MATHEMATICA MORAVICA
VOL. 2 (1998), 75–83

THEOREM OF SYNTHESIS FOR BISEMILATTICE-VALUED FUZZY SETS

V. Lazarević and A. Tepavčević

Abstract. Bisemilattice-valued fuzzy set (B-fuzzy set) is a mapping from a nonempty set to a bisemilattice. A B-fuzzy set has two families of level subsets, one for each ordering of the bisemilattice.

In this paper, necessary and sufficient conditions under which two families of subsets of a nonempty set are families of level subsets of a B-fuzzy set are given.

1. Preliminaries

A **bisemilattice** is an algebra $(B, \vee, \wedge)$ with two binary operations, such that $(B, \vee)$ and $(B, \wedge)$ are commutative and idempotent semigroups. Since a lattice is a bisemilattice satisfying the absorption laws, bisemilattices are a generalization of lattices.

Bisemilattice was introduced by J. Plonka in [4] under the name of quasi-lattice, and Padmanabhan in [3] called it bisemilattice.

Ordering relations $\leq_\vee$ and $\leq_\wedge$ on the semilattices $(B, \vee)$ and $(B, \wedge)$ of a bisemilattice $(B, \vee, \wedge)$ (respectively) are defined by:

$x \leq_\vee y$ if and only if $x \vee y = y$ and

$x \leq_\wedge y$ if and only if $x \wedge y = x$.

In addition, $x \geq_\vee y$ iff $y \leq_\vee x$, and analogously $x \geq_\wedge y$ iff $y \leq_\wedge x$.

A bisemilattice can be defined as a relational system $(B, \leq_\vee, \leq_\wedge)$, in which $(B, \leq_\vee)$ is a join-semilattice, i.e., a poset in which every two-element subset has a join and $(B, \leq_\wedge)$ is a meet-semilattice, a poset in which each two-element subset has a meet.

AMS Mathematics Subject Classification 1991. Primary: 04A72, 06A12.
Key words and phrases: fuzzy sets, bisemilattices.

A diagram of a bisemilattice consists of two Hasse diagrams, one for each ordering. We use the following convention: if $x \leq_\vee y$ and $z \leq_\wedge t$, then x is below y and z below t in the corresponding diagrams.

A **bisemilattice valued fuzzy set** (B-fuzzy set) is a mapping $\overline{A} : X \longrightarrow B$ from a nonempty set X to a bisemilattice $\mathcal{B} = (B, \vee, \wedge)$.

For each $p \in B$, there are two level functions defined as follows:

$$\overline{A}_p^\vee(x) = 1 \text{ if and only if } \overline{A}(x) \geq_\vee p$$

and

$$\overline{A}_p^\wedge(x) = 1 \text{ if and only if } \overline{A}(x) \geq_\wedge p.$$

The corresponding level subsets are denoted by: $A_p^\vee$ and $A_p^\wedge$.

Thus, for a B-fuzzy set $\overline{A} : X \longrightarrow B$, there are two families of level subsets:

$$A_B^\vee = \{A_p^\vee \mid p \in B\} \quad \text{and} \quad A_B^\wedge = \{A_p^\wedge \mid p \in B\}.$$

Recall that the bottom element in an ordered set, if such an element exists, is denoted by $\mathbf{0}$, and the top, if it exists, by $\mathbf{1}$. The following three theorems were proved in [2] and [6].

Theorem 1. [2] *Let $\overline{A} : X \to B$ be a bisemilattice-valued fuzzy set (B-fuzzy set) on X. Then*

(1) $A_0^\wedge = X$ ($(B, \wedge)$ is supposed to have the bottom element $\mathbf{0} = \bigwedge^\wedge(p \mid p \in B)$);

(2) if $p \leq_\vee q$, then $A_q^\vee \subseteq A_p^\vee$ and if $p \leq_\wedge q$, then $A_q^\wedge \subseteq A_p^\wedge$;

(3) for every $x \in X$,
$\overline{A}(x) = \bigvee^\vee\{p \in B \mid \overline{A}_p^\vee(x) = 1\}$ *and*
$\overline{A}(x) = \bigvee^\wedge\{p \in B \mid \overline{A}_p^\wedge(x) = 1\}$
(i.e., supremum on the right exists in B for both families and it is equal to $\overline{A}(x)$). □

