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Some Results for Fuzzy Maps Under Nonexpansive Type Condition

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Abstract. In this paper, we have proved some results for fuzzy maps satisfying non-expansive type condition.

1. Introduction

A mapping $T : X \to X$ is called non-expansive if its Lipschitz constant $k(T)$ does not exceed 1. Thus, this class of mappings includes the contraction and strictly contractive mappings; moreover it contains all isometries (including the identity).

A map $T : X \to X$ is said to be non-expansive if

$$d(Tx, Ty) \leq d(x, y), forall x, y \in X.$$

Ciric studied the following non-expansive type condition in his paper [1] and [2] for a self map, T of X:

$$\begin{aligned} d(Tx, Ty) \leq a \max\{d(x, y), d(x, Tx), d(y, Ty)\} \\ + b \max\{d(x, Tx), d(y, Ty)\} \\ + c[d(x, Ty) + d(y, Tx)] \end{aligned}$$

$$d(Tx, Ty) \leq ad(x, y) + b \max\{d(x, Tx), d(y, Ty)\} + c[d(x, Ty), d(y, Tx)]$$

for all $x, y \in X$, where $a, b, c \geq 0$ such that $a + b + 2c = 1$.

The fuzzy set was introduced by L. Zadeh [9] in 1965. In this paper we shall use the terminology and notation of Heilpern [3]. Heilpern gave some fundamental results related to fuzzy map. Since that time a substantial literature has developed on this subject. In some earlier work Rhoades and Bruce Watson [7,8] proved several fixed point theorems involving a very general contractive condition, for fuzzy maps on complete linear metric space.

Definition 1. A fuzzy set A in complete metric space X is a function from X into $[0, 1]$. If $x \in X$, the function value $A(x)$ is called the grade of member

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of x in A. The α-level set of A, denoted by

$$A_\alpha = \{x : A(x) \geq \alpha\} \quad \text{if} \quad \alpha \in (0,1],$$
$$A_0 = \{x : A(x) > 0\}.$$

Definition 2. A fuzzy set A is said to be an approximate quantity iff A_α is compact and convex for each $\alpha \in [0,1]$, and $\sup_{x \in X} A(x) = 1$.

When A is an approximate quantity and $A(x_0) = 1$ for some $x_0 \in X$, A is identified with an approximation of x_0.

The collection of all fuzzy sets in X is denoted by $F(X)$ and $W(X)$ is the sub-collection of all approximate quantities.

Definition 3. Let $A, B \in W(X), \alpha \in [0,1]$. Then

$$D_\alpha(A,B) = \inf_{x \in A_\alpha,\, y \in B_\alpha} d(x,y),$$
$$D(A,B) = \sup_\alpha D_\alpha(A,B),$$
$$H_\alpha(A,B) = \text{dist}(A_\alpha, B_\alpha),$$

where "dist" is the Hausdorff distance.

Definition 4. Let $A, B \in W(X)$, then A is said to be more accurate than B, denoted by $A \subset B$ iff $A(x) \leq B(x)$ for each $x \in X$.

The relation "$\subset$" induces a partial ordering on the family $W(X)$.

Definition 5. Let X and Y be two complete linear metric spaces. F is called a fuzzy mapping if and only if F is a mapping from the set X into $W(Y)$.

A fuzzy mapping F is a fuzzy subset of $X \times Y$ with membership function $F(x,y)$. The function value $F(x,y)$ is the grade of membership of y in $F(x)$. Each fuzzy mapping is a set valued mapping.

Lee [4] proved the following Lemma.

Lemma 1. *Let (X,d) be a complete linear metric space, F is a fuzzy map from X into $W(X)$ and $x_0 \in X$ then there exists an $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

The following two lemmas are due to Heilpern [3].

Lemma 2. *Let $A, B \in W(X)$, $\alpha \in [0,1]$, and $D_\alpha(A,B) = \inf_{x \in A_\alpha,\, y \in B_\alpha} d(x,y)$, where $A_\alpha = \{x : A(x) \geq \alpha\}$, then $D_\alpha(x,A) \leq d(x,y) + D_\alpha(y,A)$ for each $x, y \in X$.*

Lemma 3. *Let $H_\alpha(A,B) = \text{dist}(A_\alpha, B_\alpha)$ where "dist" is the Hausdorff distance. If $\{x_0\} \subset A$ then $D_\alpha(x_0,B) \leq H_\alpha(A,B)$ for each $B \in W(X)$.*

Rhoades [5] proved the following common fixed point theorem involving a very general contractive condition, for fuzzy maps on complete linear metric space.

