MATHEMATICA MORAVICA
VOL. 2 (1998), 9–19

IMPLEMENTATION OF MULTI-CRITERION DECISION MAKING MODELS RECOMMENDATIONS

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Abstract. This paper gives some introductory notes on multi-criterion decision making. It addresses to the known division into models of multi-purpose decision making and multi-attribution decision making or multi-criterion analysis, general notion of models, terminology, taxonomy of methods as well as significance of interactive methods and the role of a decision maker in the procedure of solution making. The first out of a set of significant facets of implementation of multi-criterion decision making models is shown further: marginal solutions and their consequences, that is an examination of a possibility of perfect solution of the said model.

1. Introductory considerations

There is a steady opinion prevailing in both theory and practice that a mono-criterion approach results into too rough and in many cases unacceptable approximation of reality thus distorting a problem and leading to an inadequate decision.

Making a choice of a decision making is mostly affected by nature of the given problem and current conditions under which it is realized. Whereas it is relatively easy to get a solution within mono-criterion models, it is not the case with multi-criterion ones.

Common features of each multicriterion problem can be seen within presence of elements as follows: (1) more criteria for decision making with

AMS Mathematics Subject Classification 1991. Primary: 11B39, 11B37.
Key words: Multi-criterion optimization, multi-purpose decision making, multi-attribution optimization, multi-criterion analysis, mathematical model, methods, software.

eventual separation of global criteria and their subcriteria, (2) criteria being conflict, (3) unequal levels of criteria signifilance, (4) more alternatives (solutions) for choice, (5) solving procedure and (6) the role of decision maker in modelling and solution making.

a) Mathematical approach to the model of multi-criterion decision making

Within multi-criterion decision making two kinds of problems are evident (See Table 1) multi-purpose decision making and multi-attribution decision making or multi-criterion analysis. The first ones are known as "well structured problems" and second ones as "poor structured problems". Problems of multi-purpose decision making can be seen as models of purpose programming showing some specific facets: deviation variables, priority factors, purpose (target) function coefficients to deviation variables having the same priority factor given in this paper furtheon.

Multi-purpose decision making model parameters being: n=number of variables, J=set of indexes (j) for variables, p=set of criterion functions, K=set number of indexes (k) for criterion functions, m=number of limitations, I=set of indexes (i) for limitations, x = n-dimensional vector of variables x_j, $f_k(x)$=criterion functions, $g_i(x)$=parts of limitations with variables, b_i=free members of limitations. Multi-attribution decision making parameters being: n=number of criteria (attributes), J=set of indexes (j) for criteria, m=number of alternatives (actions), I=set of indexes (i) for alternatives, f_j=criteria (attributes), a_i=alternative (action), A=set of each alternative (action), f_{ij}=value of j-criterion for i^{th}-action (alternative).

It is almost standard to maximize functions of criteria, since criteria of minimization can be transferred to criteria of maximization. Example being, for criterion exponents applies: $\min f_s(x) = -\max\{-f_s(x)\}$.

b) Basic ideas and terms for multi-criterion decision making

Unlike to mono-criterion problems, problems of multi-criterion decision making are known for possibility of numerous efficient solutions bearing suitable values of criteria, in case there is not a perfect solution providing extremes values for each criterion. From what has been said above it follows that terms for multi-criterion decision making are complex.