Theorem 2. [2] *If $\overline{A} : X \to B$ is a bisemilattice-valued fuzzy set (B-fuzzy set) on X, then the following holds:*

(i) for $B_1 \subseteq B$, $\bigcap(A_p^\vee \mid p \in B_1) = A^\vee{}_{\bigvee^\vee B_1}$;
if for $B_1 \subseteq B$ there is supremum of B_1 under $\leq_\wedge$, then
$\bigcap(A_p^\wedge \mid p \in B_1) = A^\wedge{}_{\bigvee^\wedge B_1}$;

(ii) $\bigcup(A_p^\vee \mid p \in B) = X$ and $\bigcup(A_p^\wedge \mid p \in B) = X$;

(iii) for every $x \in X$,

$\bigcap(A_p^{\vee} \mid x \in A_p^{\vee}) \in A_{\mathcal{B}}^{\vee}$ *and* $\bigcap(A_p^{\wedge} \mid x \in A_p^{\wedge}) \in A_{\mathcal{B}}^{\wedge}$.

Theorem 3. [6] *Let* $\overline{A} : X \to S$ *be a semilattice valued fuzzy set, where* $(S, \leq)$ *is a join (meet) semilattice. Then, the poset* $(A_S, \subseteq)$ *of levels of* $\overline{A}$ *is a meet (join) semilattice.* □

2. Theorem of synthesis for B-fuzzy sets.

An important difference between B-fuzzy sets and L-valued ones is that the collection of level sets of an L-fuzzy set is always a lattice (under inclusion), but the corresponding collection for a B-fuzzy set is not a bisemilattice. The reason is that the collections of level subsets $A_B^{\vee}$ and $A_B^{\wedge}$ of a B-fuzzy set $\overline{A}$ do not coincide; in general, they even do not have the same cardinality. Thus, the theorem of synthesis differs a lot from the one in lattice case. The theorem proved in this paper solves a problem stated at the conference PRIM 97 in Palić, Yugoslavia.

Let $\mathcal{B}$ be a family of subsets of a nonempty set X, such that

(i) $\bigcup \mathcal{B} = X$;

(ii) for all $x \in X$,

$$\bigcap(p \in \mathcal{B} \mid x \in p) \in \mathcal{B}.$$

Let $\rho_{\mathcal{B}}$ be a relation on X, defined by:

$$(x, y) \in \rho_{\mathcal{B}} \text{ if and only if } \bigcap(p \in \mathcal{B} \mid x \in p) = \bigcap(p \in \mathcal{B} \mid y \in p).$$

Lemma 1. *Let* X *be a nonempty set and* $\mathcal{B}$ *a family of subsets of* X*, such that* $(\mathcal{B}, \subseteq)$ *is* $\wedge$*-semilattice. Infimum in* $(\mathcal{B}, \subseteq)$ *is the set intersection if and only if for all* $x \in X$,

$$\bigcap(p \in \mathcal{B} \mid x \in p) \in \mathcal{B}.$$

Proof. Suppose that for all $x \in X$,

$$\bigcap(p \in \mathcal{B} \mid x \in p) \in \mathcal{B}.$$

Let $\mathcal{C} \subseteq \mathcal{B}$. We have to prove that $\bigwedge \mathcal{C} = \bigcap \mathcal{C}$. Since $\bigwedge \mathcal{C} \subseteq Z$ for all $Z \in \mathcal{C}$, we have that $\bigwedge \mathcal{C} \subseteq \bigcap \mathcal{C}$. To prove the opposite inclusion, take $a \in \bigcap \mathcal{C}$.

