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A Note on the Zeros of One Form of Composite Polynomials

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Abstract. In this paper we consider one form of composite polynomials. Several relations concerning their zeros are obtained.

Let $P(z)$ be a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0, \tag{1}$$

whose zeros $z_1, z_2, \ldots, z_n$ are arranged so that

$$|z_1| \leq |z_2| \leq \ldots \leq |z_n| \tag{2}$$

and a polynomial

$$Q(z) = P(z) - c_k z^k \tag{3}$$

where k is a fixed integer $(1 \leq k \leq n)$, c_k is an arbitarary constant and it holds that $c_n \neq a_n$.

Let $u_1, u_2, \ldots, u_n$ be the zeros of the polynomial $Q(z)$ arranged so that

$$|u_1| \leq |u_2| \leq \ldots \leq |u_n|. \tag{4}$$

Then:

(A) $\big|P(u_1)\big| \leq \big|P(u_2)\big| \leq \ldots \leq \big|P(u_n)\big|.$

(B) $\big|Q(z_1)\big| \leq \big|Q(z_2)\big| \leq \ldots \leq \big|Q(z_n)\big|.$

(C) Besides every zero u_i of the polynomial $Q(z)$ there exists at least one zero z_j of the polynomial $P(z)$ such that

$$|z_j - u_i| \leq \left(\left|\frac{c_k}{a_n}\right| |u_i|^k\right)^{\frac{1}{n}} \leq \left(\left|\frac{c_k}{a_n}\right| M_q^{\,k}\right)^{\frac{1}{n}}, \tag{5}$$

where u_q is the upper bound of the moduli of zeros of the polynomial $Q(z)$.

(D) Besides every zero z_i of the polynomial $P(z)$ there exists at least one zero u_s of the polynomial $Q(z)$ such that

$$|u_s - z_i| \leq \left(\left|\frac{c_k}{a_n}\right| |z_i|^k\right)^{\frac{1}{n}} \leq \left(\left|\frac{c_k}{a_n}\right| M_p^{\,k}\right)^{\frac{1}{n}}, \tag{6}$$

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where u_p is the upper bound of the moduli of zeros of the polynomial $P(z)$.

Before we give proofs of (A)–(D), we represent polynomials $P(z)$ and $Q(z)$ in the following form

$$P(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n), \tag{7}$$

$$Q(z) = a_n(z - u_1)(z - u_2)\cdots(z - u_n). \tag{8}$$

Proof (A). From the equation (7), it follows that

$$P(u_i) = c_k {u_i}^k,\ i = 1, 2, \dots, n,$$

that is

$$\big|P(u_i)\big| = |c_k|\,|u_i|^k,\ i = 1, 2, \dots, n, \tag{9}$$

wherefrom, because of (4), we conclude that (A) holds.

Proof (B). From the equation (3), it follows that

$$Q(z_i) = -c_k {z_i}^k,\ i = 1, 2, \dots, n,$$

that is

$$\big|Q(z_i)\big| = |c_k|\,|z_i|^k,\ i = 1, 2, \dots, n, \tag{10}$$

wherefrom, because of (2), we conclude that (A) holds.

Proof (C). The equation (9), according to (7), is reduced to the equation

$$|a_n||u_i - z_1||u_i - z_2|\cdots|u_i - z_n| = |c_k|\,|u_i|^k,\ i = 1, 2, \dots, n. \tag{11}$$

Let z_j be a zero of the polynomial $P(z)$ that is closest to the zero u_i of the polynomial $Q(z)$. Then, from (11) we get

$$|a_n||u_i - z_j|^n \le |c_k|\,|u_i|^k,$$

wherefrom (C) follows.

Proof (D). The equation (10), according to (8), is reduced to the eqaution

$$|a_n||z_i - u_1||z_i - u_2|\cdots|z_i - u_n| = |c_k|\,|z_i|^k,\ i = 1, 2, \dots, n. \tag{12}$$

Let u_s be a zero of the polynomial $Q(z)$ that is closest to the zero z_i of the polynomial $P(z)$. Then, from (12) we get

$$|a_n||z_i - u_s|^n \le |c_k|\,|z_i|^k,$$

wherefrom (D) follows.

The case where $k = 0$ and $c_0 = c$ is given in [**1**, p. 80]. The case where $k = n$ and $c_n = a_n$ is given in [**1**. p.80], and also in [**2**] and [**3**].

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