Table 1. Mathematical models and comparation of two groups of problems

Multi-criterion dec. mak. problem	Multi-purpose dec. mak.	Multi-attribution dec. mak. dec. mak. or multi-criterion analysis
Mathematical models	$\text{Max}\{f_k(x),\quad \forall k \in K\}$ p.o. $g_i(x) \le b_i, \quad \forall i \in I_1$. $g_i(x) = b_i, \quad \forall i \in I_2$ $g_i(x) \ge b_i, \quad \forall i \in I_3$ $x_j \ge 0 \quad \forall j \in J;$ where:$x = \{x_1, ..., x_n\};$ $K = \{1, \ldots, p\}, \; p \ge 2;$ $I_1 \cup I_2 \cup I_3 = I;$ $I_1 \cap I_2 \cap I_3 = \emptyset$ $I = \{1, \ldots, m\};$ $J = \{1, \ldots, n\}; \; n \ge 2$	$\text{Max}\{f_j(A),\quad \forall j \in J\}$ $\begin{matrix} & f_1 & \cdots & f_n \\ a_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_m & a_{m1} & \cdots & a_{mn} \end{matrix}$ where: $A = \{a_1, \ldots, a_m\};$ $f_{ij} = f_j(a_i), \; \forall (i,j);$ $J = \{1. \ldots, n\}, \quad n \ge 2;$ $I = \{1, \ldots, m\}, m \ge 2;$
Evidence of many criteria defined with	Targets criteria	Attributions
Target	Explicit	Implicite
Attribution	Implicit	Explicite
Limitations	Active	Inactive
Application (model solution)	Projecting (solution finding out and choice)	Choice, evolution (solutions known)

1) Variables space: Multi-purpose decision making model allows solution in the space of variables shown as $x = \{x_1, x_2, \ldots, x_n\} \subseteq R_n$, if limitations aren't contradictory and set of solutions $X = \{x | g_i(x) \le b_i,\ \forall i \in I_1;\ g_i(x) = b_i,\ \forall i \in I_2;\ g_i(x) \ge b_i,\ \forall i \in I_3;\ x \ge 0\}$ that can eventually be reduced to one solution only. Set of allowed solutions of multi-attribution decision making is represented with alternatives, $A = \{a_1, \ldots, a_m\}$.

2) Criterion function space: To each allowable solution X of multi-purpose decision making model corresponds a set of values of criteria functions, that is a vector $f(x) = [f_1(x), f_2(x), \ldots, f_p(x)]$, so that a set of allowable solutions X can be translated into criterion set $S = \{f(x)|x \in X\}$. A matrix bearing values for criteria of alternatives specified in definition of multi-attribution decision making model, so-called matrix of decision making $[f_{ij}]$ is a space of criterion function as well.

3) Marginal solutions: Optimizing each function defines marginal solutions (optimal solutions for individual criterion. Thus, for multi-purpose decision making (we consider $\forall k \in K$) applies: $x^{(k)*} \Leftrightarrow f_k(x^{(k)*}) \geq f_s(x), \forall s \in K$, where $x \in X$, $x^{(k)*} \in X$; and for multi-attribution decision making (we consider $\forall j \in J$): $x^{(j)*} = a_i \Leftrightarrow f_{ij} \geq f_{sj}, \forall s \in J$.

4) Ideal criterion value and ideal point: Marginal solutions determine corresponding ideal values of functions of criteria: $f_k^* = f_k(x^{(k)*})$, $k \in K$ (for multi-purpose decision making); that is $f_j^* = f_j(x^{(j)*})$, $j \in J$ (for multi-attribution decision making). Ideal criterion determines ideal point in criterion space S (ideal value of vector function): $f^* = [f_1^*, f_2^*, \ldots f_p^*]$ for multi-purpose decision making and $f^* = [f_1^*, f_2^*, \ldots, f_n^*]$ for multi-attributionn decision making.

5) Table of marginal solutions consequences: Marginal solution of one criterion function defines values for each other criterion which are, mostly, for all or even for one, more or less, unsuitable than their ideal values. For the example or multi-purpose decision making (any $k \in K$ considered) applies: $f_{ks} = f_s(x^{(k)*}) \leq f_s^*$, $\forall s \in K$. To analyse this type of model it is recommendable to make a table of consequences of each marginal solution (Table 2), so that the ideal criterion values are set to the major diagonal. In case any r-th criterion has multiple marginal solution, set $X^{(r)*} = \{x^{(r)*}\}$, as a consequence for each k-th criterion a best solution is determined (that one which provides the best value to k-th criterion): $f_{rk} = \max f_k(x^{(r)*})$, $\forall s \in K$, $k \neq r$. Using analogy, it's possible to take a part with marginal solutions out of a decision making matrix of multi-attribution model.

