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Related Fixed Point Theorems for Three Metric Spaces, II

Sampada Navshinde, Dr. J. Achari and Brian Fisher

Abstract. In this paper, we have proved some related fixed point theorems for three metric spaces which improve the results of Jain, Sahu and Fisher [2].

1. Introduction

The following fixed point theorem was proved by Jain, Sahu and Fisher [2].

Theorem 1. *Let (X, d), (Y, ρ) and (Z, σ) be complete metric spaces. If T is continuous mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned}
d(RSTx, RSTx') &\leq c \max\big\{d(x,x'), d(x,RSTx), d(x',RSTx'),\\
&\qquad \rho(Tx,Tx'), \sigma(STx,STx')\big\},\\
\rho(TRSy, TRSy') &\leq c \max\big\{\rho(y,y'), \rho(y,TRSy), \rho(y',TRSy'),\\
&\qquad \sigma(sy,sy'), d(RSy,RSy')\big\},\\
\sigma(STRz, STRz') &\leq c \max\big\{\sigma(z,z'), \sigma(z,STRz), \sigma(z',STRz')\\
&\qquad d(Rz,Rz'), \rho(TRz,TRz')\big\}
\end{aligned}$$

for all x, x' in X, y, y' in Y and z, z' in Z, where $0 \leq c < 1$, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = v$, and $Rw = u$.

2. Main Results

We now prove the following related fixed point theorem which improves Theorem 1.

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Theorem 2. *Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces. If T is continuous mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$d(RSTx,RSTx') \le c\max\Big\{\frac{d(x,x')[1+d(x,RSTx)]}{1+d(x,x')}, \frac{d(x',RSTx)[1+d(x,RSTx')]}{1+d(x,x')}, \frac{d(x',RSTx')[1+d(x,RSTx)]}{1+d(x,x')}, \rho(Tx,Tx'),\sigma(STx,STx')\Big\}, \tag{1}$$

$$\rho(TRSy,TRSy') \le c\max\Big\{\frac{\rho(y,y')[1+\rho(y,TRSy)]}{1+\rho(y,y')}, \frac{\rho(y',TRSy)[1+\rho(y,TRSy')]}{1+\rho(y,y')}, \frac{\rho(y',TRSy')[1+\rho(y,TRSy)]}{1+\rho(y,y')}, \sigma(Sy,Sy'),d(RSy,RSy')\Big\}, \tag{2}$$

$$\sigma(STRz,STRz') \le c\max\Big\{\frac{\sigma(z,z')[1+\sigma(z,STRz)]}{1+\sigma(z,z')}, \frac{\sigma(z',STRz)[1+\sigma(z,STRz')]}{1+\sigma(z,z')}, \frac{\sigma(z',STRz')[1+\sigma(z,STRz)]}{1+\sigma(z,z')}, d(Rz,Rz'),\rho(TRz,TRz')\Big\} \tag{3}$$

for all x,x' in X, y,y' in Y and z,z' in Z, where $0 \le c < 1$, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

Proof. Let x_0 be an arbitrary point in X. Define sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X, Y and Z respectively by

$$x_n = (RST)^n x_0, \qquad y_n = Tx_{n-1}, \qquad z_n = Sy_n$$

for $n = 1,2,\ldots$.

Applying inequality (2), we have

$$(4)\quad \begin{aligned}\rho(y_n, y_{n+1}) &= \rho(TRSy_{n-1}, TRSy_n)\\ &\leq c\max\Big\{\frac{\rho(y_{n-1}, y_n)[1+\rho(y_{n-1}, TRSy_{n-1})]}{1+\rho(y_{n-1}, y_n)},\\ &\qquad \frac{\rho(y_n, TRSy_{n-1})[1+\rho(y_{n-1}, TRSy_n)]}{1+\rho(y_{n-1}, y_n)},\\ &\qquad \frac{\rho(y_n, TRSy_n)[1+\rho(y_{n-1}, TRSy_{n-1})]}{1+\rho(y_{n-1}, y_n)},\\ &\qquad \sigma(Sy_{n-1}, Sy_n), d(RSy_{n-1}, RSy_n)\Big\},\\ &\leq c\max\big\{\rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)\big\}.\end{aligned}$$

