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TAUBERIAN THEOREMS FOR GENERALIZED ABELIAN SUMMABILITY METHODS*

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Abstract. We introduce and study a significant generalization of Abel's summability method, and their corresponding limiting process. This leads to an analogue to Hardy-Littlewood Tauberian Theorem. The first section includes an introduction to some basic concepts of summability methods and a survey of classical and neoclassical results. In the second section a general summability method is designed and some related Tauberian theorems are established. In the third section higher order of Abel's summability methods are obtained as a special case of a general summability method and the general Littlewood theorem is proved for those summability methods. Finally we give Tauberian theorems corresponding to (C, m)-summability methods and present some further convergence theorems.

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1. Introduction

A. Definitions and notations

Let $A(\triangle)$ denote all the analytic functions in the unit disc. To each $f(a)$ in $A(\triangle)$ we associate a series $\sum_{n=0}^{\infty} a_n$ of Taylor coefficients of f. Write

$$S_n(a) = \sum_{k=0}^{n} a_k \tag{1.1}$$

and

$$\sigma_n(S(a)) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(a) \tag{1.2}$$

If $\lim_{n\to\infty} \sigma_n S(a))$ exists, then we say that the sequence $\{S_n(a)\}$ is $(C,1)$-summable and $\lim_{n\to\infty} \sigma_n(S(a))$ is the $(C,1)$ limit of the sequence $\{S_n(a)\}$.

Abel [1] proved that for any $f(a) \in A(\triangle)$ if the sequence $\{S_n(a)\}$ converges then $\lim_{x\to 1-0} f(a,x)$ exists. This theorem is known as Abel's limit theorem for power series.
Let $f(a) \in A(\triangle)$. If $\lim_{x\to 1-0} f(a,x)$ exists, then the sequence $\{S_n(a)\}$ is Abel summable to $\lim_{x\to a-0} f(a,x)$.

The class of all Abel summable sequence is denoted by (A,o).

In this work we are interested in the generalizations of (A,o)-summability, which we call (A,m)-summability later. Our main concern is to study (A,m)-summability and find corresponding Tauberian theorems for the recovery of the convergence.

A summability method is regular if it sums every convergent sequence to its sum.

The $(C,1)$ and Abel summability methods are regular, that is,

$$\lim_{n\to\infty} \sigma_n(S(a)) = \lim_{n\to\infty} S_n(a) \tag{1.3}$$

and

$$\lim_{x\to 1-0} f(a,x) = \lim_{n\to\infty} S_n(a) \tag{1.4}$$

provided that $\lim_{n\to\infty} S_n(a)$ exists.
The first result above is obtained by the theorem of Cauchy and the other is obtained by Abel's theorem.
If $f(a,x) = O(1)$, $x \to 1-0$ then the sequence $\{S_n(a)\}$ is Abel bounded and we write $S_n(a) = o(1)$ (A, o).
The sequence $\{S_n(a)\}$ is said to be very slowly oscillating if

$$(1.5) \qquad \frac{1}{n+1}\sum_{k=0}^{n} S_k(a) - S_n(1) = o(1), \quad n \to \infty$$

The class of all very slowly oscillating sequence is denoted by VSO.
It is easy to see that if $\{S_n(a)\}$ is very slowly oscillating, then $V_n(a,1) = o(1), n \to \infty$.
The sequence $\{S_n(a)\}$ converges if and only if $\{S_n(a)\}$ is $(C,1)$-summable provided that $\{S_n(a)\}$ very slowly oscillating.
If $\{S_n(a)\}$ is $(C,1)$-summable, then $S_n(a) = o(n)$, $n \to \infty$. It easily follows that $a_n = o(n)$, $n \to \infty$.

Consider the summability method, which is called Holder method, applying an iteration method to the method of arithmetic means.
We write

$$(1.6) \qquad \sigma_n^{(0)}(S(a)) = S_n(a)$$

and for some integer $m \geq 1$

$$(1.7) \qquad \sigma_n^{(m)}(S(a)) = \frac{1}{n+1}\sum_{k=0}^{n} \sigma_n^{(m-1)}(S(a))$$

If $\lim_{n\to\infty} \sigma_n^{(m)}(S(a))$ exists, then we say that $\{S_n(a)\}$ is (H,m)-summable. The $(H,1)$ method and $(C,1)$ method are the same. Another method of summability which generalizes the arithmetic mean defined previously is the Cesaro summability method. For $m > -1$, and $n \geq 0$
We write

$$(1.8) \qquad E_n^{(m)} = \binom{n+m}{m}$$

and

$$S_n^m = \sum_{k=0}^{n} A_k^{m-1}, \quad A_n^0 = S_n(a) \tag{1.9}$$

If $\lim_{n\to\infty} \frac{S_n^m}{E_n^{(m)}}$ exists, then the sequence $\{S_n(a)\}$ is said to be (C, m)-summable.
We say that $\{S_n(a)\}$ is (C, m) bounded if $\{\sigma_n^{(m)}(S(a))\}$ is bounded and denoted by $S_n(a) = O(1)$ (C, m).
Notice that (C, m) and (H, m) methods are equivalent ([2]).
It is also well known that for $m > -1$ (C, m)-summability implies the Abel summability ([2]).

B. Survey of classical methods and results

This section includes some well known Tauberian theorems corresponding to the Abel summability method in the historical order. One may ask whether the converse of Abel's theorem, i.e. the statement $\lim_{x\to 1-0} f(a, x)$ exists implies that $\sum_{n=0}^{\infty} a_n$ converges also holds true. The following counterexample shows that the inverse statement of Abel's theorem is not true. For the function $f(a) = f(a, x) = \sum_{n=0}^{\infty}(-1)^n x^n$, $|x| < 1$, $\lim_{x\to 1-0} f(a, x)$ exists but the sequence $\sum_{n=0}^{\infty}(-1)^n n$ is not convergent. However the converse of Abel's theorem is valid provided that we add some condition, so called Tauberian condition. So any theorem which states the convergence of sequence follows a summability method and some Tauberian condition is said to be a Tauberian theorem. In the beginning conditions for the recovery of the convergence of the series $\{S_n(a)\}$ out of its Abel's summability were conditions on the order of magnitude of the Taylor coefficients $\{a_n\}$ of a function $f(a) = f(a, x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < 1$ analytical in the unit disc.

Theorem 1.1 *Let $f(a) = f(a, x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < a$ and $\lim_{x\to 1-0} f(a, x)$ exist. If*

$$na_n = o(1), \qquad n \to \infty$$

then the sequence converges.

Proof Let $na_n = 0(1)$, $n \to \infty$. Denote $\sigma_n = \max_{k \geq n} |ka_k|$. Then we have $\sigma \downarrow 0$. Consider

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n(1-x^n) + \sum_{n=N+1}^{\infty} a_n x^n, \qquad 0 < x < 1$$

Thus, we obtain

$$\begin{aligned} \left|\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n\right| &\leq \sum_{n=0}^{\infty} |a_n n(1-x)| + \left|\sum_{n=N+1}^{\infty} \frac{na_n}{n} x^n\right| \\ &\leq (1-x)N\sigma_0 + \frac{\sigma_{N+1}}{(N+1)(1-x)} \end{aligned}$$

Let $N = \left[\frac{\varepsilon}{1-x}\right] = N(x)$. Then $N \to \infty$ as $x \to 1-0$, and $\lim_{x\to 1-0}$ $\sup \left[\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n\right| \leq \varepsilon\sigma_0$. ε was arbitrary. This completes the proof of the theorem.

Denote $W_n(a,1) = \sum_{k=1}^{n} ka_k$. If the condition $na_n = o(1)$, $n \to 1$ can be replaced by $\frac{W_n(a,1)}{n} = o(1)$, $n \to +\infty$ in Theorem 1.1, we have the convergence of the series $\{S_n(a)\}$ out of its Abel summability.

After Tauber's theorem many significant generalizations of this theorem have been obtained. Later, Littlewood [4] lightened the original condition $na_n = o(1)$, $n \to \infty$ by $na_n = O(1)$, $n \to \infty$ and proved the following theorem.

Theorem 1.2 *Let* $f(a) = f(a,x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < 1$ *and* $\lim_{x\to 1-0} f(a,x)$ *exist. If* $na_n = 0(1)$, $n \to +\infty$ *then the sequence* $\{S_n(a)\}$ *converges.*

The main object in the rest of the section B is to represent some improvements of Tauber's Theorem.

All the conditions in Theorem 1.1 , 1.2 and the Classical Hardy-Littlewood [5] condition

$$(1.10) \quad V_n(|a|,p) = \frac{1}{n}\sum_{k=1}^{n} k^p |a_k|^p = 0(1), \qquad n \to \infty, \quad p > 1$$

(And other similar) imply that

$$(1.11) \quad S_N(a) - S_M(a) = o(1), \qquad N > M \to \infty, \ \frac{N}{M} \to 1$$

i.e. the slow oscillation of $\{S_n(a)\}$, introduced by Schmidt [6], but the converse is not rue.
This led to the generalized Littlewood [7] Tauberian theorem asserting that if the limit $\lim_{x\to 1-0} f(a,x)$ exist and (1.11) holds then the series $\{S_n(a)\}$ converges to $\lim_{x\to 1-0} f(a,x)$.
As an example of this theorem we have the following: If $\{\sigma_n^{(1)}(S(a))\}$ is slowly oscillating and Abel summable then the sequence $\{S_n(a)\}$ is $(C,1)$-summable.
The following corollary to the generalized Littlewood theorem will be used extensively later.

Corollary Let the sequence $\{S_n(a)\}$ be $(C,1)$-summable. If $\{S_n(a)\}$ is slowly oscillating, then the sequence $\{S_n(a)\}$ converges.

Proof By the remark in the section A the sequence $\{S_n(a)\}$ is Abel summable. The proof follows from the Generalized Littlewood theorem.

Hardy and Littlewoood [5] conjectured the following result and Szász [8] proved that the Classical Hardy-Littlewood condition (1.10) is a Tauberian condition to recover the convergence of the sequence of $\{S_n(a)\}$ out of its Abel summability.

Theorem 1.3 *Let* $f(a) = f(a,x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < 1$ *and* $\lim_{x\to 1-0} f(a,x)$ *exist. If (1.10) holds for some fixed* $p > 1$, *then the sequence* $\{S_n(a)\}$ *converges.*

Proof Denote $\alpha(t) = \sum_{n\le t} a_n$. Then we have

$$\alpha(t') - \alpha(t) = \sum_{t<n\le t'} a_n$$

and

$$|\alpha(t') - \alpha(t)| \le \sum_{t<n\le t'} |a_n| \ .$$

We obtain ,by Holder's inequality,

$$\sum_{k=m+1}^{n} |a_k| \le (n-m)^{1-\frac{1}{p}} \left(\sum_{k=m+1}^{n} |a_k|^p \right)^{\frac{1}{p}} \ .$$

From the last inequality it follows that

$$(n-m)^{1-\frac{1}{p}}\left(\sum_{k=m+1}^{n}|a_k|^p\right)^{\frac{1}{p}} \le (n-m)^{1-\frac{1}{p}}\frac{1}{m}\left(\sum_{k=m+1}^{n}|k|^p|a_k|^p\right).$$

By the hypotheses of the theorem we have

$$\left(\sum_{k=m+1}^{n}k^p|a_k|^p\right)^{\frac{1}{p}} \le Cn^{\frac{1}{p}}$$

where C is a constant. Thus,

$$|\alpha(t')-\alpha(t)| \le C\frac{(n-m)^{1-\frac{1}{p}}}{m}n^{\frac{1}{p}} = C\frac{n}{m}\left(1-\frac{m}{n}\right)^{1-\frac{1}{p}}$$

If $\frac{n}{m}\to 1$ we obtain that $\sum_{m<k\le n} a_k \to 0$.
Thus $\alpha(t)$ satisfies the hypotheses of the generalized Littleweood theorem. This completes the proof.