Now, $\bigcap(p \in \mathcal{B} \mid a \in p) \in \mathcal{B}$, and $\bigcap(p \in \mathcal{B} \mid a \in p) \subseteq Z$, for all $Z \in \mathcal{C}$. Thus, by the definition of infimum,

$$\bigcap(p \in \mathcal{B} \mid a \in p) \subseteq \bigwedge \mathcal{C}.$$

Since $a \in \bigcap(p \in \mathcal{B} \mid a \in p)$, we have that $a \in \bigwedge \mathcal{C}$, and hence,

$$\bigwedge \mathcal{C} = \bigcap \mathcal{C}.$$

The other implication ("only if" part of Lemma) is straightforward.□

Theorem 4 (Theorem of synthesis). *Let X be a nonempty finite set and $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{P}(X)$. Necessary and sufficient conditions under which $\mathcal{B}_1$ and $\mathcal{B}_2$ are families of level subsets of a B-fuzzy set on X, are*

(1) $(\mathcal{B}_1, \subseteq)$ is a $\wedge$-semilattice, in which the infimum is the set intersection;

$(\mathcal{B}_2, \subseteq)$ is a $\vee$-semilattice with the top element X;

(2) $\bigcup \mathcal{B}_1 = X$;

(3) for all $x \in X$,

$$\bigcap(p \in \mathcal{B}_2 \mid x \in p) \in \mathcal{B}_2;$$

(4) $\rho_{\mathcal{B}_1} = \rho_{\mathcal{B}_2}$.

Proof. Suppose that $\overline{A} : X \to B$ is a B-fuzzy set, and $\mathcal{B}_1$ and $\mathcal{B}_2$ its families of level subsets. Since $(B, \vee)$ and $(B, \wedge)$ are semilattices, $\overline{A}$ determines two semilattice-valued fuzzy sets, which are the same mappings, considered as fuzzy sets on different semilattices of the bisemilattice. Thereby, and also by Theorem 1-3 and by Lemma 1 it follows that (1)-(3) hold.

In the proof of (4), we will use Theorem 1 (3):

$\overline{A}(x) = \bigvee^{\vee}\{p \in B \mid x \in A_p^{\vee}\} = \bigvee^{\wedge}\{p \in B \mid x \in A_p^{\wedge}\}$.

Recall the definition of $\rho_{\mathcal{B}}$:

$$(x,y) \in \rho_{\mathcal{B}} \quad \text{if and only if} \quad \bigcap(p \in \mathcal{B} \mid x \in p) = \bigcap(p \in \mathcal{B} \mid y \in p).$$

Here we have two collections $\mathcal{B}_1 = \{A_p^{\vee} \mid p \in B\}$ and $\mathcal{B}_2 = \{A_p^{\wedge} \mid p \in B\}$. We show that:

$$\bigcap(A_p^{\vee} \in \mathcal{B}_1 \mid x \in A_p^{\vee}) = \bigcap(A_p^{\vee} \in \mathcal{B}_1 \mid y \in A_p^{\vee})$$

if and only if

$$\overline{A}(x) = \overline{A}(y),$$

if and only if

$$\bigcap(A_p^{\wedge} \in \mathcal{B}_2 \mid x \in A_p^{\wedge}) = \bigcap(A_p^{\wedge} \in \mathcal{B}_2 \mid y \in A_p^{\wedge}).$$

We will prove the first equivalence, the proof of the second one is similar.

Suppose that $\overline{A}(x) = \overline{A}(y)$. Then, $\overline{A}(x) \geq_{\vee} p$ if and only if $\overline{A}(y) \geq_{\vee} p$, for all $p \in B$, i.e., $x \in A_p^{\vee}$ if and only if $y \in A_p^{\vee}$. This means that the intersection of all $A_p^{\vee}$ to which x belongs is equal to the intersection of all $A_p^{\vee}$ to which y belongs.

On the other hand, suppose that

$$\bigcap(A_p^{\vee} \in \mathcal{B}_1 \mid x \in A_p^{\vee}) = \bigcap(A_p^{\vee} \in \mathcal{B}_1 \mid y \in A_p^{\vee}).$$

By Theorem 2 (iii), there is $q \in B$ such that $\bigcap(A_p^{\vee} \in \mathcal{B}_1 \mid x \in A_p^{\vee}) = A_q^{\vee}$.

It is easy to see that $x \in A_q^{\vee}$ and $y \in A_q^{\vee}$, and if $x \in A_p^{\vee}$ for an element $p \in B$, then $A_q^{\vee} \subseteq A_p^{\vee}$.