6) Perfect solution: The perfect solution is that solution x^* which maximizes each criterion function at the same time: $x^* \Leftrightarrow f_k(x^*) = f_k^*$, $\forall k \in K$ in the model of multi-purpose decision; $x^* \Leftrightarrow f_j(x^*) = f_j^*$, $\forall j \in J$ in the model of multi-attribution decision making. When there is a permissible perfect solution ($x^* \in X$ for multi-purpose decision making, $x^* \in X$

for multi-attribution dec. mak.), there isn't any problem of multi-criterion dec. mak. in fact. each marginal solution, thus, is equal, some of them being multiple solution. Models of multi-criterion decision making, however, in general, don't have any permissible perfect solution, but diversity (of all or even some) of marginal solution.

7) Efficient solution: The permissible solution x^e is an efficient solution, Paretho–optimal solution or nondominant solution, uncess there is some other permissible solution x^0 for which, using example of multi-purpose decision making model, applies: $\{f_k(x^0) \geq f_k(x^e), \forall k \in K\} \wedge \{f_s(x^0) > f_s(x^e)$ for at least one $s \in K$, $s \neq k\}$

Table 2. Consequences of marginal solutions of multi-purpose decision making model

Optimized criteria	Marginal solutions	Consequences of marginal solutions to criteria			
		f_1	f_2	$\cdots$	f_p
f_1	$x^{(1)*}$	$f_1^* = f_{11}$	f_{12}	$\cdots$	f_{1p}
f_2	$x^{(2)*}$	f_{21}	$f_2^* = f_{22}$	$\cdots$	f_{2p}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
f_p	$x^{(p)*}$	f_{p1}	f_{p2}	$\cdots$	$f_p^* = f_{pp}$
Ideals, f_k^*		f_1^*	f_2^*	$\cdots$	f_p^*

8) Inefficient (dominated) solution: If there are permissible solutions x^+ and $\underline{x}$ such that x^+ is better than $\underline{x}$ accordin to at least one criterion, and at the same time not worse according to all other criteria, then $\underline{x}$ is an inefficient solution. In that case, for multi-purpose decision making stands true: $\{f_s(x) > f_s(\underline{x})$, for at least one $s \in K\}$, $\wedge\{f_k(x) \geq f_k(\underline{x})$, $\forall k \in K$, $k \neq s\}$.

Note: an inefficient solution should be omitted from further analysis. For x^+ being better solution than $\underline{x}$, however, makes $\underline{x}$ unambiguously inefficient solution without being compulsory for x^+ to be efficient. Maybe there is a solution x' better than solution x^+, so that it is found out that x^+ is an inefficient solution no matter that x^+ is better than $\underline{x}$.

9) Compromise solution. A compromise solution is such $\underline{x}^*$ solution which provides the least possible deviation form ideal values of criterion functions. A measure of distance of ideal point f^* in linear models of multi-purpose decision making is given by any of L_r-metrica (L_r is used here instead of an usual L_p, sine "p" is a number of criteria in a model of multi-purpose decision making): $||f(x) - f^*|| = [\sum_k |f_k(x) - f_k^*|^r]^{1/r}$, $1 \leq r \leq \infty$.

10. Optimal solution of monocriterion and multicriterion models; In terms of an optimal solution in monocriterion models, which can be a multiple optimal solution, for a model of multi-purpose decision making it stands: an optimal solution of multi-purpose decision making model, in mathematical terms, is a set of all efficient solutions, $X_e = \{x^e\}$.

Note: It's clear that marginal solutions are efficient solutions at the same time vhich can be proved for a compromise solution too. More complex model of multi-purpose decision making, most frequently, has many other efficient solutions besides marginal ones.

11. Final solution - solution for application; For a concrete application in a real problem, multi-purpose decision making is used for one (efficient) solution only. It is called a solution for realization preferred solution, best solution, best compromise solution and alike. The most used term, however, is a compromise solution even if it doesn't suit to definition 9).

Regarding nature of each concrete problem of multi-attribution decision making, three basic approaches to its solution are practicable: (I) **problem of ranking**, attempting to rank a set of all variants (actions, knots, projects etc.) from "best" to "worst"; (II) **problem of an alternative choice**, there is a need to choose the "best" alternative or (III) **problem of more alternative choice**, many alternative are chosen taking into consideration some additional requirements.