This theorem is a corollary of the Generalized Littlewood theorem. The condition (1.11) implies that $\frac{S_n(a)}{n} = \hat{h}(n)$ for some $h \in H^r$, $r \ge 2$ [9]. The condition (1.10) is of considerable interest to our study. It does not only imply (1.11) but for $p \in (1,2]$ implies that $g(a) = g(a,x) = \sum_{n=0}^{\infty}\frac{a_n}{n+1}x^n$ belongs to H^q, $\frac{1}{p}+\frac{1}{q}=1$, [10]. Rényi [11] noticed that for $p=1$ $V_n(|a|,p)=O(1)$, $n\to\infty$ is no longer a Tauberian condition for the recovery of the convergence of the series out of its Abel's summability.
Rényi [11] have constructed an example that for a sequence $a=\{a_n\}$ the condition $V_n(|a|,1)=O(1)$, $n\to\infty$ cannot be replaced by (1.10).
Rényi's example satisfies the condition (1.10) but the sequence $\{S_n(a)\}$ does not converge.
He considered the sequence $a=\{a_n\}$, where a_k is 1 if $k=2^n$ and -1 if $k=2^n+1$ for $n=1,2,3,\ldots$ and 0 for other values.
Show that $V_n(|a|,1)$ is bounded.
It is sufficient to verify that for n's of the form 2^s or 2^{s+1}.
For $n=2^s$, we have

$$V_{2^s} = \frac{1}{2^s}\sum_{j=1}^{s-1}(2.2^j+1)+1 < 4$$

Similarly, V_{2^s+1} is bounded. Now verify that the sequence $\{S_n(a)\}$ is Abel summable.

Rewrite $f(x) = \sum_{k=1}^{\infty} a_k x^k$ as

$$f(x) = \sum_{k=1}^{\infty} \left(x^{2^k} - x^{2^k+1}\right) . \tag{1.12}$$

For $0 \le x < 1$, $f(x) \ge 0$. Thus $\lim_{x\to -0} \inf f(x) \ge 0$. From (1.12) We also obtain

$$\begin{aligned} f(x) &= (1-x)\sum_{k=1}^{\infty} x^{2^k} = (1-x)\left(x^2+x^4+x^8+\sum_{k=4}^{\infty} x^{2^k}\right) \\ &\le (1-x)\left(x^2+x^4+x^8+\int_0^{\infty} x^{t^2}\right) \\ &= (1-x)\left(x^2+x^4+x^8+\int_0^{\infty} e^{-\ln\frac{1}{x}t^2}\right) \\ &= (1-x)\left(x^2+x^4+x^8+\frac{1}{\sqrt{\ln\frac{1}{x}}}\int_0^{\infty} e^{-u^2}du\right) \\ &= (1-x)\left(x^2+x^4+x^8++C\left(\sqrt{\ln\frac{1}{x}}\right)^{-1}\right) \end{aligned}$$

Since $\ln\left(\frac{1}{x}\right) \sim 1-x$ as $x \to 1-o$, we have $\lim_{x\to 1-0} \sup f(x) \le 0$. Thus $\lim_{x\to 1-o} f(x) = 0$. But the sequence $\{S_n(a)\}$ diverges.

Rényi [11] also observed that if $V_n(|a|,1) = O(1)$, $n \to \infty$ is replaced by somewhat stronger condition that $\lim_{n\to\infty} V_n(|a|,1)$ exists then one can recover convergence of the series $\{S_n(a)\}$ out of its Abel's summability.

Theorem 1.4 *Let* $f(a) = f(a,x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < 1$ *and* $\lim_{x\to 1-o} f(a,x)$ *exist. If*

$$\lim_{n\to\infty} V_n(|a|,1) = V , \tag{1.13}$$

then the sequence converges.

Proof Denote $\alpha(t) = \sum_{n\le t} a_n$. Show that $\alpha(t)$ satisfies the hypotheses of the generalized Littlewood Theorem.

Let $n \geq m+1$, $m \to \infty$ and $\frac{n}{m} \to 1$.

By the condition in the theorem we have

$$\sum_{k \leq n} k|a_k| = Vn + o(n)$$

and consequently

$$\sum_{m<k\leq n} k|a_k| = V(n-m) + o(m).$$

By an Abel transformation, we obtain

$$\begin{aligned}\sum_{m<l\leq n} |a_k| &= \sum_{m<k\leq n} \frac{k|a_k|}{k} = \int_m^n \left(\sum_{m<k\leq u} k|a_k|\right)\frac{du}{u^2} + \frac{\sum_{m<k\leq n} k|a_k|}{n} \\ &= \int_m^n \frac{(u-m)V + o(m)}{u^2} du + \frac{V(n-m)}{n}.\end{aligned}$$

Let $\frac{n}{m} \to 1$. Then we have $\frac{o(m)}{n} = o(1)$ and

$$\int_m^n \frac{o(m)}{u^2} du = o\left(m\left(\frac{1}{m} - \frac{1}{n}\right)\right) = o\left(1 - \frac{m}{n}\right) = o(1).$$

From the last calculation we get

$$\begin{aligned}\sum_{m<l\leq n} |a_k| &= V\int_m^n \frac{u-m}{u^2} du + V\left(1 - \frac{m}{n}\right) + o(1) \\ &= V\int_m^n \frac{u-m}{u^2} du + o(1) \\ &= V\ln\frac{n}{m} - Vm\left(\frac{1}{n} - \frac{1}{m}\right) + o(1) = o(1).\end{aligned}$$

Rényi constructed examples to show that neither of his theorem nor Szasz's theorem includes the other.

In [12] the authors designed a general summability method generalized Abel's summability method as its special case and proved that the limiting case become a tauberian condition for those summability methods. I extend this work and present tauberian theorems for (C, m)-summability method and further convergence theorems in the last section.

The situation in Theorem 1.4 motivates the following questions:

- (i) Are there some generalized Abel's summability methods of $\{S_n(a)\}$ such that $V_n(|a|,1) = O(1)$, $n \to \infty$ or its generalization, entail some limiting information about $\{S_n(a)\}$?
- (ii) What are the conditions for recovering Abel's summability out of those general summability methods, regular with respect to Abel's summability?

The answers of these questions will be the main point of this work.

To search the answers to the questions (i) and (ii) consider the following variant of Borel summability method.

Let $P(x) = \sum_{n=0}^{\infty} p_n x^n$ be an analytical function on (0,1) such that P is not polynomial on (0,1) and $P(x) \to \infty$, $x \to 1-o$.

If

$$\lim_{x\to 1-o} \frac{\sum_{n=0}^{\infty} S_n(a) p_n x^n}{P(x)}$$

exists, then $\{S_n(a)\}$ is (K, P)-summable.

The method (K, P) is regular since the convergence of $\{S_n(a)\}$ implies the existence of $\lim_{x\to -o} f(a,x)$ For

$$P(x) = \frac{1}{1-x} = \sigma_1(x)$$

we get

$$\frac{\sum_{n=0}^{\infty} S_n(a) p_n x^n}{P(x)} = (1-x) \sum_{n=0}^{\infty} S_n(a) x^n = f(a,x) \ .$$

For this choice of $P(x)$ if $\{S_n(a)\}$ is (K, σ_1)-summable then it is Abel summable. If $p_n = \frac{1}{n+1}$, then from the definition above it follows that the method (K, P) includes the (A, o) method.

Instead $\{S_n(a)\}$ we may consider

$$\sigma_n^{(1)}(a) = \sigma_n(a) = \sigma_n(S(a)) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(a) = \frac{1}{n+1} \sum_{k=0}^{n} \sigma_k^{(o)}(S(a))$$

or any other sequence $\{A_n(S(a))\}$ generated by $\{S_n(a)\}$. In general we define

$$K(f, P, A(S(a)), x) = \frac{1}{P(x)} \sum_{n=0}^{\infty} A_n(S(a)) p_n x^n \ .$$

For $P(x) = \delta_1$ and $A_n(S(a)) = \sigma_n(a)$,

$$K(f,\sigma_1,\sigma,x) = \frac{\sum_{n=0}^{\infty}\sigma_n(a).1.x^n}{\delta_1(x)} = (1-x)\sum_{n=0}^{\infty}\sigma_n(a)x^n = \sum_{n=0}^{\infty}(\sigma_{n+1}(a)-\sigma_n(a))x^n$$

The Taylor coefficients of (K,δ_1,σ) are $\{\sigma_{n+1}(a)-\sigma_n(a)\} = -\Delta\sigma$.

If $\{S_n(a)\}$ is (K,δ_1,σ)-summable, i.e if

$$\lim_{x\to 1-o}\frac{\sum_{n=0}^{\infty}\sigma_n(a).1.x^n}{\delta_1(x)}$$

exists, and if $V_n(|a|,1) = O(1)$, $n\to\infty$ then the series $\{S_n(a)\}$ is $(C,1)$-summable. Thus $V_n(|a|,1) = O(1)$ is a Tauberian condition for (K,δ_1,σ)-method to obtain the $(C,1)$-convergence of $\{S_n(a)\}$.

For the convergence recovery of $\{S_n(a)\}$ out of (K,δ_1,σ) - summability we need Tauberian condition: $\{V_n(a,1)\}$ is a bounded slowly oscillating sequence (or $\{S_n(a)\}$ is bounded slowly oscillating). (From any slowly oscillating sequence $\{L(n)\}$ one can construct a bounded slowly oscillating sequence: $\{\exp(iL(n))\}$ in the complex case and $\{\sin(L(n))\}$ in the real case.)

However if $\{V_n(a,1)\}$ is slowly oscillating and

$$V_n(|V|,1) = O(1), \qquad n\to\infty$$

holds we can recover the convergence of $\{S_n(a)\}$ out of its (K,δ_1,σ)-summability.

These theorems will be proved in the next sections.

This type of theorems which we obtain the convergence of the series out of (K,δ_1,σ)-summability method will be the main goal in the next sections. The general situation motivated by the above examples and remarks will be considered as well.

A nondecreasing sequence $\{R(n)\}$ of positive numbers is O-Regularly varying if for $\lambda > 1$ $\overline{\lim}_n \frac{R([\lambda n])}{R(n)}$ is finite;

If

$$V_n(|a|,p) = O(1), \qquad n \to \infty, \qquad p \in (1,2] \tag{1.14}$$

then (1.14) is equivalent to $\sum_{k=1}^{n} k^{p-1}|a_k|^p = lgR(n)$ for some O-Regulary varying sequence $R(n)$. Thus $f(a) = f(a,x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < 1$, belongs to H^q, $\frac{1}{p} + \frac{1}{q} = 1$.

Let $L^1(T)$ denote the Banach space of all complex valued Lebesgue integrable functions on the circle group $T = \frac{R}{2\pi Z}$ with the usual norm $||f||_1 = \int_T |f(t)|dt$.

The partial sums $S_n(f) = S_n(f,t) = \sum_{|k|\leq n} \hat{f}(k)e^{ikt}$ are $(C,1)$ summable in L^1 norm.

Stanojevic has modified Karamata's method [13] to obtain the condition needed for the recovery of convergence in L^1 norm. This condition is obtained in [14] and it has the form

$$\lim_{\lambda \to 1+o} \overline{\lim_n} \sum_{|k|=n+1}^{[\lambda n]} |k|^{p-1} \left|\Delta\hat{f}(k)\right|^p < \infty, \quad p \in (1,2]\,. \tag{1.15}$$

The condition (1.15) is equivalent to $V_n\left(|\Delta\hat{f}|,p\right) = O(1)$, $n \to \infty$.

