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THE MATUSZEWSKA SEQUENCES

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Abstract. In this paper we prove a representation theorem for ΔRV-sequences, which we call "Matuszewska sequences", in the Bojanić-Seneta sense. We also find a connection between the class of ΔRV-sequences and the functional Matuszewska class ERV, and the relations between the sequencial class ΔRV and the sequencial classes RVS and $*RV$

1. Introduction

In [6] W. Orlicz and W. Matuszewska introduced the functional class ERV of **extended regularly varying functions**. A function $f :$ $: [a, +\infty) \to (0, +\infty) (a > 0)$ is ERV if it is measurable and it satisfies

$$(1) \qquad \lambda^c \leq k_f^1(\lambda) \leq k_f(\lambda) \leq \lambda^d$$

for some $c, d \in \mathbf{R}$ and every $\lambda \geq 1$ where

$$k_f^1(\lambda) = \varliminf_{x\to+\infty} \frac{f(\lambda x)}{f(x)}, \qquad k_f(\lambda) = \varlimsup_{x\to+\infty} \frac{f(\lambda x)}{f(x)}.$$

The class ERV is one of classes of regularly varying functions in Karamata sense [1]. It is an important functional class of asymptotic analysis. It is well-known (see e.g. [2]) that

$$(2) \qquad RV \subsetneqq ERV \subsetneqq CRV,$$

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where RV is the class of **regularly varying functions**, and CRV is the class of **regularly varying functions which have continuous index functions**. The class ERV is also called the functional Matuszewska class.

The increasing sequence of positive numbers (c_n) which satisfy condition

$$\lim_{\substack{n\to+\infty\\ \lambda\to 1}} \frac{c_{[\lambda n]}}{c_n} = 1$$

define the class of **$*$-regularly varying sequences** (denoted $*RV$). The sequencial class $*RV$ was very widely used in the Theory of theorems of Tauberian type, and in Fourier analysis (see e.g. [7], [8]). Fundamental results about this class can be found in [3].

Before, introducing new sequencial class we shall prove the next statement.

Lemma 1. *Let (a_n) be the sequence of positive numbers and (b_n) be a decreasing sequence of positive numbers, such that $a_n \sim b_n$ $(n\to\infty)$. Then the sequence (c_n), $c_n = \sum_{k=1}^{n} a_k$ $(n\in\mathbf{N})$ is $*RV$, and there is a $d \geq 0$ such that $k_c(\lambda) \leq \lambda^d$ $(\lambda > 0)$.*

Proof. Consider the sequence (d_n) $(n\in\mathbf{N})$, where $d_n = \sum_{k=1}^{n} b_k$. If it is convergent, then the sequence (c_n) $(n\in\mathbf{N})$ also convergies, so lemma holds true. Further assume that (d_n) divergies. Since $d_{2n} \leq 2d_n$ $(n\in\mathbf{N})$, we find that $k_d(\lambda) < +\infty$ for $\lambda\in(0,2]$. Then $f(x) = d_{[x]}$ $(x\geq 1)$ satisfies $k_f(\lambda) \leq k_d(\lambda)\cdot M$ for all $\lambda\in(0,2]$ and

$$M = \lim_{\alpha\to 1+} \overline{\lim_{n\to+\infty}} \sup_{\theta\in[1,\alpha]} \frac{d_{[\theta n]}}{d_n} < +\infty\,.$$

Hence $k_f(\lambda) < +\infty$ $(\lambda > 0)$. Since $k_d(\lambda) \leq k_f(\lambda)$ $(\lambda > 0)$, we find that increasing sequence (d_n) is ORV. Besides, we have that $g(n) = d_n$ $(n\in\mathbf{N})$, $g(x)$ is linear on every interval $[n, n+1]$ $(n\in\mathbf{N})$, continuous, increasing and concave on interval $[1,+\infty)$, and it is ORV because $f(x) \leq g(x) \leq f(x+1)$ $(x\geq 1)$. Hence $k_g(\lambda) < +\infty$ $(\lambda > 0)$ and

$$k_g^1(\lambda) = \underline{\lim}_{x\to+\infty} \frac{g(\lambda x)}{g(x)} \quad (\lambda > 0)$$

is concave for every $\lambda\in(1/2,2)$. Function $k_g^1(\lambda)$ is continuous on interval $(1/2,2)$, so $k_g(\lambda) = \frac{1}{k_g^1(\frac{1}{\lambda})}$ does for every $\lambda\in(1/2,2)$. Hence $k_g(\lambda)$ is continuous for every $\lambda > 0$. This qives that function $g(x)$ $(x \geq 1)$ is CRV. Since

