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A REMARK CONCERNING ZEROS OF ONE CLASS OF POLYNOMIALS

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Abstract. In this paper the distribution of zeros of a class of real polynomials is considered. In some cases the intervals, each one containing one zero, are determined with more accuracy.

1. One considers in [2] the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (n \geq 3) \tag{1}$$

with coefficients a_k satisfying the conditions

$$a_k > 0 \ (k=0,1,2,\ldots,n) \text{ and } 0 < \frac{a_0}{a_1} < \frac{a_1}{a_2} < \cdots < \frac{a_{n-2}}{a_{n-1}} < \frac{a_{n-1}}{a_n}, \tag{2}$$

or more generally

$$|a_k| > 0 \ (k=0,1,2,\ldots,n) \text{ and } 0 < \left|\frac{a_0}{a_1}\right| < \left|\frac{a_1}{a_2}\right| < \cdots < \left|\frac{a_{n-2}}{a_{n-1}}\right| < \left|\frac{a_{n-1}}{a_n}\right|,$$

where next theorems are proved.

A) If $a_k > 0$ $(k = 0,1,2,\ldots,n)$ and if

$$\frac{a_k^2}{a_{k-1}a_{k+1}} \geq 4 \quad (k = 1,2,\ldots,n-1), \tag{3}$$

the polynomial (1) has only simple real negative zeros, one in each of the intervals

$$\left(-\frac{2a_0}{a_1}, 0\right), \left(-\frac{2a_1}{a_2}, -\frac{2a_0}{a_1}\right), \ldots, \left(-\frac{2a_{n-1}}{a_n}, -\frac{2a_{n-2}}{a_{n-1}}\right). \tag{3a}$$

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B) If $a_k > 0$ $(k = 0, 1, 2, \dots, n)$ and if

$$\frac{a_k^2}{a_{k-1}a_{k+1}} \geq 2 \quad (k = 1, 2, \dots, n-1)\,,$$

the polynomial (1) has no zeros in the domain

$$-\frac{\pi}{2} \leq \arg\{z\} \leq \frac{\pi}{2}\,.$$

C) If $|a_k| > 0$ $(k = 0, 1, \dots, n)$ and if

$$\left|\frac{a_k^2}{a_{k-1}a_{k+1}}\right| \geq 5 \quad (k = 1, 2, \dots, n-1)\,,$$

the polynomial (1) is different from zero at the boundary of every circular ring

$$\sqrt{5}\left|\frac{a_{k-2}}{a_{k-1}}\right| < |z| < \sqrt{5}\left|\frac{a_{k-1}}{a_k}\right| \qquad (k = 2, 3, \dots, n)$$

and has one zero inside each one of them.

2. In this paper we shall consider the polynomial (1), with coefficients a_k satisfying, in addition to (2), the conditions

$$\frac{a_k^2}{a_{k-1}a_{k+1}} \geq s \quad (s > 1) \quad (k = 1, 2, \dots, n-1) \tag{4}$$

where s is a constant.

The theorem A) holds for $s = 4$ and the theorem B) holds for $s = 2$.

The purpose of this paper is to establish some conditions on the coefficients a_k for which 1) some, or all, of zeros of the polynomial (1) will be real and negative also for the values of the constant $s < 4$, and 2) intervals containing zeros of the polynomial (1) in the case $s > 4$ are determined with more accuracy.

Let us first make a constatation.

If the coefficients of the polynomial

$$F_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad (a_k > 0;\ k = 0, 1, 2, 3)$$

satisfy condition

$$\frac{a_2^2}{a_1a_3} < 3\,,$$

F_3 has one real negative and two complex zeros.

If $\frac{a_2^2}{a_1 a_3} = 3$, $\frac{a_1^2}{a_0 a_2} \neq 3$, F_3 has also one real negative and two complex zeros.

If $\frac{a_2^2}{a_1 a_3} = \frac{a_1^2}{a_0 a_2} = 3$, F_3 has a triple real negative zero $x = -\frac{a_2}{3a_3}$.