2. A general summability method

Let A be the space of all analytical functions in the unit disc, or unit interval. We give the answer for the questions (i) and (ii) motivated by Theorem 1.4. The answer will depend on the choice of a summability method generalizing the Abel summability method and provides an algorithm for the construction of analytical function in some interval $[\alpha_o, 1)$, $\alpha_o \in (0,1)$, whose Taylor coefficient generates limiting processes. For the recovery of the convergence of the series out of this general summability method. We need to establish a corresponding Tauberian condition for the recovery of the convergence of series out of this general summability method.

Our approach to answer the questions (i) and (ii) depends on a method of integral transformations of the space A.

We will use the following denotation:
For $f(a) \in A$,

$$\begin{aligned} f(a) &= f(a,x) = \sum_{n=0}^{\infty} a_n x^n, \quad |x| < 1 \\ S_n(a) &= \sum_{k=0}^{n} a_k \,, \\ \sigma_n^{(m)}(S(a)) &= \sigma_n^{(m)}(a) = \frac{1}{n+1} \sum_{k=0}^{n} \sigma_k^{(m-1)}(a) \,, \end{aligned}$$

for integers $m \geq 1$. $(\sigma_n^{(1)}(a) = \sigma_n(a)$, $\sigma_n^{(o)}(a) = S_n(a))$.

To describe the class of kernels of the integral transforms of functions in A, we need the following properties of functions φ in Φ:

- (i) There exists a number $\alpha_o = \alpha_o(\Phi) \in (0,1)$ such that every $\varphi \in \Phi$ is analytical in $[\alpha_o, 1)$.
- (ii) For every $\varphi \in \Phi$,

$$\varphi(x) \to \infty, \qquad x \to 1 - o\,.$$

- (iii) Each $\varphi \in \Phi$ is zerofree in $[\alpha_o, 1)$.
- (iv) $\frac{\varphi_m(x)}{\varphi_{m-1}(x)} = o(1)$, $x \to 1 - o$, $m \geq 1$.

Then for every $f(a) \in A$ and $\varphi \in \Phi$ we define

$$\begin{aligned} M_\varphi(f(a)) &= M_\varphi(f(a),\varphi) = M(f(a),\varphi,x) = \\ &= \begin{cases} \frac{\int_{\alpha_o}^{x} f(a,t)\varphi(t)dt}{\varphi_1(x)} & , \quad x \neq \alpha_o \\ \lim_{x\to\alpha_o} M_\varphi(f(a),x) = f(\alpha_o) & , \quad x = \alpha_o \end{cases} \end{aligned}$$

or in general

$$\begin{aligned} M_\varphi(f(a)) &= M_\varphi(f(a),\varphi) = M(f(a),\varphi,x) = \\ &= \begin{cases} \frac{\int_{\alpha_o}^{x} f(a,t)\varphi_m(t)dt}{\varphi_{m+1}(x)} & , \quad x \neq \alpha_o \\ \lim_{x\to\alpha_o} M_{\varphi_m}(f(a),x) = f(\alpha_o) & , \quad x = \alpha_o \end{cases} \end{aligned}$$

for $m \geq 0$.

If for some $\varphi \in \Phi$

$$\lim_{x \to 1-o} M(f(a), \varphi_m, x) \tag{2.1}$$

exists, $\{S_n(a)\}$ is (M, φ)-summable to the limit above.

Since $\varphi'_{m+1}(x) = \varphi_m(x)$, it is clear that the existence of the limit $\lim_{x \to 1-o} f(a, x)$ implies the existence of the limit (2.1). That is, the summability method M_φ is a regular summability method.

An important subclass of Φ are functions

$$\delta_m(x) = \frac{1}{(1-x)^m}, \qquad m > 0$$

and $\alpha_o = 0$.

For $m = 1$, we have

$$M(f(a), \delta_1, x) = \frac{1}{\log\left(\frac{1}{1-x}\right)} \int_o^x \frac{1}{1-t} f(t) dt$$

For $m = 2$, we have

$$\begin{aligned} M(f(a), \delta_2, x) &= \frac{1-x}{x} \int_o^x \frac{f(a,t)}{(1-t)^2} dt = K(f(a), \delta_1, \sigma, x) = \\ &= \sum_{n=0}^{\infty} \left(\sigma_{n+1}^{(1)}(a) - \sigma_n^{(1)}(a)\right) x^n \end{aligned}$$

The function $M(f(a), \delta_2)$ is analytical in (0,1) and it is denoted by

$$f(-\Delta\sigma) = f(-\Delta\sigma, x) = A^{(1)}(f(a)) = A^{(1)}(f(a), x) .$$

Notice that $A^{(1)} f(a)$ defines a regular summability method $(A, 1)$, because the Abel summability method $(A, o) = A^{(o)}(f(a)) = f(a) = f(a, x)$ implies $(A, 1)$.

For any $\alpha \neq \beta$ we have

$$M(M(f(a), \delta_\alpha), \delta_\beta, x) = \frac{\alpha\beta}{\alpha - \beta}(M(f(a), \delta_\alpha, x) - M(f(a), \delta_\beta, x)) .$$

From the above identity we see that for any sequence $\{S_n(a)\}$ which is (M, δ_α)-summable is (M, δ_β)-summable and similarly changing the roles of α and β in the above identity we recover the inverse of the statement.

An elementary answer to the questions (i) and (ii) is provided by the next theorem.

If the $\lim_{x\to 1-o} A^{(1)}(f(a), x)$ exists then $\{S_n(a)\}$ is $(A, 1)$-summable to that limit.

As a corollary of the generalized Littlewood theorem we have

Theorem 2.1 *Let $\{S_n(a)\}$ be $(A, 1)$-summable. If*

$$V_n(|a|, 1) = O(1), \qquad n \to \infty \tag{2.2}$$

then $\{S_n(a)\}$ is (A, o)-summable.

Proof The condition (2.2) implies that $\{\sigma_n(a)\}$ is slowly oscillating. Since $\{S_n(a)\}$ is $(A, 1)$-summable, by generalized Littlewood theorem we obtain that $\{S_n(a)\}$ is $(C, 1)$-summable. Therefore, $\{S_n(a)\}$ is (A, o)-summable.

The Tauberian condition (2.2) (recall that (2.2) is not a Tauberian condition for the recovery of convergence $\{S_n(a)\}$ of out of (A, o)-summability) can be generalized all the way up to $\{\sigma_n(a)\}$ being slowly oscillating and implies the (A, o)-summability of the series $\{S_n(a)\}$ provided that $\{S_n(a)\}$ is $(A, 1)$-summable. It can even be obtained the convergence of the series $\{S_n(a)\}$ if we strength the condition (2.2).

The following theorem is the one which is closer the classical situation.

Theorem 2.2 *Let $\{S_n(a)\}$ be $(A, 1)$-summable and let $\{V_n(a, 1)\}$ be bounded and slowly oscillating. Then the series $\{S_n(a)\}$ converges to its $(A, 1)$-sum.*

In the next section we shall study more specific (M, φ)-summability methods in particular those related to various generalizations of (A, o)-summability method.

B. Tauberian theorems for (M,φ)-summability methods

In this section we give some results regarding (M, φ)-summability methods.

Theorem 2.3 *Let $\{S_n(a)\}$ be M_φ-summable and let*

$$f'(a,x) = o\left(\frac{\varphi(x)}{\varphi_1(x)}\right), \qquad x \to 1-o\,. \tag{2.3}$$

If $\{V_n(a,1)\}$ is slowly oscillating then the series $\{S_n(a)\}$ converges to its M_φ-sum.

Proof From the identity

$$M(f(a),\varphi,x) = f(a,x) - \frac{\int_{\alpha_o}^{f} f'(a,t)\varphi_1(t)dt}{\varphi_1(x)}$$

and (2.3) it follows that $\{S_n(a)\}$ is (A,o)-summable to its M_φ-sum. The (A,o)-summability implies $(A,1)$-summability. Hence $\{V_n(a,1)\}$ is (A,o)-summable and since $\{V_n(a,1)\}$ is slowly oscillating we have from the generalized Littlewood theorem that $\{V_n(a,1)\}$ converges. Therefore $\{\sigma_n^{(1)}(a)\}$ is slowly oscillating and consequently $\{S_n(a)\}$ is slowly oscillating. Recalling again the generalized Littlewood theorem we conclude that the series $\sum_{n=0}^{\infty} a_n$ converges to its M_φ-sum.

A more general result can be obtained by replacing (2.3) with

$$M(f'(a),\varphi,x) = o\left(\frac{\varphi_1(x)}{\varphi_2(x)}\right), \qquad x \to 1-o$$

The composition of two M_φ-summability methods can be defined as follows.

Let φ, $\Psi \in \Phi$. Then

$$(M_\varphi \circ M_\psi)(f(a),x) = M(M_\psi(f(a)),\varphi,x) = \frac{\int_{\alpha_o}^{x} M(f,\psi,t)\varphi(t)dt}{\varphi_1(x)}\,.$$

Now theorem 2.3 can be rewritten in the following way.

Theorem 2.4 *Let $\{S_n(a)\}$ be $(M_\psi \circ M_\varphi)$-summable and let*

$$f'(a,x) = o\left(\frac{\varphi(x)}{\varphi_1(x)}\right), \qquad x \to 1-o \tag{2.4}$$

and

$$M(f'(a)\varphi_1, x) = o\left(\frac{\psi_1(x)}{\psi_2(x)}\right), \qquad x \to 1-o \tag{2.5}$$

If $\{V_n(a,1)\}$ is slowly oscillating then the series $\sum_{n=0}^{\infty} a_n$ converges to its $M_\psi \circ M_\varphi$-sum.

Proof The iterated identity from the proof of theorem 2.3 yields

$$\begin{aligned}(M_\psi \circ M_\varphi)(f(a),x) &= M_\varphi(f(a),x) - \frac{1}{\psi_1(x)}\int_{\alpha_0}^{x} M'(f(a),\varphi,t)\psi_1(t)dt \\ &= f(a,x) - \frac{1}{\varphi_1(x)}\int_{\alpha_0}^{x} f'(t)\varphi_1(t)dt - \\ &- \frac{1}{\psi_1(x)}\int_{\alpha_0}^{x} M'(f(a),\varphi,t)\psi(t)dt\end{aligned}$$

The conditions (2.4) and (2.5) imply that is (A,o)-summable. The rest of the proof follows the lines of the proof of Theorem 2.3 it was shown that if $\{S_n(a)\}$ is (A,o)-summable, then it is (M,φ)-summable. It is natural to ask under which condition (M,φ)-summability implies (M,φ_1)-summability.

Theorem 2.5 *Let $\{S_n(a)\}$ be (M,φ)-summable. If* $\lim_{x\to 1-o} f'(a,x)$ *exists, then $\{S_n(a)\}$ is (M,φ_1)-summable.*

Proof Consider

$$M_\varphi(f(a),x) = f(a,x) + \frac{\int_{\alpha_0}^{x} f''(a,t)\varphi_2(t)dt - \varphi_2(x)f'(x)}{\varphi_1(x)}. \tag{2.6}$$

Applying M_{φ_1} in (2.6) yields

$$M_{\varphi_1}(M_\varphi(f(a)),x) = \tag{2.7}$$

$$M_{\varphi_1}(f(a),x) + \frac{\int_{\alpha_0}^{x}\left(\int_{\alpha_0}^{t} f''(u)\varphi_2(u)du - \varphi_2(t)f'(t)\right)dt}{\varphi_2(x)}.$$

Since $\{S_n(a)\}$ is (M,φ)-summable, then $\lim_{x\to 1-o} M_{\varphi_1}(M_\varphi(f(a)),x)$ exists. It follows from (2.7) that $\{S_n(a)\}$ is (M,φ_1)-summable.