Theorem A. *Let (X, d) be complete linear metric space and let F, G be fuzzy mappings from X into $W(X)$ satisfying*

$$H(Fx, Gy) \leq Q(m(x, y)) \quad \textit{of all} \quad x, y \quad \textit{in} \quad X,$$

where

$$m(x,y) = \max\Big\{d(x,y), D_\alpha(x,Fx), D_\alpha(y,Gy), \frac{1}{2}[D_\alpha(x,Gy) + D_\alpha(y,Fx)]\Big\}, \tag{1}$$

Q is a real-valued function defined on D, the closure of the range of d, satisfying the following three conditions:

(a) *$0 < Q(s) < s$ for each $s \in D \backslash \{0\}$ and $Q(0) = 0$,*
(b) *Q is non-decreasing on D, and*
(c) *$g(s) = s/s - Q(s)$ is non-increasing on $D \backslash \{0\}$.*

Then there exists a point z in X, such that $\{z\} \subset Fz \cap Gz$.

We have proved the following common fixed point theorem satisfying non-expansive condition, for fuzzy maps on complete linear metric space.

Theorem 1. *Let (X, d) be a complete linear metric space. F, G are fuzzy mappings from X into $W(X)$, T is a self-map of X, satisfying*

$$\begin{aligned} H(Fx, Gy) \leq a \max\{& d(Tx, Ty), D_\alpha(Tx, Fx), D_\alpha(Ty, Gy), \\ & \frac{1}{2}[D_\alpha(Tx, Gy) + D_\alpha(Ty, Fx)]\} \\ & + b \max\{D_\alpha(Tx, Fx), D_\alpha(Ty, Gy)\} \\ & + c[D_\alpha(Tx, Gy) + D_\alpha(Ty, Fx)], \end{aligned} \tag{2}$$

where a, b, c are non-negative real numbers such that $a + b + 2c = 1$.

If T is continuous, T is weakly commutes with S and T and there exist a sequence which is asymptotically F-regular and G-regular with respect to T, then there exists a point z in X, which is a common fixed point of maps F, G, T.

Proof. Let $x_0 \in X$, then by Lemma 1, we can choose $Tx_1 \in X$ such that $\{Tx_1\} \subset Fx_0$. Choose x_2 such that $d(Tx_1, Tx_2) \leq H(Fx_0, Gx_1)$, continuing the process we obtain a sequence $\{Tx_n\}$ such that $\{Tx_{2n+1}\} \subset Fx_{2n}$, $\{Tx_{2n+2}\} \subset Gx_{2n+1}$ and $d(Tx_{2n+1}, x_{2n+2}) \leq H(Fx_{2n}, Gx_{2n+1})$, where $n = 1, 2, 3 \ldots$

Applying (2) and using triangle inequality, we have,

$$\begin{aligned}
d(Tx_{2n},Tx_{2n+1}) &\le H(Fx_{2n-1},Gx_{2n})\\
&\le a\max\big\{d(Tx_{2n-1},Tx_{2n}),D_\alpha(Tx_{2n-1},Fx_{2n-1}),\\
&\quad D_\alpha(Tx_{2n},Gx_{2n}),\frac{1}{2}[D_\alpha(Tx_{2n-1},Gx_{2n})+D_\alpha(Tx_{2n},Fx_{2n-1})]\big\}\\
&\quad +b\max\big\{D_\alpha(Tx_{2n-1},Fx_{2n-1}),D_\alpha(Tx_{2n},Gx_{2n})\big\}\\
&\quad +c[D_\alpha(Tx_{2n-1},Gx_{2n})+D_\alpha(Tx_{2n},Fx_{2n-1})]\\
&\le a\max\big\{d(Tx_{2n-1},Tx_{2n}),d(Tx_{2n},Tx_{2n+1}),\\
&\quad \frac{1}{2}[d(Tx_{2n-1},Tx_{2n})+d(Tx_{2n},Tx_{2n+1})]\big\}\\
&\quad +b\max\big\{d(Tx_{2n-1},Tx_{2n})+d(Tx_{2n},Tx_{2n+1})\big\}\\
&\quad +c[d(Tx_{2n-1},Tx_{2n})+d(Tx_{2n},Tx_{2n+1})]\\
&\le (a+b)\max\big\{d(Tx_{2n-1},Tx_{2n}),d(Tx_{2n},Tx_{2n+1})\big\}\\
&\quad +c[d(Tx_{2n-1},Tx_{2n})+d(Tx_{2n},Tx_{2n+1})].
\end{aligned}$$