Using inequality (3), we have

$$(5)\quad \begin{aligned}\sigma(z_n, z_{n+1}) &= \sigma(STRz_{n-1}, STRz_n)\\ &\leq c\max\Big\{\frac{\sigma(z_{n-1}, z_n)[1+\sigma(z_{n-1}, STRz_{n-1})]}{1+\sigma(z_{n-1}, z_n)},\\ &\qquad \frac{\sigma(z_n, STRz_{n-1})[1+\sigma(z_{n-1}, STRz_n)]}{1+\sigma(z_{n-1}, z_n)},\\ &\qquad \frac{\sigma(z_n, STRz_n)[1+\sigma(z_{n-1}, STRz_{n-1})]}{1+\sigma(z_{n-1}, z_n)},\\ &\qquad d(Rz_{n-1}, Rz_n), \rho(TRz_{n-1}, TRz_n)\Big\}\\ &\leq c\max\big\{\rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)\big\},\end{aligned}$$

on using inequality (4).

Using inequality (1) we have

$$(6)\quad \begin{aligned}d(x_n, x_{n+1}) &= d(RSTx_{n-1}, RSTx_n)\\ &\leq c\max\big\{\rho(y_{n+1}, y_n), \sigma(z_{n+1}, z_n), d(x_{n+1}, x_n), d(x_{n-1}, x_n)\big\}\\ &\leq c\max\big\{\rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)\big\},\end{aligned}$$

on using inequalities (4) and (5).

It follows easily by induction on using inequalities (4),(5) and (6) that

$$\begin{aligned}d(x_n, x_{n+1}) &\leq c^{n-1}\max\big\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\big\},\\ \rho(y_n, y_{n+1}) &\leq c^{n-1}\max\big\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\big\},\\ \sigma(z_n, z_{n+1}) &\leq c^{n-1}\max\big\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\big\}.\end{aligned}$$

Since $c < 1$, it follows that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences with limits u, v and w in X, Y and Z respectively.

Since T and S are continuous, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = Tu = v,$$
$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} Sy_n = Sv = w.$$

Using inequality (1) again, we have

$$\begin{aligned}
d(RSTu, x_n) &= d(RSTu, RSTx_{n-1}) \\
&\leq c \max\Big\{\frac{d(u,x_{n-1})[1+d(u,RSTu)]}{1+d(u,x_{n-1})}, \frac{d(x_{n-1},RSTu)[1+d(u,RSTx_{n-1})]}{1+d(u,x_{n-1})}, \\
&\quad \frac{d(x_{n-1},RSTx_{n-1})[1+d(u,RSTu)]}{1+d(u,x_{n-1})}, \rho(Tu,Tx_{n-1}), \sigma(STu,STx_{n-1})\Big\}, \\
&\leq c \max\Big\{\frac{d(u,x_{n-1})[1+d(u,RSTu)]}{1+d(u,x_{n-1})}, \frac{d(x_{n-1},RSTu)[1+d(u,x_n)]}{1+d(u,x_{n-1})}, \\
&\quad \frac{d(x_{n-1},x_n)[1+d(u,RSTu)]}{1+d(u,x_{n-1})}, \rho(Tu,Tx_{n-1}), \sigma(STu,STx_{n-1})\Big\}.
\end{aligned}$$

Since S and T are continuous, it follows on letting $n \to \infty$ that

$$d(RSTu, u) \leq cd(RSTu, u).$$

Thus $RSTu = u$, since $c < 1$ and so u is a fixed point of RST.

We therefore have

$$TRSv = TRSTu = Tu = v$$

and so

$$STRw = STRSv = Sv = w.$$

Hence v and w are fixed points of TRS and STR respectively.