3. Generalized Abel's summability methods

In section 1 we listed several Tauberian theorem corresponding to Abel summability method and later introduced the general summability method and established some Tauberian theorems to obtain the convergence of the series out of this summability method. In that section we concentrate on the specific (M,φ)-summability methods. After Tauber's Theorem many generalizations have been obtained. It was proved that the condition $V_n(|a|,1) = O(1)$, $n \to \infty$ is not a Tauberian condition for the Abel summability method. This condition turns out to be a Tauberian condition for the higher order of Abel's summability method that we introduce and study in this chapter.

For any $f(a) \in A$, let the limit $\lim_{x\to 1-o} f(a,x)$ be exist. Then

$$\frac{f(a,x)}{1-x} = \sum_{k=0}^{\infty} S_k(a)x^k$$

and

$$\frac{f(a,x)}{(1-x)^2} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} S_j(a)\right)x^k = \sum_{k=0}^{\infty}(k+a)\sigma_k^{(1)}(S(a))x^k\,. \tag{3.1}$$

Integrating both sides of (3.1) from 0 to x, and then multiplying by $1-x$, we have

$$(1-x)\int_0^x \frac{f(a,t)dt}{(1-t)^2} = (1-x)x\sum_{k=0}^{\infty}\sigma_k^{(1)}(S(a))x^k = x\sum_{k=0}^{\infty}\Delta\sigma_k^{(1)}(S(a))x^k$$

Since $\lim_{x\to 1-o} f(a,x)$ exists, applying the L'hospital rule in the last formula we obtain that

$$\lim_{x\to 1-o}\frac{1-x}{x}\int_0^x \frac{f(t)}{(1-t)^2}dt$$

exists.

This motivates the following definition.

Definition 3.1 Let $f \in A(\Delta)$. If

$$\lim_{x\to 1-o} \frac{1-x}{x}\int_0^x \frac{f(t)}{(1-t)^2}dt \tag{3.2}$$

exists, the sequence $\{S_n(a)\}$ is $(A,1)$-summable.

Recall that $(A,1)$-summability method was obtained as a special case of the (M,φ)-summability method.

As shown in [15] $(C,1)$-summability implies (A,o)-summability. For $(A,1)$-summability method we have the following theorem.

Theorem 3.1 *If $\{S_n(a)\}$ is $(C,2)$-summable, then it is $(A,1)$-summable.*

Proof Assume $\{S_n(a)\}$ is $(C,2)$-summable, then $f(\Delta\sigma)=f(\Delta\sigma,x)= = \sum_{n=0}^{\infty}\Delta\sigma_n^{(1)}(S(a))x^n$ converges for all $|x|<1$. Since $\sum_{n=0}^{\infty}x^k$ converges absolutely to $\frac{1}{1-x}$, we get

$$\frac{f(\Delta\sigma,x)}{(1-x)^2}=\left(\sum_{n=0}^{\infty}\Delta\sigma_n^{(1)}(S(a))x^n\right)\left(\sum_{n=0}^{\infty}x^n\right)=\sum_{n=0}^{\infty}\sigma_n^{(1)}(S(a))x^n .$$

Dividing equality above by $\frac{1}{1-x}$, we obtain

$$\frac{f(\Delta\sigma,x)}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)\sigma_n^{(2)}(S(a))x^n .$$

For given $\varepsilon>0$, there exists a $M>0$ such that $\left|\sigma_n^{(2)}(S(a))-S\right|<\varepsilon$ for any $n>M$.

Then, we have

$$f(\Delta\sigma,x)-S=(1-x)^2\sum_{n=0}^{\infty}(n+1)(\sigma_n^{(2)}(S(a))-S)x^n .$$

Write the difference $f(\Delta\sigma,x)-S$ as

$$\begin{aligned} f(\Delta\sigma,x)-S &= (1-x)^2\sum_{n=0}^{M}(n+1)(\sigma_n^{(2)}(a)-S)x^n + \\ &+ (1-x)^2\sum_{n=M+1}^{\infty}(n+1)(\sigma_n^{(2)}(a)-S)x^n . \end{aligned}$$

Thus,

$$
\begin{aligned}
|f(\Delta\sigma,x)-S| &\le (1-x)^2\sum_{n=0}^{M}(n+1)\left|\sigma_n^{(2)}(a)-S\right|+\\
&+ (1-x)^2\sum_{n=M+1}^{\infty}(n+1)\left|\sigma_n^{(2)}(a)-S\right|x^n, \quad |x|<1\,.
\end{aligned}
$$

The first term on the right tends to zero as $x \to 1-o$, and the second one is less than

$$
(1-x)^2\sum_{n=M+1}^{\infty}(n+1)\varepsilon x^n \le (1-x)^2\varepsilon\sum_{n=0}^{\infty}(n+1)x^n=\varepsilon\,.
$$

It was shown that if $f\in(A,o)$ then $f\in(A,1)$. But the converse is not true. For example, although the function $f(x)=\sin\left(\frac{1}{1-x}\right)$ satisfies the condition (3.2), but it does not satisfy to be Abel summable.In general, consider the analytic functions in A of the form

$$
f(x)=g'\left(\frac{1}{1-x}\right),
$$

where g is bounded, has a first derivative in the unit disc and $\lim_{x\to1-o}f(a,x)$ does not exist. Then $\frac{1-x}{x}\int_0^x\frac{1}{(1-t)^2}g'\left(\frac{1}{1-t}\right)dt=\frac{1-x}{x}\left(g\left(\frac{1}{1-x}\right)-g(1)\right)$. Since f is bounded, this implies that $\lim_{x\to1-o}A^{(1)}(f(a),x)$ exists.

Alternately, the following theorem shows that under some condition there are some functions in A such that $\{S_n(a)\}$ is $(A,1)$-summable, but not (A,o)-summable.

Theorem 3.2 *Let $\{S_n(a)\}$ be $(A,1)$-summable. If*

$$
\lim_{x\to1-o}\int_0^x f''(t)lg(1-t)dt
$$

exists, then $\{S_n(a)\}$ is not (A,o)-summable.

Proof Observe

$$
\begin{aligned}
A^{(1)}(f(a),x) &= \frac{f(x)+(1-x)lg(1-x)f'(x)}{x}-\frac{1-x}{x}f(0)-\\
&- \frac{1-x}{x}\int_0^x f''(t)lg(1-t)dt\,.
\end{aligned}
$$

Let $G(x) = f(x) + (1-x)lg(1-x)f'(x)$. Then we obtain

$$f(x) = lg(1-x)\int_0^x \frac{G(t)}{(1-t)lg^2(1-t)}dt\,.$$

Rewrite $f(x)$ as

$$f(x) = lg(1-x)\left[\frac{1}{lg(1-x)}\right]_0^x M_{\frac{1}{(1-x)lg^2(1-x)}}(G(a),x)\,.$$

From the equality above and the hypotheses of the theorem it follows that $\lim_{x\to 1-o}\frac{f(a,x)}{lg(1-x)}$ exists. This completes the proof.

The method $(A,1)$ is a generalization of the method (A,o). Similarly we define the $(A,2)$-summability of the series $\{S_n(a)\}$ as $(A,1)$-summability of $A^{(1)}(f(a))$.

In general, for any integer $m \geq 1$

$$A^{(m)}(f(a)) = A^{(m)}(f(a),x) = A^{(1)}(A^{(m-1)}(f(a),x) = A^{(1)}(A^{(m-1)}(f(a))$$

and

$$A^{(m)}(f(a),x) = \sum_{n=0}^{\infty}\Delta\sigma_n^{(m)}(S(a))x^n\,.$$

If $\lim_{x\to 1-o}A^{(m)}(f(a))$ exists then $\{S_n(a)\}$ is called (A,m)-summable to that limit.

Notice that $A^{(m)}(f(a))$ defines a regular summability method (A,m).

For the $(C,1)$-convergence recovery of the series $\{S_n(a)\}$ out of its $(A,1)$-summability we will introduce Tauberian conditions in the following theorem.

Theorem 3.3 *Let*

$$\lim_{x\to 1-o}\int_0^x \frac{f(a,t)}{1-t}dt \tag{3.3}$$

exists. If $\left\{\sum_{k=0}^{n}\frac{S_k(a)}{k+1}\right\}$ *is slowly oscillating then the series* $\{S_n(a)\}$ *is* $(C,1)$*-summable to zero.*

Proof Since $\lim_{x\to 1-o}\int_0^x \frac{f(a,t)}{1-t}dt$ exists, the series $\left\{\sum_{k=0}^{n}\frac{S_k(a)}{k+1}\right\}$ is (A,o)-summable and since $\left\{\sum_{k=0}^{n}\frac{S_k(a)}{k+1}\right\}$ is slowly oscillating, by the generalized Littlewood theorem it converges. Set $S_n^*(a)=\sum_{k=0}^{n}\frac{S_k(a)}{k+1}$.

It follows that

$$S_n^*(a)-\sigma_n^{(1)}(S^*(a))=\sigma_n^{(1)}(S(a)) \tag{3.4}$$

This completes the proof.

The next two corollaries are obtained by replacing slow oscillation of $\left\{\sum_{k=0}^{n}\frac{S_k(a)}{k+1}\right\}$ by some stronger condition.

Corollary 3.3.1 Let

$$\lim_{x\to 1-o}\int_0^x \frac{f(a,t)}{1-t}dt$$

exists. If for some $p>1$

$$\frac{1}{n+1}\sum_{k=0}^{n}|S_k(a)|^p=O(1), \qquad n\to\infty\ , \tag{3.5}$$

then $\{S_n(a)\}$ is $(C,1)$-summable to zero.

Proof (3.5) implies that $\{S_n(a)\}$ is slowly oscillating.

Corollary 3.3.2 Let

$$\lim_{x\to 1-o}\int_0^x \frac{f(a,t)}{1-t}dt$$

exist. If

$$V_n(S(a),1)=O(1), \qquad n\to\infty \tag{3.6}$$

then $\{S_n(a)\}$ is $(C,1)$-summable to zero.

Proof (3.6) implies that $\{S_n^*(a)\}$ is slowly oscillating.

Recall that

$$f(-\Delta\sigma,x)=A^{(1)}(f(a),x)=\sum_{n=0}^{\infty}(\sigma_{n+1}^{(1)}(a)-\sigma_n^{(1)}(a))x^n\ .$$

From the above identity it is clear that the statements

(i) $\{S_n(a)\}$ is $(A,1)$-summable and (ii) $\{\sigma_n^{(1)}(S(a))\}$ is (A,o)-summable are equivalent.

In general, we have

$$f(\sigma^{(m+1)}(S(a)),x) = \frac{1-x}{x}\int_0^x \frac{f(\Delta\sigma^{(m)}(S(a)),t)dt}{(1-t)^2}$$

For $f(a) \in A$ the following denotation will be used throughout the paper.

$$V_n^{(m)}(a,1) = \frac{1}{n+1}\sum_{k=0}^{n} V_k^{(m-1)}(a,1)\,,$$

for integers $m \geq 1$. $(V_n^{(o)}(a,1) = V_n(a,1) = \frac{1}{n+1}\sum_{k=0}^{n} ka_k)$

Notice that

$$\sigma_n^{(m)}(S(a)) - \sigma_n^{(m+1)}(S(a)) = V_n^{(m)}(a,1)$$

for integers $m \geq 1$.