If $d(Tx_{2n},Tx_{2n+1}) > d(Tx_{2n-1},Tx_{2n})$ for some n, then we have,

$$\begin{aligned}
d(Tx_{2n},Tx_{2n+1}) &\le (a+b+2c)d(Tx_{2n},Tx_{2n+1})\\
&= d(Tx_{2n},Tx_{2n+1}),
\end{aligned}$$

a contradiction. Thus $d(Tx_{2n},Tx_{2n+1}) \le d(Tx_{2n-1},Tx_{2n})$.

Hence, for all positive integers n,

$$d(Tx_{2n},Tx_{2n+1}) \le d(Tx_0,Tx_1). \tag{3}$$

Again applying (2) and using (3) we get

$$\begin{aligned}
d(Tx_2,Tx_3) &\le a\max\big\{d(Tx_1,Tx_2),d(Tx_2,Tx_3),d(Tx_1,Tx_2),\\
&\quad \frac{1}{2}[d(Tx_2,Tx_2)+d(Tx_1,Tx_3)]\big\}\\
&\quad +b\max\big\{d(Tx_1,Tx_2),d(Tx_2,Tx_3)\big\}\\
&\quad +c[d(Tx_1,Tx_3)+d(Tx_2,Tx_2)]\\
&\le a\max\big\{d(Tx_0,Tx_1),d(Tx_0,Tx_1),d(Tx_0,Tx_1),\frac{1}{2}d(Tx_1,Tx_3)\big\}\\
&\quad +b\max\big\{d(Tx_0,Tx_1),d(Tx_0,Tx_1)\big\}+cd(Tx_1,Tx_3)
\end{aligned} \tag{4}$$

Applying (2) again and using (3) we have

$$
\begin{aligned}
d(Tx_1, Tx_3) \le\ & a\max\big\{d(Tx_0, Tx_1), d(Tx_2, Tx_3), d(Tx_0, Tx_2),\\
& \frac{1}{2}[d(Tx_0, Tx_3) + d(Tx_2, Tx_1)]\big\}\\
& + bmax\big\{d(Tx_0, Tx_1), d(Tx_2, Tx_3)\big\}\\
& + c[d(Tx_0, Tx_3) + d(Tx_2, Tx_1)]\\
\le\ & (2-b)d(Tx_0, Tx_1).
\end{aligned} \tag{5}
$$

Using (4) and (5), we get

$$
\begin{aligned}
d(Tx_2, Tx_3) &\le ad(Tx_0, Tx_1) + bd(Tx_0, Tx_1) + (2c - bc)d(Tx_0, Tx_1)\\
&\le (1 - bc)d(Tx_0, Tx_1).
\end{aligned}
$$

It is easy to show that

$$
d(Tx_{n+1}, Tx_n) \le (1 - bc)^{[n/2]} d(Tx_0, Tx_1),
$$

where $[n/2]$ means the greatest integer not exceeding $n/2$.

We conclude that $\{Tx_n\}$ is Cauchy sequence. Since X is complete, $\{Tx_n\}$ is convergent to the point z (say).

Since $\alpha \in [0, 1]$ then using Lemmas 2, 3 and (2) we have

$$
\begin{aligned}
D_\alpha(Tz, Fz) &\le d(Tz, GTx_n)) + D_\alpha(GTx_n, Fz)\\
&\le d(Tz, GTx_n) + H_\alpha(Fz, GTx_n)\\
&\le d(Tz, GTx_n) + H(Fz, GTx_n).
\end{aligned}
$$

Taking the limit n tends to infinity we get

$$
D_\alpha(Tz, Fz) \le \lim_{n\to\infty} H(Fz, GTx_n) \le \lim_{n\to\infty} H(Fz, GTx_n) \tag{6}
$$

Again using (2) we have

$$
\begin{aligned}
H(Fz, GTx_n) \le\ & a\max\big\{d(Tz, TTx_n), D_\alpha(Tz, FTx_n), D_\alpha(TTx_n, GTx_n),\\
& \frac{1}{2}[D_\alpha(Tz, GTx_n) + D_\alpha(TTx_n, Fz)]\big\}\\
& + b\max\big\{D_\alpha(Tz, Fz), D_\alpha(TTx_n, GTx_n)\big\}\\
& + c[D_\alpha(Tz, GTx_n) + D_\alpha(TTx_n, Fz)].
\end{aligned}
$$