$$\lim_{\substack{n\to+\infty \\ \lambda\to 1}} \frac{d_{[\lambda n]}}{d_n} = \lim_{\substack{n\to+\infty \\ \lambda\to 1}} \frac{g([\lambda n])}{g(n)} = 1 ,$$

we find that (d_n) is $*RV$. Since the index function

$$k_g(\lambda) = \overline{\lim_{x\to+\infty}} \frac{g(\lambda x)}{g(x)} \quad (\lambda > 0)$$

is continuous and increasing, we find that left Matuszewska index of $g(x)$ $(x \geq 1)$ is $\underset{\sim}{k}_g = k'_{g-}(1) \geq 0$ (see [2]). The right Matuszewska index of the some function is:

$$\begin{aligned}
\tilde{k}_g = k'_{g+}(1) &= \lim_{\lambda\to 1+} \frac{k_g(\lambda) - 1}{\lambda - 1} \\
&= \lim_{\lambda\to 1+} \frac{\frac{1}{k_g^1(\frac{1}{\lambda})} - 1}{\lambda - 1} \\
&= \lim_{\lambda\to 1+} \frac{1 - k_g^1(\frac{1}{\lambda})}{1 - \frac{1}{\lambda}} \cdot \lim_{\lambda\to 1+} \frac{k_g(\lambda)}{\lambda} = \lim_{t\to 1-} \frac{k_g^1(t) - 1}{t - 1} \\
&= k_{g-}^{1'}(1) \leq C < +\infty .
\end{aligned}$$

Hence, $g(x)$ $(x \geq 1)$ belongs to the functional Masuszewska class ([2]). Hence, there is a $d \geq 0$ such that $k_g(\lambda) \leq \lambda^d$ for every $\lambda \geq 1$. This implies

$$k_d(\lambda) = \overline{\lim_{n\to+\infty}} \frac{d_{[\lambda n]}}{d_n} = \overline{\lim_{n\to+\infty}} \frac{g([\lambda n])}{g(n)} \leq \overline{\lim_{n\to+\infty}} \frac{g(\lambda n)}{g(n)} \leq k_g(\lambda) \leq \lambda^d$$

for all $\lambda \geq 1$. Further we consider functions $r(x) = a_{[x]}$, $s(x) = b_{[x]}$ $(x \geq 1)$. Then $r(x) \sim s(x)$, $x \to +\infty$, and integrals $\int_1^x r(t)dt$, $\int_1^x s(t)dt$ divergies as $x \to +\infty$. We find that $\int_1^x r(t)dt \sim \int_1^x s(t)dt$ as $x \to +\infty$, which implies $\sum_{k=1}^n a_k \sim \sum_{k=1}^n b_k$ as $n \to \infty$. Hence (c_n) is $*RV$. So for some $d \geq 0$ we have $k_c(\lambda) \leq \lambda^d$ $(\lambda \geq 1)$.

This completes the proof. □

Corollary 1. *Let (a_n) be a sequence of positive numbers, and (λ_n) its sequence of exponents of the slow variability [9], such that the sequence*

(b_n), $b_n = a_n^{\lambda_n} \sim c_n$ as $n \to +\infty$. If (c_n) is a decreasing sequence of positive numbers, then the sequence (s'_n), $s'_n = \sum_{k=1}^{n} a_k^{\lambda_k(1+\mu)}$ $(n \in \mathbf{N})$ where $\mu \in [-1, 0)$ is $*RV$, and there holds $k_{s'}(\lambda) \leq \lambda^d$ for every $\lambda \geq 1$ and some $d \geq 0$.

In some analogy with the functional class ERV, we can define the sequentional class ΔRV. We call that a sequence (c_n) $(n \in \mathbf{N})$ is ΔRV if it is positive, increasing, which also satisfies

$$k_c(\lambda) \leq \lambda^d \tag{1'}$$

for some $d \in \mathbf{R}_0^+$ and every $\lambda \geq 1$, where

$$k_c(\lambda) = \overline{\lim_{n \to +\infty}} \frac{c_{[\lambda n]}}{c_n}$$

If RVS is the class of all increasing regularly varying sequences [1], and $*RV$ is the class of all $*$-regularly sequences [3], then it is known that

$$RVS \subseteq \Delta RV \subseteq *RV\ . \tag{2'}$$

Sequencial classes RVS and $*RV$ have a great importance in the Fourier analysis (see e.g. [8]), and in particular in the theory of Tauberian type theorems (see e.g. [5]).