From the last identity we obtain that

$$A^{(m)}(f(a),x) - A^{(m+1)}(f(a),x) = f(\Delta V^{(m)}(a,1),x)\,.$$

It is straightforward that if $\{S_n(a)\}$ is $(A, m+1)$-summable then it is (A,m)-summable provided that $\{V_n^{(m)}(a,1)\}$ is (A,o)-summable.

The analogue of the generalized Littlewood theorem for (A,m)-summability method is given in the next theorem.

If $\{\sigma_n^{(m)}(S(a))\}$ is slowly oscillating, the series is called (C,m)-slowly oscillating.

Theorem 3.4 *Let $\{S_n(a)\}$ be (A,m)-summable. If $\{S_n(a)\}$ is (C,m)-slowly oscillating then $\{S_n(a)\}$ is (C,m)-summable to its (A,m)-sum.*

The next three corollaries are obtained by replacing the slow oscillation of $\{\sigma_n^{(m)}(S(a))\}$ by some stronger conditions.

Corollary 3.4.1 For some integer $m \geq 1$ let $\{S_n(a)\}$ be (A,m)-summable. If $\{V_n^{(m-1)}(a,1)\}$ is bounded, then $\{S_n(a)\}$ is (C,m)-summable to its (A,m)-sum.

Proof From the identity

$$\sigma_n^{(m)}(Sa)) = \sum_{k=1}^{n} \frac{V_k^{(m-1)}(a,1)}{k}$$

and $V_n^{(m-1)}(a,1) = O(1)$, $n \to 1$ it follows that $\{\sigma_n^{(m)}(S(a))\}$ is slowly oscillating.

Corollary 3.4.2 Let $\{S_n(a)\}$ be (A,m)-summable. If for some $p > 1$

$$\frac{1}{n+1}\sum_{k=1}^{n}\left|V_k^{(m-1)}(a,1)\right|^p = O(1), \qquad n \to \infty \tag{3.7}$$

then the series $\{S_n(a)\}$ is (C,m)-summable to its (A,m)-sum.

Proof Rewritten form of (3.7)

$$\frac{1}{n+1}\sum_{k=1}^{n} k^p \left|\frac{V_k^{(m-1)}(a,1)}{k}\right|^p = O(1), \qquad n \to \infty$$

shows that $\left\{\sum_{k=1}^{n} \frac{V_k^{(m-1)}(a,1)}{k}\right\}$ is slowly oscillating.

Corollary 3.4.3 Let $\{S_n(a)\}$ be (A,m)-summable. If for some integer $m \geq 1$

$$V_n^{(o)}\left(V^{(m-1)}(a,1),1\right) = O(1), \qquad n \to \infty \tag{3.8}$$

the series $\{S_n(a)\}$ is (C,m)-summable to its (A,m)-sum.

Proof From the condition (3.8) it follows that $\sigma_n^{(m)}(S(a)) = = \sum_{k=1}^{n} \frac{\beta_k}{k} - \sum_{k=1}^{n} \frac{k+1}{k^2}\beta_{k+1}$ for some bounded sequence $\{\beta_n\}$. Since $\{S_n(a)\}$ is bounded, the last identity implies that $\{\sigma_n^{(m)}(S(a))\}$ is slowly oscillating.

For the higher order of Abel's summability method (A,m) we have the generalized Littlewood Theorem. To recover the convergence of the series $\{S_n(a)\}$ out of its (A,m)-summability method we have to assume an extra condition on $\{S_n(a)\}$. As seen in the next theorem weakening the summability method and strengthening the Tauberian condition makes the series $\{S_n(a)\}$ convergent to its (A,m)-sum.

Theorem 3.5 *For some integer $m \geq 1$ let $\{S_n(a)\}$ be (A,m)-summable. If $\{S_n(a)\}$ is bounded slowly oscillating then the series $\{S_n(a)\}$ converges to its (A,m)-sum.*

Proof If $\{S_n(a)\}$ is (A,m)-summable, $\lim_{x\to 1-o}\sum_{n=0}^{\infty}\Delta\sigma_n^{(m)}(S(a))x^n$ exists. Since $S_n(a)=O(1)$, $n\to\infty$ the series $\{\sigma_n^{(m)}(S(a))\}$ is slowly oscillating. By Theorem 3.4 the series $\{S_n(a)\}$ is (C,m)-summable to its (A,m)-sum. (C,m)-summability implies (A,o)-summability. Therefore $\{S_n(a)\}$ converges to its (A,m)-sum by the generalized Littlewood theorem.

In theorem 3.4 the Tauberian condition can be replaced by a weaker condition.

Theorem 3.6 *For some integer $m \geq 1$ let $\{S_n(a)\}$ be (A,m)-summable. If $\{V_n^{(m)}(a,1)\}$ is bounded and slowly oscillating then the series $\{S_n(a)\}$ converges to its (A,m)-sum.*

Proof Assume that $\{V_n^{(o)}(a,1)\}$ is bounded. Then for each integer $m \geq 1$, $\{V_n^{(m)}(a,1)\}$ is bounded. Therefore $\{\sigma_n^{(m)}(S(a))\}$ is slowly oscillating. Since $\{S_n(a)\}$ is (A,m)-summable we obtain that $\{S_n(a)\}$ is (C,m)-summable by Theorem 3.4.

The condition $V_n^{(o)}(a,1)=O(1)$, $n\to\infty$ implies that $\{\sigma_n^{(1)}(S(a))\}$ is slowly oscillating. Hence $\{S_n(a)\}$ is slowly oscillating. Recalling again the generalized Littlewood theorem we conclude that the series $\{S_n(a)\}$ converges to its (A,m)-sum.

In the generalized Littlewood theorem the Tauberian condition can be replaced by the condition $\{V_n^{(0)}(a,1)\}$ being slowly oscillating as follows.

Theorem 3.7 *Let $\{S_n(a)\}$ be (A,o)-summable. If $\{V_n^{(0)}(a,1)\}$ is slowly oscillating then the series $\{S_n(a)\}$ converges to its (A,o)-sum.*

Proof Since (A,o)-summability implies $(A,1)$-summability, is (A,o)-summable. By the generalized Littlewood theorem $\{V_n^{(0)}(a,1)\}$ converges. This implies that $\{S_n(a)\}$ is $(C,1)$-slowly oscillating. Since $\{V_n^{(0)}(a,1)\}$ is slowly oscillating, $\{S_n(a)\}$ is slowly oscillating. Again by the generalized Littlewood theorem we conclude that $\{S_n(a)\}$ converges.

Theorem 3.8 *Let $\{S_n(a)\}$ be $(A,2)$-summable. If $\{V_n^{(0)}(a,1)\}$ very slowly oscillating, then $\{S_n(a)\}$ converges.*

Proof Recall that every very slowly oscillating sequence is slowly oscillating. Since $\{V_n^{(0)}(a,1)\}$ is very slowly oscillating, it is slowly oscillating. By the definition of Very slowly oscillation, $V_n^{(0)}(a,1) - V_n^{(1)}(a,1) = o(1)$, $n \to \infty$. Therefore $\{V_n^{(1)}(a,1)\}$ is slowly oscillating. From the identity $S_n(a) - \sigma_n(S(a)) = V_n^{(0)}(a,1)$, we obtain that $\{\sigma_n(S(a))\}$ is slowly oscillating. Since $\sigma_n^{(1)}(S(a)) - \sigma_n^{(2)}(S(a)) = V_n^{(1)}(a,1)$, it follows that $\{\sigma_n^{(2)}(S(a))\}$ is slowly oscillating. By generalized Littlewood theorem, $\{S_n(a)\}$ is $(C,2)$-summable. Then $\{S_n(a)\}$ is (A,o)-summable. Again by generalized Littlewood theorem, $\{S_n(a)\}$ converges.

For the recovery of the convergence of $\{S_n(a)\}$ out of its generalized Abel summability method we need more than one condition. The following theorem is like an analogue of generalized Littlewood theorem for $(A,1)$-summability method.

Theorem 3.9 *Let $\{S_n(a)\}$ be $(A,1)$-summable. If $\{S_n(a)\}$ is very slowly oscillating then the sequence $\{S_n(a)\}$ converges.*

Proof Assume that $\{S_n(a)\}$ is very slowly oscillating. Then it is slowly oscillating and $V_n^{(0)}(a,1) = o(1)$, $n \to \infty$. Since $\{V_n^{(0)}(a,1)\}$ is slowly oscillating, from the identity $S_n(a) - \sigma_n^{(1)}(S(a)) = V_n^{(0)}(a,1)$ it follows that $\{S_n(a)\}$ is slowly oscillating. Since $\{S_n(a)\}$ is $(A,1)$ summable ,i.e. $\{\sigma_n^{(1)}(S(a))\}$ is (A,o) summable, the sequence $\{\sigma_n^{(1)}(S(a))\}$ converges by the generalized Littlewood theorem.

From the (A,m)-summability of $\{S_n(a)\}$ one obtains the convergence of the series $\{S_n(a)\}$ under some conditions as in next two theorems.

Theorem 3.10 *For some integer $m \geq 1$ let $\{S_n(a)\}$ be (A,m)-summable. If $\{V_n^{(0)}(a,1)\}$ is (A,o)-summable, then $\{S_n(a)\}$ is (A,o)-summable. If $\{S_n(a)\}$ is slowly oscillating, then it converges.*

Proof For any k, $0 \leq k < \infty$

$$A^{(k)}(f,x) - A^{(k+1)}(f,x) = \sum_{n=0}^{\infty} \Delta V_n^{(k)}(a,1)x^n \tag{3.9}$$

Since $\{V_n^{(0)}(a,1)\}$ is (A,o)- summable, $\{V_n^{(k)}(a,1)\}$ is (A,o)-summable for $k=0,1,2,\ldots$ Then by (3.9) we obtain that $\{S_n(a)\}$ is (A,o)-summable. If $\{S_n(a)\}$ is slowly oscillating we complete the proof.

The following theorem is a straightforward generalization of Rényi's theorem.

Theorem 3.11 *Let $\{S_n(a)\}$ be (A,m) summable. If $\{V_n^{(0)}(a,1)\}$ converges, then $\{S_n(a)\}$ converges.*

Proof It is assumed that $\{V_n^{(0)}(a,1)\}$ converges. Then for any integer $m \geq 0$ $\{V_n^{(m)}(a,1)\}$ is bounded. Therefore $\{S_n(a)\}$ is (C,m)-slowly oscillating. Since $\{S_n(a)\}$ is (A,m)-summable, it is (C,m)-summable by the generalized Littlewood theorem. It is well known that Cesaro summability of any order implies Abel summability. Thus $\{S_n(a)\}$ is (A,o)-summable. This completes the proof.

Theorem 3.12 *Let $\{S_n(a)\}$ be $(A,2)$-summable and let*

$$\frac{1}{n}\sum_{k=1}^{n}\left|V_k^{(0)}(a,1)\right|^p = O(1), \qquad n\to\infty, \quad p>1\,.$$

If $\{V_n^{(0)}(a,1)\}$ is slowly oscillating, then $\{S_n(a)\}$ converges.

Proof From the hypotheses of the theorem it follows that $\{V_n^{(1)}(a,1)\}$ is bounded. Since $\sigma_n^{(2)}(S(a)) = \sum_{k=1}^{n}\frac{V_k^{(1)}(a,1)}{k}$, the sequence $\{\sigma_n^{(2)}(S(a))\}$ is slowly oscillating. By the generalized Littlewood's theorem $\{S_n(a)\}$ is $(C,2)$-summable. Therefore $\{S_n(a)\}$ is (A,o)-summable. Thus $\{V_n^{(0)}(a,1)\}$ is (A,o)-summable. Again by the generalized Littlewood's Theorem $\{V_n^{(0)}(a,1)\}$ converges. So $\{\sigma_n^{(1)}(S(a))\}$ is slowly oscillating. $\{\sigma_n^{(1)}(S(a))\}$ converges. Finally, it follows $\{S_n(a)\}$ converges.