Letting n tend to infinity, we have

$$
\begin{aligned}
\lim_{n\to\infty} H(Fz, GTx_n) \le\ & a\max\big\{d(Tz, Tz), d(Tz, Fz), D_\alpha(Tz, Gz),\\
& \frac{1}{2}[d(Tz, Gz) + d(Tz, Fz)]\big\}\\
& + b\max\{d(Tz, Fz), d(Tz, Gz)\big\}\\
& + c[d(Tz, Gz) + d(Tz, Fz)]
\end{aligned}
$$

$$\lim_{n\to\infty} H(Fz, GTx_n) \le (a+b+2c)\max\{d(Tz,Fz), d(Tz,Gz)\} \\ = d(Tz,Fz). \tag{7}$$

Using (6) and (7) we have

$$D_\alpha(Tz,Fz) \le d(Tz,Fz),$$

a contradiction. Hence we must have $D_\alpha(Tz,Fz)=0$. Since α is arbitrary number in $[0,1]$. It follows that $D(Tz,Fz)=0$, which implies that $Tz=Fz$ Similarly it can be shown that $Tz=Gz$.

$$\begin{aligned} H(Fx_n, GTx_n) \le\ & a\max\big\{d(Tx_n,TTx_n), D_\alpha(Tx_n,Fx_n), D_\alpha(TTx_n,GTx_n), \\ & \frac{1}{2}[D_\alpha(Tx_n,GTx_n)+D_\alpha(TTx_n,Fx_n)]\big\} \\ & + b\max\big\{D_\alpha(Tx_n,Fx_n), D_\alpha(TTx_n,GTx_n)\big\} \\ & + c[D_\alpha(Tx_n,GTx_n)+D_\alpha(TTx_n,Fx_n)] \end{aligned}$$

Letting n tend to infinity and supposing T is continuous, T weakly commutes with S and T and there exist a sequence which is asymptotically F-regular and G-regular with respect to T, than we have

$$\le ad(z,Tz) + 2cd(z,Tz)$$
$$d(z,Tz) \le (1-b)d(z,Tz),$$

which implies $z = Tz$.

Hence z is a common fixed point of maps G, F, T. □

Corollary 1. *Let (X,d) be a complete linear metric space. F, G are fuzzy mappings from X into $W(X)$ satisfying*

$$\begin{aligned} H(Fx,Gy) \le\ & a\max\big\{d(x,y), D_\alpha(x,Fx), D_\alpha(y,Gy), \\ & \frac{1}{2}[D_\alpha(x,Gy)+D_\alpha(y,Fx)]\big\} \\ & b\max\big\{D_\alpha(x,Fx), D_\alpha(y,Gy)\big\} \\ & + c[D_\alpha(x,Gy)+D_\alpha(y,Fx)] \end{aligned}$$

where a, b, c are non-negative real numbers such that $a+b+2c=1$.

Then there exists a point z in X, which is a common fixed point of maps F and G, i.e., $\{z\} \subset Fz \cap Gz$.

Proof. Taking T is identity map of X in Theorem 1. □

Rhoades [6], generalized the result of Theorem A for sequence of fuzzy maps on complete linear metric space. He proved the following theorem.

Theorem B. *Let g be a non-expansive self mapping of a complete linear metric space (X,d). Let $\{F_i\}$ be a sequence of fuzzy mappings from X into $W(X)$. For each pair of fuzzy mappings F_i, F_j and for any $x \in X$, $\{u_x\} \subset$*

$F_i(x)$, there exists a $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that $D(\{u_x\}, \{v_y\}) \leq Q(m(x, y))$, where

$$m(x,y) = \max\big\{d(g(x), g(u_x)), d(g(y), g(v_y)), d(g(x), g(y)), \frac{1}{2}[d(g(x), g(v_y)) + d(g(y), g(u_x))]\big\} \tag{8}$$

where Q satisfying the conditions (a)-(c) of Theorem A. Then there exists $\{p\} \subset \bigcap_{i \in N} F_i(p)$.

We prove the result of above for common fixed point for sequence of fuzzy mappings of non-expansive condition.