If $\overline{A}(x) \geq_{\vee} p$ then $x \in A_p^{\vee}$ and by the previous, $A_q^{\vee} \subseteq A_p^{\vee}$. Since $y \in A_q^{\vee}$, we have that $y \in A_p^{\vee}$ and $\overline{A}(y) \geq_{\vee} p$.

Suppose that $\overline{A}(x) = r$ and $\overline{A}(y) = s$. We have that $\overline{A}(x) \geq_{\vee} r$ and thus $\overline{A}(y) = s \geq_{\vee} r$. Similarly, $r \geq_{\vee} s$, and

$$\overline{A}(x) = \overline{A}(y).$$

Now, suppose that X is a nonempty set and $\mathcal{B}_1, \mathcal{B}_2$ two families of subsets of X, satisfying (1)-(4). We have to prove that there exists a B-fuzzy set on X such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are its families of level subsets.

First, we are going to construct a bisemilattice to serve as a codomain of the required fuzzy set.

We choose the one of $\mathcal{B}_1$ and $\mathcal{B}_2$ with greater cardinality, for instance, $\mathcal{B}_2$. Let $B = \mathcal{B}_2$ be the underlying set of the bisemilattice. The ordering relation to determine one of the semilattices of the bisemilattice is dual to inclusion, i.e. $x \leq_{\wedge} y$ if and only if $y \subseteq x$. The second one, $\leq_{\vee}$ is obtained by using the poset which is dual to $(\mathcal{B}_1, \subseteq)$. Namely, if $|\mathcal{B}_2| = |\mathcal{B}_1 \cup \mathcal{C}|$ for some set $\mathcal{C} = \{x_i | i \in I\}$ with $\mathcal{C} \cap \mathcal{B}_1 = \emptyset$, we add a chain $\{x_i \mid i \in I\}$ below a minimum element of the Hasse-diagram of the dual of $\mathcal{B}_1$.

Let $B_1(x) = \bigcap(p \in \mathcal{B}_1 \mid x \in p)$ and $B_2(x) = \bigcap(p \in \mathcal{B}_2 \mid x \in p)$ be mappings from X to $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Let $B_1(X)$ and $B_2(X)$ be sets of images of these mappings. Since $(x, y) \in \rho_{\mathcal{B}_i}$ is equivalent with $B_i(x) = B_i(y)$, for $i = 1, 2$, and since by (4) $\rho_{\mathcal{B}_1} = \rho_{\mathcal{B}_2}$, sets $B_1(X)$ and $B_2(X)$ have cardinalities. By the construction of set $\{x_i \mid i \in I\}$, we have that sets $\mathcal{B}_2 \setminus B_2(X)$ and $\mathcal{B}_1 \cup \{x_i \mid i \in I\} \setminus B_1(X)$ also have same cardinalities. Let $\mathcal{G}$ be a bijection,

$$\mathcal{G} : \mathcal{B}_2 \setminus B_2(X) \longrightarrow \mathcal{B}_1 \cup \{x_i \mid i \in I\} \setminus B_1(X).$$

Further on, we define a mapping $\varphi : \mathcal{B}_2 \to \mathcal{B}_1 \cup \{x_i \mid i \in I\}$ in the following way:

If $p \in \mathcal{B}_2$ is the image of an $x \in X$ by B_2, i.e., if $B_2(x) = p$ for an $x \in X$, then let

$$\varphi(p) := B_1(x) \in \mathcal{B}_1.$$

If $p \in \mathcal{B}_1$ is not the image of any $x \in X$, then let

$$\varphi(p) := \mathcal{G}(p).$$

Mapping φ is well defined by condition (4), and it is straightforward that it is a bijection.

Now, we are ready to define relation $\leq_\vee$ on B:

$$x \leq_\vee y \text{ if and only if } \varphi(y) \subseteq \varphi(x).$$

Since this $\leq_\vee$ is defined by inclusion, it is an ordering relation, and $\mathcal{B} = (B, \leq_\vee, \leq_\wedge)$ is a bisemilattice.