Theorem 3.13 *Let $\{S_n(a)\}$ be $(A,2)$-summable. If $\{V_n^{(1)}(a,1)\}$ is bounded slowly oscillating, then*

(i) $\{S_n(a)\}$ *is $(C,1)$-summable,*

(ii) $\left\{\sum_{k=1}^{n}\frac{S_k(a)}{k}\right\}$ *is slowly oscillating,*

(iii) $S_n(a) = o(n), \qquad n \to \infty$

(iv) $\left\{\sum_{k=1}^{n} \frac{a_k}{k}\right\}$ *converges.*

Proof (i) From the hypotheses it follows that $\{V_n^{(1)}(a,1)\}$ is bounded. This implies $\{\sigma_n^{(2)}(S(a))\}$ is slowly oscillating. Since $\{S_n(a)\}$ is $(A,2)$-summable, by generalized Littlewood's theorem $\{S_n(a)\}$ is $(C,2)$-summable. From the equality

$$\sigma_n^{(1)}(S(a)) - \sigma_n^{(2)}(S(a)) = V_n^{(1)}(a,1)\ ,$$

it follows that $\{\sigma_n^{(1)}(S(a))\}$ is slowly oscillating. Therefore, $\{\sigma_n^{(1)}(S(a))\}$ converges. This completes the part (i).

(ii)+(iii) From $S_n(a) - \sigma_n^{(1)}(S(a)) = V_n^{(o)}(a,1)$, we obtain that

$$\begin{aligned}\sum_{k=1}^{n} \frac{S_k(a)}{k} &= \sum_{k=1}^{n} \frac{\sigma_k^{(1)}(S(a))}{k} + \sum_{k=1}^{n} \frac{V_k^{(o)}(a,1)}{k} = \\ &= \sum_{k=1}^{n} \frac{\sigma_k^{(1)}(S(a))}{k} + \sigma_n^{(1)}(S(a))\ .\end{aligned}$$

It is clear that $\left\{\sum_{k=1}^{n} \frac{S_k(a)}{k}\right\}$ is slowly oscillating and $S)n(a) = o(n)$, $n \to \infty$.

(iv) Observe that $\sum_{k=1}^{n} \frac{a_k}{k} = \sum_{k=1}^{n} \frac{S_k(a)}{k(k+1)} + \frac{1}{n} S_n(a)$. By (ii) and (iii) $\{\sum_{k=1}^{n} \frac{a_k}{k}\}$ converges.

Theorem 3.14 *Let $\{S_n(a)\}$ be (A,m)-summable and let $\{V_n^{(k)}(a,1)\}$ be bounded for some $0 \le k \le m-1$. Then $\{S_n(a)\}$ is (A,o)-summable. If $\{S_n(a)\}$ is slowly oscillating then it converges.*

Proof The condition $V_n^{(k)}(a,1) = O(1)$, $n \to \infty$, $0 \le k \le m-1$ implies that $\{S_n(a)\}$ is (C,m) slowly oscillating. Since $\{S_n(a)\}$ is (A,m)-summable, then it is (C,m)-summable. Therefore it is (A,o)-summable. This completes the proof.

The $(C,1)$-sum of $\{V_k^{(m-1)}(a,1)\}_{k=0}^{n}$, for some integer $m \ge 1$, is denoted by $V_n^{(m)}(a,1)$.

In the next theorem it will be shown that for any integer $m \geq 0$, assuming that the condition $V_n^{(k)}(a,1) = O(1)$, $n \to \infty$, for some $0 \leq k \leq m-1$, is enough to recover (A,o)-summability of out of $\{S_n(a)\}$ its (A,m)-summability.

Theorem 3.15 *Let $\{S_n(a)\}$ be (A,m)-summable. If for some k, $0 \leq k \leq m-1$*

$$V_n^{(k)}(a,1) = O(1), \qquad n \to \infty \tag{3.10}$$

then

- *(i) $\{S_n(a)\}$ is (A,o)-summable.*
- *(ii) If $\{S_n(a) - \sigma_n^{(k+1)}(S(a))\}$ is slowly oscillating then $\{S_n(a)\}$ converges to its (A,m)-sum.*

Proof (i) The condition (3.10) implies that $\{S_n(a)\}$ is (C,m)-slowly oscillating. By Theorem 3.4 the series $\{S_n(a)\}$ is (C,m)-summable to its (A,m)-sum. Then it is (A,o)-summable.

ii) Since $V_n^{(k)}(a,1) = O(1)$, $n \to \infty$ then $\{V_n^{(j)}(a,1)\}$ is slowly oscillating for $j = k+1, k+2, \ldots, m-1$. (A,o)-summability of $\{S_n(a)\}$ implies that then $\{V_n^{(j)}(a,1)\}$ is (A,o)-summable for $j = k+1$, $k+2, \ldots, m-1$. Combining what we have and observing

$$\begin{aligned} S_n(a) &= \sigma_n^{(m)}(S(a)) + \sum_{j=0}^{m-1} V_n^{(j)}(a,1) \\ &= \sigma_n^{(m)}(S(a)) + \sum_{j=0}^{k} V_n^{(j)}(a,1) + \sum_{j=k+1}^{m-1} V_n^{(j)}(a,1) \end{aligned}$$

and

$$\sum_{j=0}^{k} V_n^{(j)}(a,1) = S_n(a) - \sigma_n^{(k+1)}(S(a))$$

we obtain that $\{S_n(a)\}$ is slowly oscillating. By the generalized Littlewood theorem we conclude that $\{S_n(a)\}$ converges.

It is a well known fact that if $\{S_n(a)\}$ is (C,o)-summable, then it is (A,o)-summable. It is natural to ask under which condition the inverse of the statement is valid. The following theorem is the answer for this.

Theorem 3.16 *Let $\{S_n(a)\}$ be Abel summable. If for some integer $m \geq 0$ $\{V_n^{(m)}(a,1)\}$ is slowly oscillating, then the sequence $\{S_n(a)\}$ is (C,m)-summable.*

Proof Take the difference of the identity

$$\sigma_n^{(m)}(S(a)) - \sigma_m^{(m+1)}(S(a)) = V_n^{(m)}(a,1) \tag{3.11}$$

to obtain

$$\Delta\sigma_n^{(m)}(S(a)) - \Delta\sigma_n^{(m+1)}(S(a)) = \Delta V_n^{(m)}(a,1) \tag{3.12}$$

From (3.12) we have

$$f(\Delta\sigma^{(m)}(S(a)),x) - f(\Delta\sigma^{(m+1)}(S(a)),x) = f(\Delta V_n^{(m)}(a,1),x) \tag{3.13}$$

Since the sequence $\{S_n(a)\}$ is (A,o)-summable, then $\{\sigma_n^{(m)}(S(a))\}$ and $\{\sigma_n^{(m+1)}(S(a))\}$ are. Therefore, it follows from (3.13) that $\{V_n^{(m)}(a,1)\}$ is Abel summable. Hence $\{V_n^{(m)}(a,1)\}$ converges. This implies that $\{\sigma_n^{(m+1)}(S(a))\}$ is slowly oscillating. Since it is also Abel summable, it converges by Theorem 3.4. By (3.11) it is seen that $\{S_n(a)\}$ is (C,m)-summable.

Theorem 3.17 *For some integer $m \geq 0$ let $\{V_n^{(m)}(a,1)\}$ be $(A,1)$-summable. If $\{V_n^{(m+1)}(a,1)\}$ is slowly oscillating, then $S_n(a) = O(\log n)$ $(C,m+1)$, $n \to \infty$.*

Proof Since $\{V_n^{(m)}(a,1)\}$ is $(A,1)$-summable, then

$$\lim_{x \to 1-o} \frac{1-x}{x} \int_0^x \frac{f(\Delta V^{(m)}(a,1),t)dt}{(1-t)^2}$$

exits. This is equivalent to saying that

$$\{V_n^{(m+1)}(a,1)\} = \left\{\frac{1}{n+1}\sum_{k=0}^{n} V_k^{(m)}(a,1)\right\}$$

is (A,o)-summable. Therefore by Theorem 3.4 $\{V_n^{(m+1)}(a,1)\}$ converges.

Applying summation by parts we obtain

$$\begin{aligned}
\sigma_n^{(m+1)}(S(a)) &= \sum_{k=1}^{n} \frac{V_k^{(m)}(a,1)}{k} \\
&= \frac{1}{n}\sum_{k=1}^{n} V_k^{(m)}(a,1) + \sum_{k=1}^{n-1} \frac{\sum_{j=1}^{k} V_j^{(m)}(a,1)}{k(k+1)} \\
&= V_n^{(m+1)}(a,1) + \sum_{k=1}^{n-1} \frac{V_k^{(m+1)}(a,1)}{k+1} .
\end{aligned}$$

From the last equality it follows that $S_n(a) = O(\log n)$ $(C, m+1)$, $n \to \infty$.

So far we recovered the convergence of the series out of its generalized Abel's summability method and a Tauberian condition corresponding to that method. However the asymptotic behavior of the series $\{S_n(a)\}$ and the generalized Abel's summability method can be related.

Appell [16] proved that if

$$(3.14) \qquad S_n(a) \sim An^{\gamma}, \qquad \gamma \geq 0, \quad \text{A is costant}$$

then $\lim_{x \to 1-o}(1-x)^{\gamma} f(a,x)$ exists.

The symbol $\sim$ is used in the following sense: $a_n \sim b_n$ means $\frac{a_n}{b_n} \to 1$, $n \to \infty$.

It easily follows from Appell's theorem that if $\{S_n(a)\}$ is $(C,1)$-summable, then it is (A,o)-summable. In general, if $\{S_n(a)\}$ is (C,k)-summable for any positive integer k, then $\{S_n(a)\}$ is (A,k)-summable.

It is well known that if $\{S_n(a)\}$ is $(C,1)$ summable then it is (A,o)-summable. Using Appell's theorem one shows that if $\{S_n(a)\}$ is $(C,2)$-summable then it is $(A,1)$-summable.

We now use Appell's theorem to prove the (A,m)-summability of the sequence $\{S_n(a)\}$ when we know the asymptotic behavior of the corresponding partial sums.

From Appell's theorem we conclude that if (3.14) holds for $\gamma = 2$, then $\lim_{x \to 1-o}(1-x)^2 f(a,x)$ exists.

Let $F(x) = (1-x)^2 f(x)$. Then the partial sums corresponding to $F(x)$ is $(A,1)$-summable, i.e. $\lim_{x \to 1-o} A^{(1)}((1-x)^2 f(a,x), \delta_2, x)$ exists.

But

$$A^{(1)}((1-x)^2 f(a,x), \delta_x, x) = \frac{1-x}{x} \int_0^x f(t)dt.$$

Thus it follows that $\lim_{x \to 1-o} \frac{1-x}{x} \int_0^x f(t)dt$ exists. This means that $\left\{\frac{a_n}{n+1}\right\}$ is (A, o)-summable.

If $\{a_n\}$ is slowly oscillating, then $\left\{\frac{a_n}{n+1}\right\}$ converges and thus $\lim_{x 1-o} \frac{1-x}{x} \int_0^x f(t)dt$ exists.