Theorem 2. *Let g be a non-expansive self mapping of a complete linear metric space (X, d). Let $\{F_i\}$ be a sequence of fuzzy mappings from X into $W(X)$. For each pair of fuzzy mapping F_i, F_j and for any $x \in X$, $\{u_x\} \subset F_i(x)$, there exists a $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that*

$$\begin{aligned} D(\{u_x\}, \{v_y\}) \leq a \max\Big\{& d(g(x), g(u_x)), d(g(y), g(v_y)), d(g(x), g(y)), \\ & \frac{1}{2}\big[d(g(x), g(v_y)) + d(g(y), g(u_x))\big]\Big\} \\ & + b \max\big\{d(g(x), g(u_x)), d(g(y), g(v_y))\big\} \\ & + c\big[d(g(x), g(v_y)) + d(g(y), g(u_x))\big] \end{aligned} \tag{9}$$

where a, b, c are non-negative real numbers such that $a + b + 2c = 1$.

Then there exists $\{p\} \subset \bigcap_{i \in N} F_i(p)$, i.e., p is a common fixed point of sequence of fuzzy mappings.

Proof. Let $x_0 \in X$, then by Lemma 1, we can choose $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$. Similarly for $x_1 \in X$ we can choose $x_2 \in X$ such that $\{x_2\} \subset F_2(x_1)$. In general, $\{x_{n+1}\} \subset F_{n+1}(x_n)$.

Applying (9) and using triangle inequality we have

$$\begin{aligned} d(x_n, x_{n+1}) &= D(\{x_n\}, \{x_{n+1}\}) \\ &\leq a \max\big\{d(g(x_{n-1}), g(x_n)), d(g(x_n), g(x_{n+1})), d(g(x_{n-1}), g(x_n)), \\ &\quad \frac{1}{2}\big[d(g(x_n), g(x_n)) + d(g(x_{n-1}), g(x_{n+1}))\big]\big\} \\ &\quad + b \max\big\{d(g(x_{n-1}), g(x_n)), d(g(x_n), g(x_{n+1}))\big\} \\ &\quad + c\big[d(g(x_n), g(x_n)) + d(g(x_{n-1}), g(x_{n+1}))\big]. \end{aligned}$$

Since g is non-expansive and $D(\{x_n\},\{x_{n+1}\}) = d(x_n, x_{n+1})$, we get

$$
\begin{aligned}
d(x_n, x_{n+1}) \le\ & a \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n),\\
&\quad \frac{1}{2}[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]\Big\}\\
&+ b \max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}\\
&+ c[d(x_n, x_n) + d(x_{n-1}, x_{n+1}]\\
\le\ & a \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\Big\}\\
&+ b \max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}\\
&+ c\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big].
\end{aligned}
$$

If $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$ for some n, then we have

$$
\begin{aligned}
d(x_n, x_{n+1}) <\ & a \max\Big\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), \frac{1}{2}[d(x_n, x_{n+1}) + d(x_n, x_{n+1})]\Big\}\\
&+ b \max\big\{d(x_n, x_{n+1}), d(x_n, x_{n+1})\big\}\\
&+ c\big[d(x_n, x_{n+1}) + d(x_n, x_{n+1})\big]\\
=\ & (a + b + 2c) d(x_n, x_{n+1})\\
=\ & d(x_n, x_{n+1}),
\end{aligned}
$$

a contradiction. Thus $d(x_n, x_{n+1}) \le d(x_{n-1}, x_n)$.

Hence, for all positive integers n

$$
d(x_n, x_{n+1}) \le d(x_0, x_1). \tag{10}
$$

Again applying (9) and using (10), we get

$$
\begin{aligned}
d(x_2, x_3) = D(x_2, x_3) \le\ & a \max\Big\{d(g(x_1), g(x_2)), d(g(x_2), g(x_3)), d(g(x_1), g(x_2)),\\
& \frac{1}{2}\big[d(g(x_1), g(x_3)) + d(g(x_2), g(x_2))\big]\Big\}\\
&+ b \max\big\{d(g(x_1), g(x_2)), d(g(x_2), g(x_3))\big\}\\
&+ c\big[d(g(x_1), g(x_3)) + d(g(x_2), g(x_2))\big].
\end{aligned}
$$