Having in mind the preceding constatation about F_3, we conclude that for the value $s \leq 3$ the polynomial (1) has not necessarily all zeros simple, real and negative. In order to have, in the polynomial (1), all zeros simple and negative, there must be $s > 3$ in (4). In relation to that we shall demonstrate next theorems.

Theorem 1. *If in (1) $a_k > 0$ $(k = 0, 1, 2, \ldots, n)$ and if*

$$\frac{a_1^2}{a_0 a_2} \geq \frac{a_2^2}{a_1 a_3} \geq \cdots \geq \frac{a_{n-2}^2}{a_{n-3} a_{n-1}} \geq \frac{a_{n-1}^2}{a_{n-2} a_n} \geq \frac{10}{3}, \tag{5}$$

the polynomial (1) has at least $n-2$ real negative zeros, at least one in each of the intervals

$$\left(-\frac{2a_0}{a_1}, 0\right), \left(-\frac{2a_1}{a_2}, -\frac{2a_0}{a_1}\right), \ldots, \left(-\frac{2a_{n-3}}{a_{n-2}}, -\frac{2a_{n-4}}{a_{n-3}}\right). \tag{6}$$

Theorem 2. *If in (1) $a_k > 0$ $(k = 0, 1, 2, \ldots, n)$ and if*

$$\frac{a_1^2}{a_0 a_2} \geq \frac{a_2^2}{a_1 a_2} \geq \cdots \geq \frac{a_{n-2}^2}{a_{n-3} a_{n-1}} \geq \frac{a_{n-1}^2}{a_{n-2} a_n} \geq \frac{10}{3}, \tag{7}$$

and further

$$0 \leq \frac{a_{n-2}^2}{a_{n-3} a_{n-1}} - \frac{a_{n-1}^2}{a_{n-2} a_n} \leq \frac{1}{5}, \tag{7a}$$

the polynomial (1) has all zeros real and negative, one in each of the intervals

$$\begin{gathered} \left(-\frac{2a_0}{a_1}, 0\right), \left(-\frac{2a_1}{a_2}, -\frac{2a_0}{a_1}\right), \ldots, \left(-\frac{2a_{n-3}}{a_{n-2}}, -\frac{2a_{n-4}}{a_{n-3}}\right), \\ \left(-\frac{3a_{n-2}}{2a_{n-1}}, -\frac{2a_{n-3}}{a_{n-2}}\right), \left(-\frac{3a_{n-1}}{2a_n}, -\frac{3a_{n-2}}{2a_{n-1}}\right). \end{gathered} \tag{8}$$ [1]

[1] For $n = 3$ one takes $a_{-1} = 0$ in (6) and (8).

Proof of Theorem 1. As in [2], consider the polynomial

$$Q(z) = 1 + \frac{z}{r_1} + \frac{z^2}{r_1 r_2} + \cdots + \frac{z^n}{r_1 r_2 \ldots r_n} \quad (n \geq 3) \tag{9}$$

where $0 < r_1 < r_2 < \cdots < r_n$ and moreover

$$\frac{r_2}{r_1} \geq \frac{r_3}{r_2} \geq \cdots \geq \frac{r_{n-1}}{r_{n-2}} \geq \frac{r_n}{r_{n-1}} > 3 \,. \tag{10}$$

Let

$$\frac{r_2}{r_1} = 3 + c_1, \; \frac{r_3}{r_2} = 3 + c_2, \ldots, \frac{r_n}{r_{n-1}} = 3 + c_{n-1} \,, \tag{11}$$

with

$$c_1 \geq c_2 \geq \cdots \geq c_{n-1} > 0 \,. \tag{12}$$

The modulus of the k-th term in (9) is maximal for $r_k < |z| < r_{k+1}$ $(k = 1, 2, \ldots, n-1)$. For these values of z, the moduli of the terms of the polynomial (9) continuously increase from the initial term 1 to the maximal one, then decrease from the maximal term to the last one [1] (volume I, part I, problem 117).