We know that if $\{S_n(a)\}$ is $(C, 1)$-summable, then $\{S_n(a)\}$ is (A, o)-summable. Similarly, we can ask the following question for the $(A, 1)$-summability:
$\{\sigma_n^{(m-2)}(S(a))\}$ is $(A, 1)$-summable. If $\{S_n(a)\}$ is (C, m)-summable ? The answer is affirmative and the theorem is as follows.

Theorem 3.18 *Let $\{S_n(a)\}$ is (C, m)-summable. Then for some $m \geq 2$ $\{\sigma_n^{(m-2)}(S(a))\}$ is $(A, 1)$-summable.*

Proof If $\{\sigma_n^{(m-2)}(S(a))\}$ is $(A, 1)$-summable, then $\{\sigma_n^{(m-1)}(S(a))\}$ is (A, o)- summable and vice versa.

It is enough to show that $\{\sigma_n^{(m-2)}(S(a))\}$ is Abel summable. Consider

$$\sum_{n=0}^{\infty} \Delta \sigma_n^{(m-1)}(S(a)) x^n . \tag{3.15}$$

Dividing (3.15) by $(1-x)^2$ we obtain

$$\sum_{n=0}^{\infty} \Delta \sigma_n^{(m-1)}(S(a)) x^n = (1-x)^2 \sum_{n=0}^{\infty} (n+1) \sigma_n^{(m)}(S(a)) x^n .$$

Since $\{S_n(a)\}$ is (C, m)-summable, we get $(n+1)\sigma_n^{(m)}(S(a)) \sim An$, $n \to \infty$. By a Theorem in [3]

$$\sum_{n=0}^{\infty} (n+1) \sigma_n^{(m)}(S(a)) x^n \sim A \sum_{n=0}^{\infty} n x^n = A \frac{1}{(1-x)^2}$$

Hence, $\lim_{x \to 1-o} \sum_{n=0}^{\infty} \Delta \sigma_n^{(m-1)} (S(a)) x^n$ exists or equivalently $\{\sigma_n^{(m-1)}(S(a))\}$ is (A, o)-summable.

We say that $\{S_n(a)\}$ is $(A^{(2)}, V^{(0)}, \Delta V^{(0)})$-summable if $\{S_n(a)\}$ is $(A, 2)$-summable and $V_n^{(o)}(\Delta V^{(o)}, 1) = O(1)$, $n \to \infty$.

Theorem 3.19 *Let $\{S_n(a)\}$ be $(A^{(2)}, V^{(0)}, \Delta V^{(0)})$-summable. If $\{S_n(a)\}$ is $(C,1)$-bounded then it is $(C,1)$-summable.*

Proof Observe that $V_n^{(o)}(\Delta V^{(o)}, 1) = V_n^{(0)}(a,1) - V_n^{(1)}(a,1)$. Since

$$V_n^{(1)}(a,1) = \sum_{k=1}^{n} \frac{V_k^{(o)}(\Delta V^{(o)}, 1)}{k},$$

$\{V_n^{(1)}(a,1)\}$ is slowly oscillating. By the assumption $\{S_n(a)\}$ is $(C,1)$-bounded, we have $\sigma_n^{(2)}(S(a)) = \sum_{k=1}^{n} \frac{\sigma_k^{(1)}(S(a)) - \sigma_k^{(2)}(S(a))}{k}$ is slowly oscillating. Since $\{S_n(a)\}$ is $(A,2)$-summable, it converges by Theorem 3.4. From the identity $V_n^{(1)}(a,1) = \sigma_n^{(1)}(S(a)) - \sigma_n^{(2)}(S(a))$ it follows that $\{S_n(a)\}$ is $(C,1)$-slowly oscillating. Hence $\{S_n(a)\}$ is $(C,1)$-summable.

The condition in Theorem 3.19 can be replaced by the condition $\{S_n(a)\}$ is $(C,1)$-slowly oscillating.

From the $(A,2)$-summability of the series $\{S_n(a)\}$ with an appropriate Tauberian condition we obtain how the series $\{\sigma_n^{(1)}(S(a))\}$ behaves.

Theorem 3.20 *Let $\{S_n(a)\}$ be $(A,2)$-summable. If*

$$\frac{1}{n+1} \sum_{k=0}^{n} V_k^{(0)}(a,1) = O(1), \qquad n \to \infty$$

then for $\gamma \in (0,1)$ $\sum_{n=0}^{\infty} \frac{\sigma_n^{(1)}(S(a))}{(n+1)^{\gamma+1}}$ converges.

Proof By summation by parts we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\sigma_n^{(1)}(S(a))}{(n+1)^{(\gamma+1)}} &= \frac{1}{(n+1)^{\gamma}} \sum_{k=0}^{n} \sigma_k^{(1)}(S(a)) \\
&+ \sum_{k=0}^{n-1} \left(\frac{1}{(k+1)^{(\gamma+1)}} - \frac{1}{(k+1)^{(\gamma+1)}} \right) \sum_{j=0}^{k} \sigma_j^{(1)}(S(a)) \\
&= \frac{n}{(n+1)^{(\gamma+1)}} \sigma_n^{(2)}(S(a)) + \\
&+ \sum_{k=0}^{n-1} k \frac{(k+1)^{(\gamma+1)} - (k+1)^{(\gamma+1)}}{(k+1)^{(\gamma+1)}(k+1)^{(\gamma+1)}} \sigma_k^{(2)}(S(a)) .
\end{aligned}$$

The first term exists as $n \to \infty$, and the second sequence exists as $n \to \infty$.

Under the conditions of the previous theorem we are able to show that $\{\sigma_n^{(1)}(S(a))\}$ is moderately divergent. It is enough to show that $\sigma_n^{(1)}(S(a)) = o(n^{r-1})$, $n \to \infty$ for every $r > 1$.

From

$$\sigma_n^{(1)}(S(a)) = \sum_{k=1}^{n} \frac{V_k^{(o)}(a,1)}{k} = \frac{1}{n}\sum_{k=1}^{n} V_k^{(0)}(a,1) + \sum_{k=1}^{n-1} \frac{\frac{1}{k}\sum_{j=1}^{k} V_j^{(o)}(a,1)}{k+1},$$

we have

$$\frac{\sigma_n^{(1)}(S(a))}{n^{r-1}} = \frac{1}{n^{r-1}}\frac{1}{n}\sum_{k=1}^{n} V_k^{(0)}(a,1) + \frac{1}{n^{r-1}}\sum_{k=1}^{n-1} \frac{\frac{1}{k}\sum_{j=1}^{k} V_j^{(o)}(a,1)}{k+1}$$

for every $r > 1$. Since $\frac{1}{n+1}\sum_{k=0}^{n} V_k^{(0)}(a,1) = O(1)$, $n \to \infty$, then the first term goes to zero as $n \to \infty$ and the second term behave like $\frac{\log n}{n^{r-1}}$ as $n \to \infty$.

Hence we have $\sigma_n^{(1)}(S(a)) = o(n^{r-1})$, $n \to \infty$ for every $r > 1$. So $\{\sigma_n^{(1)}(S(a))\}$ is moderately divergent.

The assumption made in the following theorem imply that $\{S_n(a)\}$ is $(C,1)$-summable. To get the convergence of the series we give a necessary and sufficient conditions.

Theorem 3.21 *Let*

$$\lim_{x \to 1-o} \int_0^x \frac{f(t)dt}{1-t} \tag{3.16}$$

exists and the sequence $\{S_n(a)\}$ is $(C,1)$-slowly oscillating. Then the sequence $\{S_n(a)\}$ converges if and only if

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=2}^{n} a_k \log k \tag{3.17}$$

exists.

Proof The sequence $\left\{\sum_{k=0}^{n} \frac{S_k(a)}{k+1}\right\}$ is Abel summable by the condition (3.16). We obtain that $\{S_n(a)\}$ is $(A,1)$-summable. Since the sequence $\{S_n(a)\}$ is $(C,1)$-slowly oscillating, it follows that $\{S_n(a)\}$ is $(C,1)$-summable. Applying summation by parts one obtains that

$$\sum_{k=0}^{n}\frac{S_k(a)}{k+1}=\sigma_n^{(1)}(S(a))+\sum_{k=0}^{n-1}\frac{\sigma_k^{(1)}(S(a))}{k+1}. \tag{3.18}$$

The last identity shows that $\left\{\sum_{k=0}^{n}\frac{S_k(a)}{k+1}\right\}$ is slowly oscillating. Together with the condition $\left\{\sum_{k=0}^{n}\frac{S_k(a)}{k+1}\right\}$ converges.

Applying summation by parts we have

$$\frac{1}{\log n}\sum_{k=2}^{n}a_k\log k=\frac{1}{\log n}\sum_{k=2}^{n-1}S_k(a)\log\left(\frac{k+1}{k}\right)+S_n(a)$$

The conclusion of the theorem follows the identity above if we show the first term on the right converges under the hypotheses.

If we apply the summation by parts for the first term above we get

$$\begin{aligned}\frac{1}{\log n}\sum_{k=2}^{n}S_k(a)\log\left(\frac{k+1}{k}\right)&= \qquad (3.19)\\ =\ &\frac{1}{\log n}\sum_{k=2}^{n}\frac{S_k(a)}{k}\log\left(\frac{k+1}{k}\right)\\ =\ &\frac{1}{\log n}\sum_{k=2}^{n-1}\left(\sum_{j=2}^{k}\frac{S_j(a)}{j}\right)\triangle\left(\log\left(1+\frac{1}{k}\right)^k\right)+\\ +\ &\frac{\left(\sum_{k=2}^{n}\frac{S_k(a)}{k}\right)\log\left(1+\frac{1}{n}\right)^n}{\log n}.\end{aligned}$$

The second term of (3.19) converges to zero as $n\to\infty$.

Set $A(n)=\log\left(1+\frac{1}{n}\right)^n$. Then

$$\begin{aligned}A(n)-A(n-1)&=\log\left(1+\frac{1}{n}\right)^n-\log\left(1+\frac{1}{n+1}\right)^{(n+1)}\\&=\log\left(\frac{(n+2)^{(n+1)}n^n}{(n+1)^{(n+1)}(n+1)^n}\right).\\&=\log\left(\frac{(n+2)\left(1+\frac{1}{n+1}\right)^{(n+1)}}{(n+1)\left(1+\frac{1}{n}\right)^n}\right)\sim\log\left(1+\frac{1}{n+1}\right)\end{aligned}$$

It follows that $\Delta A(n) \sim \frac{1}{n+1}$ as $n \to \infty$. This shows that the first term converges.

Theorem 3.22 *Let $\{a_n\}$ be a sequence of real numbers and let*

$$\lim_{x \to 1-o} \int_0^x \frac{f(t)dt}{1-t}$$

exists. If the sequence $\{S_n(a)\}$ is $(C,1)$-slowly oscillating, then

$$S_n(a) = \frac{1}{\log n} \sum_{k=2}^{n} a_k \log k + O(1), \qquad n \to \infty .$$

Denote by L the class of sequences $\{a_n\}$ such that $na_n = O(1)$, $n \to \infty$.

Then we have the following corollary.

Corollary 3.22.1 Let $\{a_n\}$ and

$$\lim_{x \to 1-o} \int_0^x \frac{f(a,t)dt}{1-t}$$

exists. If

$$V_n^{(0)}(a,1) = O(1), \qquad n \to \infty$$

then $S_n(a) = O(\log n) + O(1)$, $n \to \infty$.

4. Tauberian Theorems for (C, m)-summability methods and further convergence theorems

In the previous section we recovered the convergence of the series out of its generalized summability assuming some conditions on the series. We now give the Tauberian theorems for the Cesaro summability.