2. Results

Proposition 1. *Let (c_n) be an increasing sequence of positive numbers, then the following statements are equivalent to each other:*

(a) the sequence $(c_n) \in \Delta RV$;

(b) the function $f(x) = c_{[x]}$ $(x \geq 1)$ is ERV.

Proof. $(a) \Rightarrow (b)$. The function $f(x) = c_{[x]}$ $(x \geq 1)$ is positive and measurable. Besides, it satisfies

$$1 \leq k_f^1(\lambda) \leq k_f(\lambda) \leq k_c(\lambda) \cdot k_c(1+\delta)$$

for every $\lambda \geq 1$ and every $\delta > 0$. Hence

$$1 \leq k_f^1(\lambda) \leq k_f(\lambda) \leq k_c(\lambda) \leq \lambda^d$$

for every $\lambda \geq 1$ and some $d \geq 0$. In other words we have that $f(x)$ $(x \geq 1)$ is ERV.

$(b) \Rightarrow (a)$. Since (c_n) $(n \in \mathbf{N})$ is an increasing sequence of positive numbers and

$$k_c(\lambda) \leq k_f(\lambda) \leq \lambda^d$$

for every $\lambda \geq 1$ and some $d \geq 0$, we find that sequence (c_n) $(n \in \mathbf{N})$ is $\triangle RV$. □

The next Corollary follows immediately from the proof of Proposition 1.

Corollary 2. *If a sequence (c_n) $(n \in \mathbf{N})$ is $\triangle RV$ and $f(x) = c_{[x]}$ $(x \geq 1)$, then $k_c(\lambda) = k_f(\lambda)$ for every $\lambda > 0$.*

Using the Proposition 1, Corollary 2, and some results from papers [2] and [4], we conclude that index function $k_c(\lambda)$ $(\lambda > 0)$ of an arbitrary $\triangle RV$-sequence (c_n) $(n \in \mathbf{N})$ is continuous, and there holds $k'_{c+}(\lambda), k'_{c-}(\lambda) \in \mathbf{R}_0^+$ for every $\lambda > 0$. Hence the function $k_c(\lambda)$ $(\lambda > 0)$ is non-differentiable at the most countably many points.

The next result is in fact a **Representation Theorem.**

Theorem 1. *Let (c_n) $(n \in \mathbf{N})$ be an arbitrary increasing sequence of positive numbers. Then the next conditions are equivalent to each other:*

(a) the sequence (c_n) $(n \in \mathbf{N})$ is $\triangle RV$;

(b) there is an $n_o \in \mathbf{N}$ such that

$$c_n = \exp\left\{\varepsilon_n + \sum_{k=n_o}^{n} \frac{\xi_k}{k}\right\} \quad (n \geq n_o),$$

where $\varepsilon_n \to c \in \mathbf{R}$ as $n \to \infty$, and (δ_n) $(n \in \mathbf{N})$ is a bounded sequence.

Proof. $(a) \Rightarrow (b)$. Assume that sequence (c_n) is $\triangle RV$. Then by Proposition 1 the function $f(x) = c_{[x]}$ $(x \geq 1)$ is ERV. Then, by a result from [1], then is a $b \geq 1$ such that

$$c_n = f(n) = \exp\left\{\varepsilon_1(n) + \int_b^n \frac{\delta(t)}{t} dt\right\} \quad (n \geq b)\,.$$

Here $\delta(t)$ is a bounded, measurable function in interval $[b,+\infty)$ $(b \geq 1)$, and $\varepsilon_1(x)$ is a bounded, measurable function in the interval $[b,+\infty)$ $(b \geq 1)$, such that $\varepsilon_1(x) \to c_1 \in \mathbf{R}$ as $x \to +\infty$.