2. Significance of marginal solutions of multi-purpose decision making model and their consequences

Adopting a need to define more criteria and establish appropriate models of multi-purpose decision making models in real assignments, we pass to the problem of choice a solution method. There isn't any recommendation deriving from theory, but a common consent is that interactive approach allows the best possible analysis of a problem. The question of a suitable method choice, however, still remains.

An initial analysis, using marginal solutions to gain more clear data on criterion space (and the range of possible values of individual criterion) and find out a possible perfect solution, is not so prominent within small size models, mainly present in literature, especially when there are two variables only thus being easily noticable araphically.

General recommendation that in case of multiple optimal solution of any k-th criterion, there is a need to define the most suitable solution to each unoptimized s-th criterion, that is the max value f_{ks}, retaining optimal value f_k^*, isn't explained in details by the majority of authors.

Regarding the said above and the problem of establishing multiple solutions, we recommend a clear algorithm, based upon lexicographic procedure, for determination of ideal criteria and their consequences in the model of multi-purpose decision making. A part of Table 3 shows that the example of three criteria only requires even fifteen models solving.

Table 3. Results of solving models of multi-purpose decision making

Solving procedure		Solutions	f_1	f_2	f_3
Lexico-graphic procedure	$\text{opt}(f_1, f_2, f_3)$	$x^{(1,1)*}$	$f_1^* = f_{111}$	f_{121}	f_{131}
	$\text{opt}(f_1, f_3, f_2)$	$x^{(1,2)*}$	$f_1^* = f_{112}$	f_{122}	f_{132}
	$\text{opt}(f_2, f_1, f_3)$	$x^{(2,1)*}$	f_{211}	$f_2^* = f_{221}$	f_{231}
	$\text{opt}(f_2, f_3, f_1)$	$x^{(2,2)*}$	f_{212}	$f_2^* = f_{222}$	f_{232}
	$\text{opt}(f_3, f_1, f_2)$	$x^{(3,1)*}$	f_{311}	f_{321}	$f_3^* = f_{331}$
	$\text{opt}(f_3, f_2, f_1)$	$x^{(3,2)*}$	f_{312}	f_{322}	$f_3^* = f_{332}$
Ideals f_k^*			f_1^*	f_2^*	f_3^*
Methods of multi-purpose decision making		$x^{(7)}$	f_{41}	f_{42}	f_{43}
		$x^{(8)}$	f_{51}	f_{52}	f_{53}
		...	...	...	...
		...	...	...	...

Analysis of above gained solutions allows us to see if there is a perfect solution providing optimal values for all criteria or a solution providing optimal values for more criteria (not all). If there is a perfect solution, it is, of course, a final one and we lack a multi-criterion problem in a true sense of word. On the contrary, when there isn't any perfect solution, any marginal solution can be regarded as a final one or it is possible to try to find new solutions using any method of multi-purpose decision making as shown in a lower part of Table 3.

In the end, we have to chose the final solution proven to be the most acceptable one which can be any of the previously reached solutions from the set of marginal or new solutions.

3. Conclusion

Multi-criterion decision making and optimization considered to be a necessity to support a stationary strategy or decision making. We are to expect multi-criterion approach in sequential decision making applicable to dynamic systems of decision making support.

Supplement

Example 1. We illustrate a space of permissible values of variables and criterion space (X and S, respectively) for problem of multi-purpose decision making having $p = 2$ criteria, $n = 2$ variables and $m = 3$ limitations (figure 1 and 2).