Let us first consider the case when $n > 3$ in (9). Let $(r_1 r_2 \ldots r_k)^{-1} \cdot (-x)^k$ be the maximal term in (9) for $z = -x$ $(x > 0)$. One then obtains from (9)

$$\begin{aligned} Q(-x)(r_1 r_2 \ldots r_k)(-x)^{-k} &= \left(1 - \frac{x}{r_{k+1}} + \frac{x^2}{r_{k+1} r_{k+2}} - \frac{x^3}{r_{k+1} r_{k+2} r_{k+3}} + \cdots \right. \\ &\left. \cdots + (-1)^{n-k} \frac{x^{n-k}}{r_{k+1} r_{k+2} r_{k+3} \cdots r_n} \right) - \\ &- \frac{r_k}{x} + \frac{r_k r_{k-1}}{x^2} - \cdots + (-1)^k \frac{r_k r_{k-1} \cdots r_1}{x^k} \,, \end{aligned} \tag{13}$$

with

$$\begin{aligned} &Q(-x)(r_1 r_2 \ldots r_k)(-x)^{-k} \geq \\ &\geq 1 - \frac{x}{r_{k+1}} + \frac{x^2}{r_{k+1} r_{k+2}} - \frac{x^3}{r_{k+1} r_{k+2} r_{k+3}} - \frac{r_k}{x} \,. \end{aligned} \tag{14}$$

For $x = 2r_k$ $(k = 1,2,\dots,n-3)$, having in mind (11) and (12), one obtains from (14)

$$(15)\qquad Q(-2r_k)(r_1r_2\dots r_k)(-2r_k)^{-k} \geq$$
$$\frac{1}{2} - \frac{2r_k}{r_{k+1}} + \frac{1}{2}\left(\frac{2r_k}{r_{k+1}}\right)^2\left(\frac{2r_{k+1}}{r_{k+2}}\right) - \left(\frac{2r_k}{r_{k+1}}\right)^3\left(\frac{r_{k+1}}{r_{k+2}}\right)^2\left(\frac{r_{k+2}}{r_{k+3}}\right) =$$
$$\frac{1}{2} - \frac{2}{3+c_k} + \frac{1}{2}\left(\frac{2}{3+c_k}\right)^2\left(\frac{2}{3+c_{k+1}}\right) - \left(\frac{2}{3+c_k}\right)^3\left(\frac{1}{3+c_{k+1}}\right)^2\left(\frac{1}{3+c_{k+2}}\right) >$$
$$\frac{1}{2} - \frac{2}{3+c_k} + \frac{1}{2}\left(\frac{2}{3+c_k}\right)^3 - \frac{1}{27}\left(\frac{2}{3+c_{k+1}}\right)^3 = \frac{1}{2} - \frac{2}{3+c_k} + \frac{25}{54}\left(\frac{2}{3+c_k}\right)^3 .$$

If we put

$$(16)\qquad \begin{aligned} \frac{2}{3+c_k} &= t_k , \\ \frac{1}{2} - t_k + \frac{25}{54}t_k^3 &= h(t_k) \end{aligned}$$

we obtain from (15)

$$(17)\qquad Q(-2r_k)(r_1r_2\dots r_k)(-2r_k)^{-k} > h(t_k).$$

The function $h(t_k)$ has zeros $u_1 = \frac{-3-\sqrt{189}}{10}$, $u_2 = \frac{3}{5}$ and $u_3 = \frac{-3+\sqrt{189}}{10}$, with $h(t_k) \geq 0$ for $t_k \leq \frac{3}{5}$. The condition $t_k \leq \frac{3}{5}$, by (16), reduces to $\frac{2}{3+c_k} \leq \frac{3}{5}$, wherefrom one obtains $c_k \geq \frac{1}{3}$. Now, by (12), conditions (11) reduce to

$$(18)\qquad \frac{r_2}{r_1} \geq \frac{r_3}{r_2} \geq \cdots \geq \frac{r_{n-1}}{r_{n-2}} \geq \frac{r_n}{r_{n-1}} \geq \frac{10}{3} .$$