We now prove the uniqueness of the fixed point u. Suppose that RST has a second fixed point u'. Then using inequality (1), we have

$$\begin{aligned}
d(u,u') &= d(RSTu, RSTu') \\
&\leq c \max\Big\{\frac{d(u,u')[1+d(u,RSTu)]}{1+d(u,u')}, \frac{d(u',RSTu)[1+d(u,RSTu')]}{1+d(u,u')}, \\
&\quad \frac{d(u',RSTu')[1+d(u,RSTu)]}{1+d(u,u')}, \rho(Tu,Tu'), \sigma(STu,STu')\Big\}, \\
&\leq c \max\Big\{\frac{d(u,u')[1+d(u,u)]}{1+d(u,u')}, \frac{d(u',u)[1+d(u,u')]}{1+d(u,u')}, \\
&\quad \frac{d(u',u')[1+d(u,u)]}{1+d(u,u')}, \rho(Tu,Tu'), \sigma(STu,STu')\Big\}, \\
&= c \max\big\{\rho(Tu,Tu'), \sigma(STu,STu')\big\}.
\end{aligned}$$

Further, using inequality (2), we have

$$\begin{aligned}\rho(Tu,Tu') &= \rho(TRSTu,TRSTu') \\ &\leq c\max\Big\{\frac{\rho(Tu,Tu')[1+\rho(Tu,TRSTu)]}{1+\rho(Tu,Tu')}, \\ &\quad \frac{\rho(Tu',TRSTu)[1+\rho(Tu,TRSTu')]}{1+\rho(Tu,Tu')}, \\ &\quad \frac{\rho(Tu',TRSTu')[1+\rho(Tu,TRSTu)]}{1+\rho(Tu,Tu')}, \\ &\quad d(RSTu,RSTu'),\sigma(STu,STu')\Big\}, \\ &\leq c\max\Big\{\frac{\rho(Tu,Tu')[1+\rho(Tu,Tu)]}{1+\rho(Tu,Tu')},\frac{\rho(Tu',Tu)[1+\rho(Tu,Tu')]}{1+\rho(Tu,Tu')}, \\ &\quad \frac{\rho(Tu',Tu')[1+\rho(Tu,Tu)]}{1+\rho(Tu,Tu')},d(u,u'),\sigma(STu,STu')\Big\}, \\ &= c\max\big\{d(u,u'),\sigma(STu,STu')\big\}.\end{aligned}$$

Hence we have

$$d(u,u') \leq c\sigma(STu,STu').$$

Finally, on using inequality (3), we have

$$\begin{aligned}d(u,u') &\leq c\sigma(STu,STu') \\ &\leq c\sigma(STRSTu,STRSTu') \\ &\leq c^2\max\Big\{\frac{\sigma(STu,STu')[1+\sigma(STu,STRSTu)]}{1+\sigma(STu,STu')}, \\ &\quad \frac{\sigma(STu',STRSTu)[1+\sigma(STu,STRSTu')]}{1+\sigma(STu,STu')}, \\ &\quad \frac{\sigma(STu',STRSTu')[1+\sigma(STu,STRSTu)]}{1+\sigma(STu,STu')}, \\ &\quad d(RSTu,RSTu'),\rho(TRSTu,TRSTu')\Big\} \\ &= c^2 d(u,u').\end{aligned}$$

Since $c < 1$, it follows that $u = u'$ and the uniqueness of u follows.

Similarly, it can be proved that v is the unique fixed point of TRS and w is the unique fixed point of STR.

We finally prove that we also have $Rw = u$. To do this, note that

$$Rw = R(STRw) = RST(Rw)$$

and so Rw is a fixed point of RST. Since u is the unique fixed point of RST, it follows that $Rw = u$. This completes the proof of the theorem. □

We now prove an analogous result for compact metric spaces.