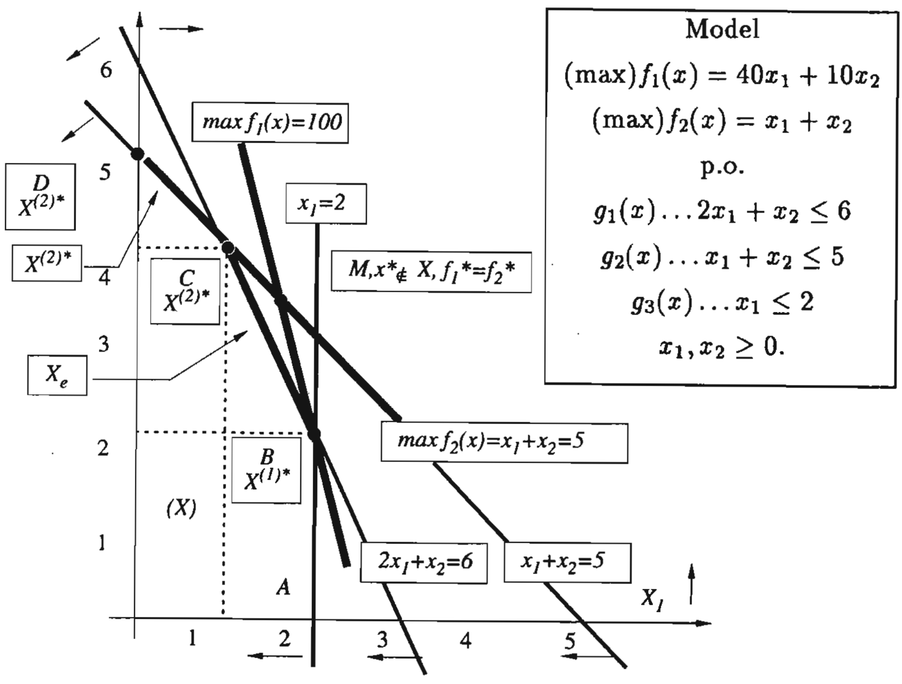


Figure 1. Space of variables (X)

The ideal values of criteria are $f_1^* = 100$ and $f_2^* = 5$ when the first criterion has a unique marginal solution $x^{(1)*} = (2, 2)$ at the point $B(2, 2)$, and the second one multiple marginal solution $X^{(2)*} = (x_1 = 5 - x_2, 0 \leq x_2 \leq 5)$ along a segment CD with $C(1, 4)$ and $D(0, 5)$.

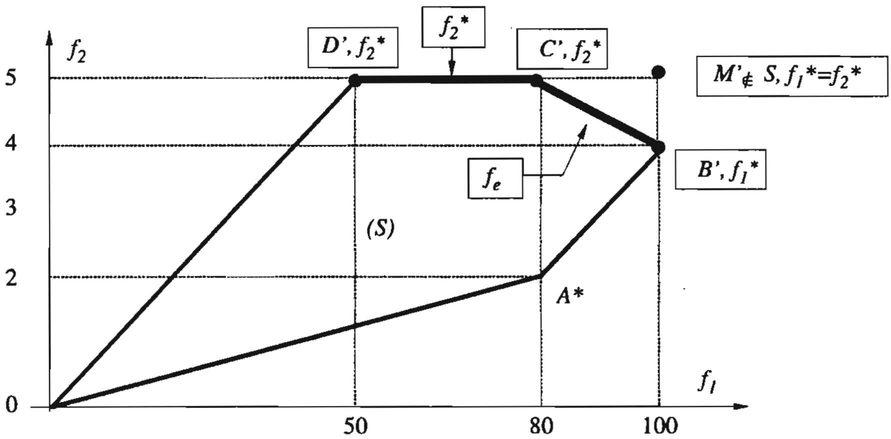


Figure 2. Space of criteria (S)

The implications of marginal solutions to criteria are given in the Table 4.

The perfect solution $x^* = (5/3, 10/3)$ isn't a member of a set of permissible solutions X, so that the ideal point $f^* = (100, 50)$ doesn't belong to a criterion set S.

To a set of efficient solutions, $X_e = (1 \leq x_1 \leq 5,\ x_2 = 6 - 2x_1)$, being at a segment BC with $C(2,2)$ corresponds a set of efficient values of criteria of a set $f_e = 60 - 20x_1,\ 6 - x_1)$ along a segment $B'C'$ with $B'(100, 4)$ and $C'(80, 5)$.