Now, the required B-fuzzy set (i.e., such that its level sets are collections $\mathcal{B}_1$ and $\mathcal{B}_2$), is $\overline{A} : X \longrightarrow B$, defined by:

$$\overline{A}(x) := B_2(x).$$

It is just a technical exercise to verify that families of level sets of $\overline{A}$ coincide with $\mathcal{B}_1$ and $\mathcal{B}_2$. □

The following example illustrates the theorem.

Example 1.

Let $X = \{1,2,3,4,5\}$ and let $\mathcal{B}_1 = \{\emptyset, \{4\}, \{2,3\}, \{4,5\}, \{2,3,4\}, \{1,2,3\}, \{1,2,3,4\}\}$, $\mathcal{B}_2 = \{\{1\}, \{2,3\}, \{4,5\}, \{5\}, \{1,2,3,4,5\}\}$. Posets $(\mathcal{B}_1, \subseteq)$ and $(\mathcal{B}_2, \subseteq)$ are given in Figure 1. It is not difficult to see that the conditions (1)-(3) from Theorem 4 are satisfied. Since $\rho_{\mathcal{B}_1}$ and $\rho_{\mathcal{B}_2}$ are relations determined by the following (same) equivalence classes: $\{\{1\}, \{2,3\}, \{4\}, \{5\}\}$, condition (4) is also fulfilled.

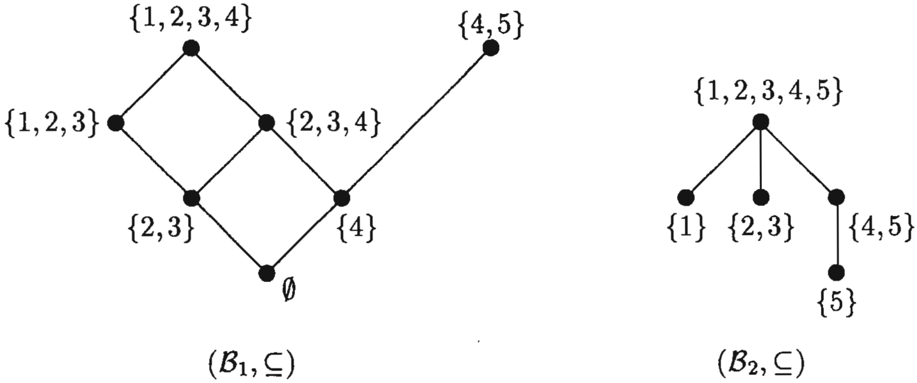

Figure 1.

Further on, we have that:

$$B_1(x) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \{1,2,3\} & \{2,3\} & \{2,3\} & \{4\} & \{4,5\} \end{pmatrix}$$

and

$$B_2(x) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \{1\} & \{2,3\} & \{2,3\} & \{4,5\} & \{5\} \end{pmatrix}.$$

Accordingly, $B_1(X) = \{\{1,2,3\}, \{2,3\}, \{4\}, \{4,5\}\}$ and $B_2(X) = \{\{1\}, \{2,3\}, \{4,5\}, \{5\}\}$.

Since $\mathcal{B}_1$ has greater cardinality than $\mathcal{B}_2$ and $|\mathcal{B}_1| - |\mathcal{B}_2| = 2$, we consider a chain with 2 elements: $C = \{x_1, x_2\}$, with $x_2 < x_1$. Let $(\mathcal{B}_1, \leq_\vee)$ and $(\mathcal{B}_2, \leq_\wedge)$ be the posets dual to $(\mathcal{B}_1, \subseteq)$ and $(\mathcal{B}_2, \subseteq)$, respectively. According to Theorem 4, we consider posets $(\mathcal{B}_1, \leq_\vee)$ and $C \oplus (\mathcal{B}_2, \leq_\wedge)$, where $\oplus$ is a denotation for the linear sum of posets.