Since g is non-expansive, we have

$$\begin{aligned} d(x_2,x_3) &\le a\max\Big\{d(x_1,x_2), d(x_2,x_3), d(x_1,x_2), \\ &\qquad \frac{1}{2}[d(x_1,x_3)+d(x_2,x_2)]\Big\} \\ &\quad + b\max\big\{d(x_1,x_2), d(x_2,x_3)\big\} + c\big[d(x_1,x_3)+d(x_2,x_2)\big] \\ &\le a\max\Big\{d(x_0,x_1), d(x_0,x_1), d(x_0,x_1), \frac{1}{2}d(x_1,x_3)\Big\} \qquad (11) \\ &\quad + b\max\big\{d(x_0,x_1), d(x_0,x_1)\big\} + cd(x_1,x_3) \\ &\le a\max\Big\{d(x_0,x_1), \frac{1}{2}d(x_1,x_3)\Big\} \\ &\quad + bd(x_0,x_1), +cd(x_1,x_3). \end{aligned}$$

Applying (9) again and using (10) we have

$$\begin{aligned} d(x_{1,}x_3) &= D(\{x_1\},\{x_3\}) \\ &\le a\max\Big\{d(g(x_0),g(x_1)), d(g(x_2),g(x_3)), d(g(x_0),g(x_2)), \\ &\qquad \frac{1}{2}\big[d(g(x_0),g(x_3))+d(g(x_2),g(x_1))\big]\Big\} \\ &\quad + b\max\big\{d(g(x_0),g(x_1)), d(g(x_2),g(x_3))\big\} \\ &\quad + c\big[d(g(x_0),g(x_3))+d(g(x_2),g(x_1))\big] \\ &\le a\max\Big\{d(x_0,x_1), d(x_2,x_3), [d(x_0,x_1)+d(x_1,x_2)], \qquad (12) \\ &\qquad \frac{1}{2}\big[d(x_0,x_1)+d(x_1,x_2)+d(x_2,x_3)+d(x_2,x_1)\big]\Big\} \\ &\quad + b\max\big\{d(x_0,x_1), d(x_2,x_3)\big\} \\ &\quad + c\big[d(x_0,x_1)+d(x_1,x_2)+d(x_2,x_3)+d(x_2,x_1)\big] \\ &\le (2a+b+4c)d(x_0,x_1) \\ &= (2-b)d(x_0,x_1). \end{aligned}$$

Using (11) and (12), we get

$$\begin{aligned} d(x_2,x_3) &\le a\max\Big\{d(x_0,x_1), \frac{1}{2}[(2-b)d(x_0,x_1)]\Big\} \\ &\quad + bd(x_0,x_1) + c(2-b)d(x_0,x_1) \\ &\le (1-bc)d(x_0,x_1). \end{aligned}$$

It is easy to show that,

$$d(x_{n+1},x_n) \le (1-bc)^{[n/2]}d(x_0,x_1),$$

where $[n/2]$ means the greatest integer not exceeding $n/2$. Since $bc < 1$, $\{x_n\}$ is a Cauchy sequence and hence the sequence $\{x_n\}$ converges to the limit p (say).

Let F_m be an arbitrary member of $\{F_i\}$. Since $\{x_n\} \subset F_m(x_{n-1})$, by Lemma 1, there exists a $v_n \in X$ such that $\{v_n\} \subset F_m(p)$ for all n.

Applying (9) again and using (10) we have

$$\begin{aligned} d(x_n,v_n) = D(\{x_n\},\{v_n\}) \le a \max\Big\{ & d(x_{n-1},x_n), d(p,v_n), d(x_{n-1},p) \\ & \frac{1}{2}[d(x_{n-1},v_n) + d(x_p,x_n)]\Big\} \\ & + b \max\big\{d(x_{n-1},x_n), d(p,v_n)\big\} \\ & + c\big[d(x_{n-1},v_n) + d(x_p,x_n)\big] \end{aligned}$$

If $\lim_{n\to\infty} v_n \neq p$, then letting n tend to infinity, we have

$$\begin{aligned} d(p,v_n) \le \text{taking a } \max\big\{ & d(p,p), d(p,v_n), d(p,p), \\ & \frac{1}{2}[d(p,v_n) + d(p,p)]\big\} \\ & + b\max\{d(p,p), d(p,v_n)\} \\ & + c[d(p,v_n) + d(p,p)] \\ \le\ & (a+b+c) d(p,v_n) \\ <\ & d(p,v_n), \end{aligned}$$

a contradiction. Hence $\lim v_n = p$.

Since F_m is arbitrary, then $\{p\} \subset \bigcap_{i=1}^{n} F_i(p)$. □

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