When conditions (18) are satisfied, with $r_1 > 0$, for $x = 2r_{n-2}$ one obtains from (13)

$$(19)\qquad Q(-2r_{n-2})(r_1r_2\dots r_{n-2})(-2r_{n-2})k)^{-(n-2)} \geq$$
$$\frac{1}{2} - \frac{2r_{n-2}}{r_{n-1}} + \frac{1}{2}\left(\frac{2r_{n-2}}{r_{n-1}}\right)^2\left(\frac{2r_{n-1}}{r_n}\right) \geq \frac{1}{2} - \frac{2r_{n-2}}{r_{n-1}} + \frac{1}{2}\left(\frac{r_{n-2}}{r_{n-1}}\right)^3 > 0 .$$

For $n = 3$, one obtains from (19)

$$(20)\qquad Q(-2r_1)(r_1)(-2r_1)^{-1} > 0 .$$

Having in mind (15), (17), (19) and (20), we conclude that

$$(21)\quad Q(-2r_k)(r_1r_2\dots r_k)(-2r_k)^{-k}>0 \quad (k=1,2,\dots,n-2).$$

As $Q(0)>0$ and as by (21) $Q(-2r_1)<0$, $Q(-2r_2)>0,\dots,(-1)^{n-2}\cdot$ $\cdot Q(-2r_{n-2})>0$, we conclude that the polynomial (9) has at least $n-2$ real negative zeros, at least one in each of the intervals

$$(22)\qquad (-2r_1,0),\ (-2r_2,-2r_1),\cdots,(-2r_{n-2},-2r_{n-3})\ ,^{2)}$$

if conditions (18) are satisfied and $r_1>0$.

Dividing (1) by a_0 one obtains the polynomial

$$\frac{P(z)}{a_0}=1+\frac{1}{\frac{a_0}{a_1}}+\frac{z^2}{\frac{a_0}{a_2}}+\cdots+\frac{z^n}{\frac{a_0}{a_n}},$$

which can be written in the form

$$(23)\qquad \frac{P(z)}{a_0}=1+\frac{1}{\frac{a_0}{a_1}}+\frac{z^2}{\frac{a_0}{a_1}\frac{a_1}{a_2}}+\cdots+\frac{z^n}{\frac{a_0}{a_1}\frac{a_1}{a_2}\cdots\frac{a_{n-1}}{a_n}}\,.$$

If putting in (23)

$$(23a)\qquad \frac{a_0}{a_1}=r_1,\ \frac{a_1}{a_2}=r_2,\cdots,\frac{a_{n-1}}{a_n}=r_n,\ \frac{P(z)}{a_0}=Q(z)$$

one should obtain the polynomial (9). Conditions (18) then reduce to (5) and the intervals (22) reduce to (6), which completes the proof of Theorem 1.

Proof of Theorem 2. Let us consider the polynomial (9) with $r_1>0$,

$$(24)\qquad \frac{r_2}{r_1}\geq\frac{r_3}{r_2}\geq\cdots\geq\frac{r_{n-1}}{r_{n-2}}\geq\frac{r_n}{r_{n-1}}\geq\frac{10}{3}$$

and

$$(24a)\qquad 0\leq\frac{r_{n-1}}{r_{n-2}}-\frac{r_n}{r_{n-1}}\leq\frac{1}{5}\,.$$

[2] For $n=3$ the intervals (22) reduce to $(-2r_1,0)$.

From (24) and (24a) one obtains

$$\frac{r_{n-2}}{r_{n-1}} \geq \frac{r_{n-1}}{r_n} - \frac{9}{500}\,. \tag{25}$$

Under conditions (24) relations (21) hold. For $k = n-1$ and $x = = \frac{3}{2}r_{n-1}$, by (24) and (25), one obtains, from (13)