Furthermore, $\mathcal{G}$ is a bijection from $\mathcal{B}_1 \backslash B_1(X)$ to $\mathcal{B}_2 \cup \{x_1, x_2\} \backslash B_2(X)$,

$$\mathcal{G}(x) = \begin{pmatrix} \emptyset & \{2,3,4\} & \{1,2,3,4\} \\ x_1 & \{1,2,3,4,5\} & x_2 \end{pmatrix}.$$

φ is a bijection from $\mathcal{B}_1$ to $\mathcal{B}_2 \cup \{x_1, x_2\}$ defined as in Theorem 4:

$$\varphi(x) = \begin{pmatrix} \emptyset & \{4\} & \{2,3\} & \{4,5\} & \{2,3,4\} & \{1,2,3\} & \{1,2,3,4\} \\ x_1 & \{4,5\} & \{2,3\} & \{5\} & \{1,2,3,4,5\} & \{1\} & x_2 \end{pmatrix}.$$

By the construction described in Theorem 4, we obtain the following fuzzy set:

$$\overline{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & b & c & d \end{pmatrix}.$$

where we denote $\{1,2,3\}$ by a, $\{2,3\}$ by b, $\{4\}$ by c, $\{4,5\}$ by d, $\{1,2,3,4\}$ by g, $\{2,3,4\}$ by e and $\emptyset$ by f.

The obtained fuzzy set is $\overline{A} : X \longrightarrow B$, where $B = \{a, b, c, d, e, f, g\}$ and the bisemilattice $\mathcal{B}$ is given in Figure 2. $\overline{A}$ has $\mathcal{B}_1$ and $\mathcal{B}_2$ as the families of level subsets.

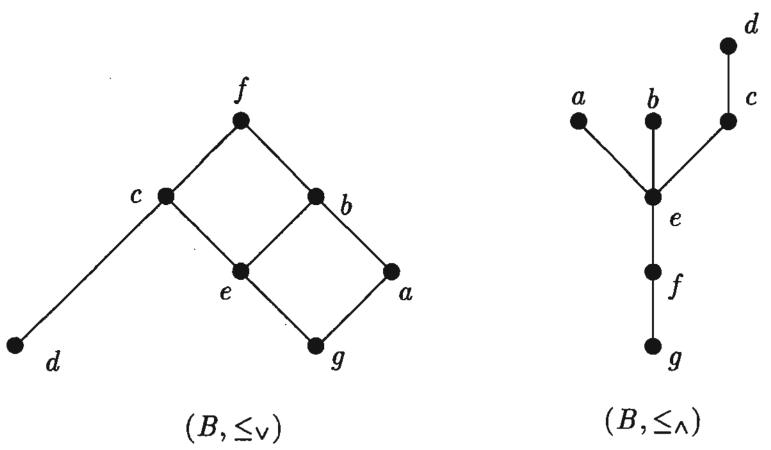

Figure 2.

3. References

[1] B. A. Davey, H. A. Priestly: *Introduction to lattices and order*, Cambridge University press, 1990.

[2] V. Lazarević, B. Šešelja, A. Tepavčević: *Bisemilattice-valued fuzzy sets*, Novi Sad J. Math, Vol. **28**, No. 3 (1998) 105–114.

[3] R. Padmanabhan: *Regular identities in lattices*, Trans. Amer. Math. Soc., **158** (1971) 179–188.

[4] J. Płonka: *On distributive quasi-lattices*, Fund. Math., **60** (1967) 191–200.

[5] B. Šešelja, A. Tepavčević: *Representation of finite semilattices by meet-irreducibles*, Indian J. Pure Appl. Math., **27** (11): 1093–1100.

[6] B. Šešelja, A. Tepavčević: *Semilattices and fuzzy sets*, The Journal of Fuzzy Mathematics Vol. **5**, No. 4, 1997.

[7] B. Šešelja, A. Tepavčević: *Relational-valued fuzzy sets*, Fuzzy sets and Systems, **52** (1992), 217-222.

[8] A. Tepavčević: *On Special Elements of Bisemilattices*, Novi Sad, J. Math. (to appear).

Faculty of Technical Sciences,
University of Kragujevac
Svetog Save 65,
32000 Čačak, Yugoslavia

Institute of Mathematics,
University of Novi Sad
Trg D. Obradovića 4,
21000 Novi Sad, Yugoslavia
e-mail: andreja@unsim.ns.ac.yu

Received May 21, 1998.