Let $n_o = [b]+1$, $s = \int_b^{n_o} \frac{\delta(t)}{t} dt$, we have that function $\varepsilon(t) = \varepsilon_1(t)+s$ is bounded and measurable on the interval $[b,+\infty)$ $(b \geq 1)$, and $\varepsilon(t) \to c_1 + \rho = C \in \mathbf{R}$ as $t \to +\infty$. Hence

$$c_n = \exp\left\{\varepsilon(n) + \sum_{k=n_o}^{n} \frac{\delta_k}{k}\right\} \quad (n \geq n_o),$$

where $\varepsilon(n) = \varepsilon_n \to c$, as $n \to \infty$ and

$$\delta_k = k \int_{k-1}^{k} \frac{\delta(t)}{t} dt$$

for $b \geq n_o + 1$ and $\sigma_{n_o} = 0$. Hence we finally get

$$|\delta_k| = k \left| \int_{k-1}^{k} \frac{\delta(t)}{t} dt \right| \leq k \cdot \sup_{t \geq k-1} |\delta(t)| \cdot \log\left(1 + \frac{1}{k-1}\right) \leq 2 \cdot \sup_{t \geq k-1} |\delta(t)| < M$$

for every $b \geq n_o + 1$, because the function $\delta(t)$ is bounded on the interval $[b,+\infty)$.

$(b) \Rightarrow (a)$. If $\lambda \geq 1$ and $n \geq n_o$ then

$$\frac{c_{[\lambda n]}}{c_n} = \exp\left\{\varepsilon_{[\lambda n]} - \varepsilon_n + \sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k}\right\}$$

Since $\overline{\lim}_{n \to +\infty}(\varepsilon_{[\lambda n]} - \varepsilon_n) = 0$ and

$$\left| \sum_{k=n+1}^{[\lambda n]} \frac{\delta k}{k} \right| \leq \sup_{k \geq n+1} |\delta_k| \cdot \int_{n+1}^{[\lambda n]+1} \frac{dt}{t-1} =$$

$$= \sup_{k \geq n+1} |\delta_k| \cdot \log \frac{[\lambda n]}{n} \quad (n \geq n_o)$$

which implies $\log k_c(\lambda) \leq d \cdot \log \lambda$, $(\lambda \geq 1)$, and so we find that $k_c(\lambda) \leq \lambda^d$ $(\lambda \geq 1)$.

Hence the sequence (c_n) $(n \in \mathbf{N})$ is ΔRV. This completes the proof.
□

Proposition 2. *Let a function $f(x)$ $(x \geq 1)$ be increasing and ERV. Then there hold:*

(a) *the sequence* (c_n) $(n \in \mathbf{N})$, $c_n = f(n)$, *is* $\triangle RV$;

(b) $k_f(\lambda) = k_c(\lambda)$ $(\lambda > 0)$.

Proof. (*a*) Since (c_n) $(n \in \mathbf{N})$ is an increasing sequence of positive numbers and

$$k_c(\lambda) = \overline{\lim_{n\to+\infty}} \frac{c_{[\lambda n]}}{c_n} = \overline{\lim_{n\to+\infty}} \frac{f([\lambda n])}{f(n)} \le \overline{\lim_{n\to+\infty}} \frac{f(\lambda n)}{f(n)} \le$$
$$\le \overline{\lim_{x\to+\infty}} \frac{f(\lambda x)}{f(x)} \le k_f(\lambda) \le \lambda^d$$

for every $\lambda \ge 1$, and some $d \in \mathbf{R}$, which implies that this sequence is $\triangle RV$.

(b) By relation (a) we have that $k_c(\lambda) \le k_f(\lambda)$ for every $\lambda > 0$. On the other hand, for every $\lambda > 0$ we have

$$k_f(\lambda) = \overline{\lim_{x\to+\infty}} \frac{f(\lambda x)}{f(x)} \le \overline{\lim_{x\to+\infty}} \frac{f([\lambda[x]])}{f([x])} \cdot \overline{\lim_{x\to+\infty}} \frac{f(\lambda x)}{f([\lambda[x]])} \cdot$$
$$\cdot \overline{\lim_{x\to+\infty}} \frac{f([x])}{f(x)} = k_c(\lambda).$$

because $f \in CRV$ (see e.g. [2]). Hence, we get $k_f(\lambda) = k_c(\lambda)$ for every $\lambda > 0$. □

By Proposition 2 follows

$$RVS \subsetneq \triangle RV, \tag{2*}$$

where RVS is the class of increasing regularly varying sequences.

The next proposition is an analogous to the Proposition 2, so we omit the proof.

Proposition 3. *Let* $f(x)$ $(x \ge 1)$ *be an increasing function from the class* CRV. *Then*

(a) *the sequence* (c_n) $(n \in \mathbf{N})$, $c_n = f(n)$, *is* $*RV$;

(b) *there holds* $k_f(\lambda) = k_c(\lambda)$ $(\lambda > 0)$.

Now Proposition 3 we have that

$$\triangle RV \subsetneq *RV. \tag{2**}$$

So improved that: $RVS \subsetneq \triangle RV \subsetneq *RV$.

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