Theorem 3. *Let (X,d), (Y,ρ) and (Z,σ) be compact metric spaces. If T is continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a continuous mapping of Z into X satisfying the inequalities*

$$d(RSTx,RSTx') < \tag{7}$$
$$< c\max\Big\{\frac{d(x,x')[1+d(x,RSTx)]}{1+d(x,x')}, \frac{d(x',RSTx)[1+d(x,RSTx')]}{1+d(x,x')}, \frac{d(x',RSTx')[1+d(x,RSTx)]}{1+d(x,x')}, \rho(Tx,Tx'), \sigma(STx,STx')\Big\},$$

$$\rho(TRSy,TRSy') < \tag{8}$$
$$< c\max\Big\{\frac{\rho(y,y')[1+\rho(y,TRSy)]}{1+\rho(y,y')}, \frac{\rho(y',TRSy)[1+\rho(y,TRSy')]}{1+\rho(y,y')}, \frac{\rho(y',TRSy')[1+\rho(y,TRSy)]}{1+\rho(y,y')}, \sigma(Sy,Sy'), d(RSy,RSy')\Big\},$$

$$\sigma(STRz,STRz') < \tag{9}$$
$$< c\max\Big\{\frac{\sigma(z,z')[1+\sigma(z,STRz)]}{1+\sigma(z,z')}, \frac{\sigma(z',STRz)[1+\sigma(z,STRz')]}{1+\sigma(z,z')}, \frac{\sigma(z',STRz')[1+\sigma(z,STRz)]}{1+\sigma(z,z')}, d(Rz,Rz'), \rho(TRz,TRz')\Big\}$$

for all distinct x,x' in X, all distinct y,y' in Y and all distinct z,z' in Z. Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu=v$, $Sv=w$ and $Rw=u$.

Proof. Let us denote the right-hand side of inequalities (7), (8) and (9) by $h(x,x')$, $k(y,y')$ and $p(z,z')$ respectively.

Suppose first of all that there exist u,u' in X such that $h(u,u')=0$. Then it follows immediately that $u=u'$ and $RSTu=u$. Then on putting $Tu=v$, $Sv=w$, we have

$$\begin{aligned} RSv=u \quad &\Rightarrow \quad TRSv=Tu=v,\\ STRSv=STRw=Sv=w \quad &\Rightarrow \quad RSv=Rw=u. \end{aligned}$$

The result of the theorem therefore holds in this case.

Similarly, if there exist v,v' in Y such that $k(v,v')=0$ or if there exist w,w' in Z such that $p(w,w')=0$, then the results of the theorem also hold.

Now suppose that $h(x,x')\neq 0$ for all x,x' in X, $k(y,y')\neq 0$ for all y,y' in Y and $p(z,z')\neq 0$ for all z,z' in Z. Define the function f on X^2 by

$$f(x,x')=\frac{d(RSTx,RSTx')}{h(x,x')}.$$

Then f is continuous and since $X \times X$ is compact, f attains its maximum value c_1. Because of inequality (7), $c_1 < 1$ and so

$$d(RSTx, RSTx') \leq c_1 h(x, x'),$$

for all x, x' in X.

Similarly, there exist c_2, $c_3 < 1$ such that

$$\rho(TRSy, TRSy') \leq c_2 k(y, y'),$$

for all y, y' in Y and

$$\sigma(STRz, STRz') \leq c_3 p(z, z'),$$

for all z, z' in Z. It follows that the conditions of Theorem 2 are satisfied with $c = \max(c_1, c_2, c_3)$ and so the results of the theorem are again satisfied.

The uniqueness of u, v, and w follows easily. □

References

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Sampada Navshinde
H.No. 104, "Atrey", Konark Vihar
Farande Nagar, Wadi(bk)
Nanded-431605 (Maharashtra)
India
E-mail address: snavshinde@gmail.com

Dr. J. Achari
Ramanand Nagar, Pawdewadi Naka
Nanded (Maharashtra)
India

Brian Fisher
Department of Mathematics
University of Leicester
Leicester, LE1 7RH
U.K.
E-mail address: fbr@le.ac.uk