Table 4. Consequences of marginal solutions

	$x^{(k)*}$	f_1	f_2
$\max f_1$	$x^{(1)*} = (2,2)$	$f_1^* = 100$	$f_{12} = 4$
$\max f_2$	$x^{(2)*} = (1,4)$	$f_{21} = 80$	$f_2 = 5$

Example 2. Let us have a problem of multi-attribution decision making:

$$\max\{f_1(x), f_2(x) | x \in A = [a_1, a_2, a_3, a_4]$$

with attribution evolution from the Table 5. Values of criteria for appropriate alternatives are shown graphically in the Figure 1.

To perform an analysis, we should consider each criterion separately, then both criteria together bearing in mind appropriate graphs of dominance of each pair of action.

Table 5. Data on multi-attribution decision making model

	max	max
	f_1	f_2
a_1	10	**525**
a_2	30	400
a_3	**50**	210
a_4	30	350

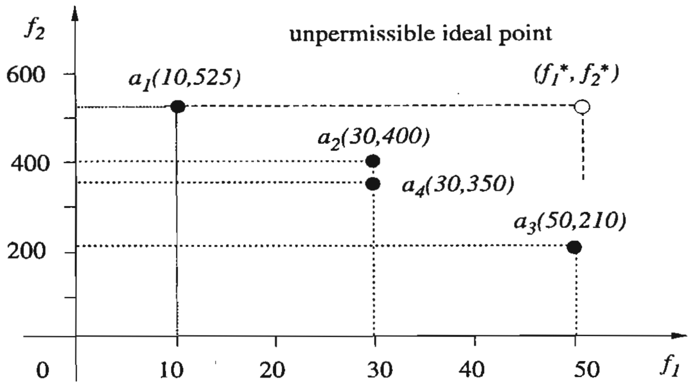
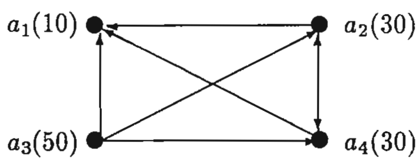


Figure 3. Graphic of multi-attribution decision making model

$$\max\{f_1(x) \mid x \in A\}$$
$$x^{(1)*} = \arg\max_i\{f_1(a_i)\} = a_3$$

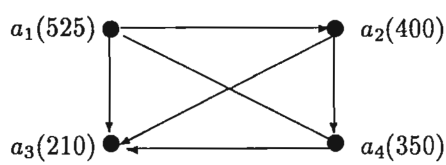
marginal solution.
$f_1^* = 50$, ideal value of criterion



a_3 dominates each alternative

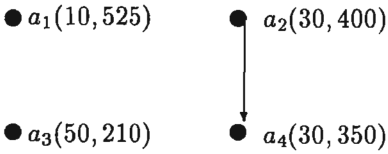
$$\max\{f_2(x) \mid x \in A\}$$
$$x^{(2)*} = \arg\max_i\{f_2(a_i)\} = a_1$$

marginal solution.
$f_2^* = 525$, ideal value of criterion



a_1 dominates each alternative

$$\max\{f_1(x), f_2(x) | \ x \in A\}$$
a_1, a_2, a_3 = efficient solutions
a_4 = inefficient solution



a_2 dominates a_4

It's obvious that the first and second criteria have their optimal-marginal solution. It is said that a_3 dominates each other action in the first case, and a_1, each other action in the second case. Ideal values of criteria are $f_1^* = 50$ and $f_2 = 525$ respectively.

If we observe both criteria simultaneously, however, we find out that only a_2 in relation to a_4, derives the same value for the first criterion and higher value for the second criterion. Other alternatives are better according to onecaiterion and worse according to other. Thus we conclude: (1) a_2 dominates a_4, (2) other actions aren't comparative, (3) there isn't a perfect solution - ideal point ($f_1 = 50$, $f_2 = 525$) isn't permissible, no alternative suits it and (4) there are three different solutions to the problem (a_1, a_2, a_3), them are efficient solutions for the model of multi-attribution decision making. A solution should be chosen among them for a concrete application (or perhaps rank alternatives if it is required).

4. References

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Received January 7, 1998.