *is bounded, then* T *has a unique fixed point* $\xi \in X$ *and all sequences of Picard iterates defined by* T *converges to* ξ.

We notice that a proof of Theorem 4 may be found in 1986 year by Tasković [6, p. 171]. Also, a proof of Theorem 4 may be found by Tasković [9, p. 52].

Main Annotations. We notice also that main result of asymptotic contractions by W. Kirk [J. Math. Anal. Appl. **277** (2003), 645-650, Theorem 2.1. p. 647] has been for the first time proved 17 years ago in Tasković [6, p. 170]. For this fact see: Tasković [9, p. 51].

In this sense a result by J. Jachymski and I. Jóźwik [4] for asymptotic contractions has been for the first time proved 18 years ago in Tasković [6, p. 170] and also a result by Y.-Z. Chen [2] has been for the first time proved 19 years ago in Tasković [6, p. 170].

Further facts. On the other hand, in connection with the statement by I. Aranđelović, we notice that Theorem 3 is, de facto, only an interesting example of the main statement by Tasković [7, Theorem 1, p. 48] or only an example of the second statement by Tasković [7, Theorem 2, p. 49], on characterizations of the class of contraction type mappings. Also see: Tasković [8, p. 454].

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