$$\begin{aligned}
&Q\left(-\frac{3}{2}r_{n-1}\right)(r_1r_2\dots r_{n-1})\left(-\frac{3}{2}r_{n-1}\right)^{-(n-1)} \geq \\
&\frac{1}{3} - \frac{3}{2}\frac{r_{n-1}}{r_n} + \frac{4}{9}\frac{r_{n-2}}{r_{n-1}} - \frac{8}{27}\left(\frac{r_{n-2}}{r_{n-1}}\right)^2\left(\frac{r_{n-3}}{r_{n-2}}\right) \geq \frac{1}{3} - \frac{3}{2}\frac{r_{n-1}}{r_n} + \\
&+\frac{4}{9}\left(\frac{r_{n-1}}{r_n} - \frac{9}{500}\right) - \frac{8}{27}\left(\frac{3}{10}\right)^3 = \frac{1}{3} - \frac{19}{18}\frac{r_{n-1}}{r_n} - \frac{4}{500} - \frac{1}{125} \geq \\
&\geq \frac{1}{3} - \frac{19}{18}\cdot\frac{3}{10} - \frac{4}{500} - \frac{1}{125} = \frac{1}{1500}\,.
\end{aligned} \tag{26}$$

For $k = n$ and $x = \frac{3}{2}r_n$, there follows from (13)

$$Q\left(-\frac{3}{2}r_n\right)(r_1r_2\dots r_n)\left(-\frac{3}{2}r_n\right)^{-n} \geq \frac{1}{3}\,. \tag{27}$$

We see, from the polynomial (9) and the relations (21), (26) and (27), that $Q(0) > 0$, $Q(-2r_1) < 0$, $Q(-2r_2) > 0, \dots, Q(-2r_{n-2})(-1)^{n-2} > 0$, $Q\left(-\frac{3}{2}r_{n-1}\right)(-1)^{n-1} > 0$, $Q\left(-\frac{3}{2}r_n\right)(-1)^n > 0$, wherefrom we conclude that the polynomial (9) has only simple real negative zeros, one in each of the intervals

$$\begin{gathered}
(-2r_1, 0),\ (-2r_2, -2r_1), \cdots, (-2r_{n-2}, -2r_{n-3}), \\
\left(-\frac{3}{2}r_{n-1}, -2r_{n-2}\right),\ \left(-\frac{3}{2}r_n, -\frac{3}{2}r_{n-1}\right)^{3)}
\end{gathered} \tag{28}$$

when conditions (24), (24a) are satisfied and $r_1 > 0$.

By (23) and (23a), conditions (24) and (24a) reduce to conditions (7) and (7a), and intervals (28) to intervals (8), by which the proof of Theorem 2 is completed.

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3) For $n = 3$ intervals (28) reduce to intervals $(-2r_1, 0)$, $\left(-\frac{3}{2}r_2, -2r_1\right)$, $\left(-\frac{3}{2}r_3, -\frac{3}{2}r_2\right)$, and intervals (8) reduce to intervals $\left(-\frac{2a_0}{a_1}, 0\right)$, $\left(-\frac{3a_1}{2a_2}, -\frac{2a_0}{a_1}\right)$, $\left(-\frac{3a_2}{2a_3}, -\frac{3a_1}{2a_2}\right)$.

Theorem 3. *If in (1)* $a_k > 0$ $(k = 0,1,2,\dots,n)$ *and if*

$$\frac{a_k^2}{a_{k-1}a_{k+1}} \geq 4 + 2h \ (h > 0), \quad (k = 1,2,\dots,n-1)\,, \tag{29}$$

the polynomial (1) is different from zero in each of the intervals

$$\left[-\frac{(2+h)a_{k-1}}{a_k},\ -\frac{2a_{k-1}}{a_k}\right] \quad (k = 1,2,\dots,n-1)\,. \tag{30}$$

Proof. Consider the polunomial (9) where $r_1 > 0$ and where

$$\frac{r_{k+1}}{r_k} \geq 4 + 2h \ (h > 0), \quad (k = 1,2,\dots,n-1)\,, \tag{31}$$

If dividing (9) by $\frac{z^k}{r_1 r_2 \dots r_k}$ $(z \neq 0)$, one obtains

$$\begin{aligned} Q(z)(r_1 r_2 \dots r_k)z^{-k} = \\ = \left(\frac{1}{2} + \frac{z}{r_{k+1}} + \frac{z^2}{r_{k+1}r_{k+2}} + \dots + \frac{z^{n-k}}{r_{k+1}r_{k+2}\dots r_n}\right) + \\ + \left(\frac{1}{2} + \frac{r_k}{z} + \frac{r_k r_{k-1}}{z^2} + \dots + \frac{r_k r_{k-1}\dots r_1}{z^k}\right) \end{aligned} \tag{32}$$

Let $z = |z|e^{\theta i}$ $(0 \leq \theta \leq 2\pi)$. Then the real part in the first bracket, at the right hand side of (32), is

$$A = \frac{1}{2} + \frac{|z|}{r_{k+1}}\cos\theta + \frac{|z|^2}{r_{k+1}r_{k+2}}\cos 2\theta + \dots + \frac{|z|^{n-k}}{r_{k+1}r_{k+2}\dots r_n}\cos(n-k)\theta \tag{33}$$

and the real part in the second bracket at the right hand side of (32) is

$$B = \frac{1}{2} + \frac{r_k}{|z|}\cos\theta + \frac{r_k r_{k-1}}{|z|^2}\cos 2\theta + \dots + \frac{r_k r_{k-1}\dots r_1}{|z|^k}\cos k\theta\,. \tag{34}$$

We shall prove that

$$A + B > 0$$

for every θ and every z for which

$$2r_k \leq |z| \leq (2+h)r_k, \quad (k = 1,2,\dots,n-1)\,, \tag{35}$$

when conditions (31) are satisfied and where $n \geq 3$.

Let

$$S = c_0 + c_1 \cos\theta + c_2 \cos 2\theta + \cdots + c_m \cos m\theta \, . \tag{36}$$

If we put

$$S_0 = 1, \quad S_1 = 1 + \cos\theta, \quad S_2 = 1 + \cos\theta + \frac{\cos 2\theta}{2}, \ldots$$
$$\ldots, S_v = 1 + \cos\theta + \frac{\cos 2\theta}{2} + \cdots + \frac{\cos v\theta}{v} \, ,$$

(36) can be written in the form

$$S = (c_0 - c_1)S_0 + (c_1 - 2c_2)S_1 + \cdots + [(m-1)c_{m-1} - mc_m]\, S_{m-1} + mc_m S_m . \tag{37}$$

By [1] (vol. II, part VI, problem 28), for every θ and every $v = 2, 3, \ldots$, there is

$$S_v = 1 + \cos\theta + \frac{\cos 2\theta}{2} + \cdots + \frac{\cos v\theta}{v} > 0 \, , \tag{38}$$

whereas

$$1 + \cos\theta \geq 0 \, . \tag{39}$$

If we have in (36)

$$c_0 \geq c_1 > 2c_2 > 3c_3 > \cdots > (m-1)c_{m-1} > mc_m > 0 \, , \tag{40}$$

then, by (37), because of (38) and (39), there is

$$S \geq 0 \, .$$

If one puts in (36)

$$c_0 = \frac{1}{2}, \quad c_1 = \frac{|z|}{r_{k+1}}, \quad c_2 = \frac{|z|^2}{r_{k+1} r_{k+2}}, \cdots$$
$$\cdots, c_m = c_{n-k} = \frac{|z|^{n-k}}{r_{k+1} r_{k+2} \cdots r_n} \, ,$$

one obtains $S = A$. Further

$$c_{v+1} = c_v \cdot \frac{|z|}{r_{k+v+1}}\,, \qquad (v = 1, 2, \ldots, n-k-1)\,.$$

By (31) and (35), for every $v = 1, 2, \ldots, n-k-1$, one has

$$\begin{aligned} vc_v - (v+1)c_{v+1} &= c_v\left[v - (v+1)\frac{|z|}{r_{k+v+1}}\right] \geq c_v\left[v - (v+1)\frac{(2+h)r_k}{r_{k+v+1}}\right] = \\ &= c_v\left[v - (v+1)(2+h)\frac{r_k}{r_{k+1}} \cdot \frac{r_{k+1}}{r_{k+2}} \cdots \frac{r_{k+v}}{r_{k+v+1}}\right] \geq \\ &\geq c_v\left[v - (v+1)(2+h)\frac{1}{2^{v+1}(2+h)^{v+1}}\right] = \\ &= c_v\left[v - (v+1)\frac{1}{2^{v+1}(2+h)^v}\right] > c_v\left(v - \frac{v+1}{2^{2v+1}}\right) > 0\,. \end{aligned}$$

As we have further

$$c_0 - c_1 = \frac{1}{2} - \frac{|z|}{r_{k+1}} \geq \frac{1}{2} - \frac{(2+h)r_k}{r_{k+1}} \geq \frac{1}{2} - \frac{2+h}{2(2+h)} = 0\,,$$

conditions (40) hold, and therefrom $A \geq 0$.

If putting in (36)

$$c_0 = \frac{1}{2}, \quad c_1 = \frac{r_k}{|z|}, \quad c_2 = \frac{r_k r_{k-1}}{|z|^2}, \cdots, c_m = c_k = \frac{r_k r_{k-1} \cdots r_1}{|z|^k}\,,$$

one shall obtain $S = B$. In this case

$$c_{v+1} = c_v \cdot \frac{r_{k-v}}{|z|}\,, \qquad (v = 1, 2, \ldots, k-1)\,.$$

By (31) and (35), for every $v = 1, 2, \ldots, k-1$, one has

$$\begin{aligned} vc_v - (v+1)c_{v+1} &= c_v\left[v - (v+1)\frac{r_{k-v}}{|z|}\right] \geq c_v\left[v - (v+1)\frac{r_{k-v}}{2r_k}\right] = \\ &= c_v\left[v - \frac{v+1}{2}\frac{r_{k-1}}{r_k} \cdot \frac{r_{k-2}}{r_{k-1}} \cdots \frac{r_{k-v}}{r_{k-v+1}}\right] \geq \\ &\geq c_v\left[v - \frac{v+1}{2}\frac{1}{2^v(2+h)^v}\right] > \\ &> c_v\left(v - \frac{v+1}{2^{2v+1}}\right) > 0\,. \end{aligned}$$

As we have further

$$c_0 - e_1 = \frac{1}{2} - \frac{r_k}{|z|} \geq \frac{1}{2} - \frac{r_k}{2r_k} = 0 ,$$

conditions (40) hold also in this case; therefrom $B \geq 0$.

For $n \geq 3$ one has $A \geq 0$ and $B \geq 0$, but

$$A + B > 0 . \tag{41}$$

Having in mind (32), (33) and (34), we conclude, from (41), that the polynomial $Q(z)$ in (9) is different from zero in the domains (35) when conditions (31) are satisfied for $n \geq 3$. The domains (35) now become

$$\frac{2a_{k-1}}{a_k} \leq |z| \leq \frac{(2+h)a_{k-1}}{a_k} , \quad (k = 1, 2, \ldots, n-1) . \tag{42}$$

We conclude, from (42), that the polynomial (1) is different from zero in every of the intervals (30), which completes the proof of Theorem 3.

3. The Theorem A) holds also under conditions (29), which means that the polynomial (1) has also only simple real negative zeros, one in each of the intervals (3a). By Theorem 3, the polynomial (1) has no zeros in the intervals (30). These two facts will be stated by the next theorem.

Theorem 4. *If in (1)* $a_k > 0$ $(k = 0, 1, 2, \ldots, n)$ *and if*

$$\frac{a_k^2}{a_{k-1}a_{k+1}} \geq 4 + 2h \quad (h > 0), \quad (k = 1, 2, \ldots, n-1) ,$$

the polynomial (1) has only simple real negative zeros, one in each of the intervals

$$\left(-\frac{2a_0}{a_1}, 0\right), \left(-\frac{2a_1}{a_2}, -\frac{(2+h)a_0}{a_1}\right), \left(-\frac{2a_2}{a_3}, -\frac{(2+h)a_1}{a_2}\right), \ldots,$$
$$\left(-\frac{2a_{n-1}}{a_n}, -\frac{(2+h)a_{n-2}}{a_{n-1}}\right) .$$

4. References

1. G. Pólya und G. Szegö: *Aufgaben und Lehrsätze aus der Analysis*, Berlin 1954.
2. D. M. Simeunović: *Sur la répartition des zéros d'une classe de polynômes*, Publ. Inst. Math., **28 (42)**, 1980, 187-194.

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