Recovering the lower order Cesaro summability of the sequence from the higher order Cesaro summability using the Karamata's techniques modified by Stanojevic will be presented in that chapter.

The first theorem shows that $(C,1)$-summability and $(C,2)$-summability are equivalent provided that $\left\{\sum_{k=0}^{n} \frac{S_k(a)}{k+1}\right\}$ is slowly oscillating.

Theorem 4.1 *The sequence* $\{S_n(a)\}$ *is* $(C,1)$*-summable if and only if*

i) is $(C,2)$*-summable,*

ii) $\left\{\sum_{k=0}^{n} \frac{S_k(a)}{k+1}\right\}$ *is slowly oscillating.*

Proof Let $\{S_n(a)\}$ be $(C,1)$-summable. Then it is clear that $\{S_n(a)\}$ is $(C,2)$-summable. From the identity

$$\sum_{k=0}^{n} \frac{S_k(a)}{k+1} = \sigma_n^{(1)}(S(a)) + \sum_{k=0}^{n-1} \frac{\sigma_k^{(1)}(S(a))}{k+2}$$

using that $\{S_n(a)\}$ is $(C,1)$-summable one obtains that $\left\{\sum_{k=0}^{n} \frac{S_k(a)}{k+1}\right\}$ is slowly oscillating.

Conversely, assume i) and ii) hold. Applying the summation by parts to the second term on the last identity and then rearranging terms we have

$$\begin{aligned}
\sigma_n^{(1)}(S(a)) &= \sum_{k=0}^{n} \frac{S_k(a)}{k+1} - \sum_{k=0}^{n-1} \frac{\sigma_k^{(1)}(S(a))}{k+2} \\
&= \sum_{k=0}^{n} \frac{S_k(a)}{k+1} - \frac{1}{n+1}\sum_{k=0}^{n-1} \sigma_k^{(1)}(S(a)) - \sum_{k=0}^{n-2} \frac{\sum_{j=1}^{k} \sigma_j^{(1)}(S(a))}{(k+2)(k+3)} \\
&= \sum_{k=0}^{n} \frac{S_k(a)}{k+1} - \frac{1}{n+1}\sigma_{n-1}^{(2)}(S(a)) - \sum_{k=0}^{n-2} \frac{k}{(k+2)(k+3)}\sigma_n^{(2)}(S(a)).
\end{aligned}$$

The conditions i) and ii) imply that $\{S_n(a)\}$ is $(C,1)$-slowly oscillating. Since $\{S_n(a)\}$ is $(C,2)$-summable. $\{S_n(a)\}$ is $(C,1)$-summable. This completes the proof of the theorem.

We now give Tauberian theorems for the recovery of (C,m)-summability out of (C,r)-summability, where $r > m$.

Define for $\lambda > 1$

$$\tau_n(S(a),\lambda) = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} S_k(a)$$

Here, $[x]$ denotes the greatest integer not exceeding x.

The sequence $\{S_n(a)\}$ is called to be (C,m)-slowly oscillating if $\{\sigma_n^{(m)}(S(a))\}$ is slowly oscillating.

The next theorem is a corollary to the generalized Littlewood theorem. Prove it using the Karamata's techniques as proved below.

Theorem 4.2 *Let $\{S_n(a)\}$ be (C,m)-summable for some $m \geq 1$. If $\{S_n(a)\}$ is (C,m)-slowly oscillating, then the sequence $\{S_n(a)\}$ is $(C,m-1)$-summable.*

Proof Consider the difference

$$\begin{aligned}\sigma_n^{(m)}(S(a)) - \sigma_n^{(m-1)}(S(a)) &= \sigma_n^{(m)}(S(a)) - \tau_n(\sigma^{(m-1)}(S(a)),\lambda) - \\ &- \sigma_n^{(m-1)}(S(a)) + \tau_n(\sigma^{(m-1)}(S(a)),\lambda)\end{aligned}$$

Hence we have

$$\begin{aligned}\left|\sigma_n^{(m)}(S(a)) - \sigma_n^{(m-1)}(S(a))\right| &\leq \left|\tau_n(\sigma^{(m-1)}(S(a)),\lambda) - \sigma_n^{(m)}(S(a))\right| \\ &+ \left|\tau_n(\sigma^{(m-1)}(S(a)),\lambda) - \sigma_n^{(m-1)}(S(a))\right|\end{aligned}$$

The first term on the right hand side of the inequality above is

$$\frac{[\lambda n]+1}{[\lambda n]-1}\left|\sigma_{[\lambda n]}^{(m)}(S(a)) - \sigma_n^{(m)}(S(a))\right| . \tag{4.1}$$

For the second term we have the estimate

$$\begin{aligned}&\left|\tau_n\left(\sigma^{(m-1)}(S(a)),\lambda\right) - \sigma_n^{(m-1)}(S(a))\right| \leq \\ &\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left|\sigma_k^{(m-1)}(S(a)) - \sigma_n^{(m-1)}(S(a))\right| \leq \\ &\max_{n+1\leq k\leq[\lambda n]}\left|\sigma_k^{(m-1)}(S(a)) - \sigma_n^{(m-1)}(S(a))\right| .\end{aligned} \tag{4.2}$$

Taking the limsup of (4.2) we obtain

$$\limsup_n \left|\tau_n\left(\sigma^{(m-1)}(S(a)),\lambda\right) - \sigma_n^{(m-1)}(S(a))\right| = 0 .$$

Since $\{S_n(a)\}$ is $(C, m-1)$-slowly oscillating,

$$\lim_{\lambda\to 1+o} \limsup_n \left|\tau_n\left(\sigma^{(m-1)}(S(a))\right) - \sigma_n^{(m-1)}(S(a))\right| = 0$$

This completes the proof of the theorem.

Corollary 4.2.1 Let $\{S_n(a)\}$ be $(C,1)$-summable. If $\{S_n(a)\}$ is slowly oscillating, then the series $\{S_n(a)\}$ converges.

Theorem 4.3 *Let $\{S_n(a)\}$ be (C,m)-summable. If*

$$\limsup_n \left(\sum_{k=n+1}^{[\lambda n]} \frac{\left|V_k^{(m-2)}(a,1)\right|^p}{k}\right)^{\frac{1}{p}} = \tag{4.3}$$

$$= o\left(\frac{1}{(\lambda-1)^{\frac{1}{q}}}\right), \quad \lambda \to 1+o, \ \frac{1}{p}+\frac{1}{q} = 1,$$

then $\{S_n(a)\}$ is $(C, m-1)$-summable.

Proof As in theorem 4.2 consider the same difference. It is enough to estimate the second term under the condition of this theorem.

For the second term we have the estimate

$$\left|\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]} \sigma_k^{(m-1)}(S(a)) - \sigma_n^{(m-1)}(S(a))\right|$$

$$\leq \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k} \frac{\left|\sigma_j^{(m-2)}(S(a)) - \sigma_j^{(m-1)}(S(a))\right|}{j}$$

$$\leq \sum_{k=n+1}^{[\lambda n]} \frac{\left|\sigma_k^{(m-1)}(S(a)) - \sigma_k^{(m-1)}(S(a))\right|}{k}$$

$$= \sum_{k=n+1}^{[\lambda n]} \frac{\left|V_k^{(m-1)}(a,1)\right|}{k}$$

$$= ([\lambda n]-n)\left(\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]} \frac{\left|V_k^{(m-2)}(a,1)\right|}{k}\right).$$

Using the Jensen -Petrović inequality we obtain

$$\le \quad ([\lambda n]-n)\left(\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\frac{\left|V_k^{(m-2)}(a,1)\right|^p}{k^p}\right)^{\frac{1}{p}}$$

$$\le \quad (\lambda-1)^{\frac{1}{q}}n^{\frac{1}{q}}\left(\sum_{k=n+1}^{[\lambda n]}\frac{\left|V_k^{(m-2)}(a,1)\right|^p}{k^p}\right)^{\frac{1}{p}}$$

$$\le \quad (\lambda-1)^{\frac{1}{q}}\left(\sum_{k=n+1}^{[\lambda n]}\frac{\left|V_k^{(m-2)}(a,1)\right|^p}{k}\right)^{\frac{1}{p}}$$

$$\le \quad (\lambda-1)^{\frac{1}{q}}\left(\sum_{k=n+1}^{[\lambda n]}\frac{k^{p-1}}{k^p}\left|V_k^{(m-2)}(a,1)\right|^p\right)^{\frac{1}{p}}$$

$$\le \quad (\lambda-1)^{\frac{1}{q}}\left(\sum_{k=n+1}^{[\lambda n]}\frac{\left|V_k^{(m-2)}(a,1)\right|^p}{k}\right)^{\frac{1}{p}}.$$

By (4.3)

$$\limsup_n\left|\tau_n\left(\sigma^{(m-1)}(S(a),\lambda)\right)-\sigma_n^{(m-1)}(S(a))\right|=o(1),\quad \lambda\to 1+o\,.$$

This completes the proof of the theorem.

Theorem 4.4 *Let $\{S_n(a)\}$ be $(C,2)$-summable. If then $\{S_n(a)\}$ is $(C,1)$- summable.*

Consider the difference $\sigma_n^{(2)}(S(a))-\sigma_n^{(1)}(S(a))$. Adding and subtracting the term $\tau_n(\sigma^{(1)}(S(a)),\lambda)$ to the difference we have the estimate

$$(4.4)\quad \left|\sigma_n^{(2)}(S(a))-\sigma_n^{(1)}(S(a))\right| \le \left|\sigma_n^{(2)}(S(a))-\tau_n(\sigma_n^{(1)}(S(a)),\lambda)\right|+ + \left|\tau_n(\sigma^{(1)}(S(a)),\lambda)-\sigma_n^{(1)}(S(a))\right|.$$

For the second quantity on the right above we have

$$\left|\tau_n(\sigma^{(1)}(S(a)),\lambda)-\sigma_n^{(1)}(S(a))\right|$$

$$(4.5) \qquad \le \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left|\sigma_k^{(1)}(S(a))-\sigma_n^{(1)}(S(a))\right|$$

$$= \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left|\sum_{j=n+1}^{k}\frac{\sigma_j^{(0)}(S(a))-\sigma_j^{(1)}(S(a))}{j}\right|$$

$$\le \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\frac{\left|V_j^{(0)}(a,1)\right|}{j}.$$

For the second term on the above inequality we have

$$(4.6) \qquad \left|\tau_n(\sigma^{(1)}(S(a)),\lambda)-\sigma_n^{(2)}(S(a))\right| =$$

$$= \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(2)}(S(a))-\sigma_n^{(2)}(S(a))\right).$$

Taking limsup of both sides of the inequality (4.4) and using the estimations on (4.5) and (4.6) we obtain

$$(4.7) \qquad \limsup_n \left|\sigma_n^{(2)}(S(a))-\sigma_n^{(1)}(S(a))\right| \le \limsup_n \sum_{j=n+1}^{[\lambda n]}\frac{\left|V_j^{(0)}(a,1)\right|}{j}$$

$$+ \limsup_n \frac{[\lambda n]+1}{[\lambda n]-n}\limsup_n \left(\sigma_{[\lambda n]}^{(2)}(S(a))-\sigma_n^{(2)}(S(a))\right)$$

Since $\{S_n(a)\}$ is $(C,1)$-summable, $\limsup_n \left(\sigma_{[\lambda n]}^{(2)}(S(a))-\sigma_n^{(2)}(S(a))\right)=0$.

For the first term we have the estimate

$$\limsup_n \sum_{j=n+1}^{[\lambda n]}\frac{\left|V_j^{(0)}(a,1)\right|}{j} \le$$

$$\le \limsup_n \frac{[\lambda n]-n}{n}\limsup_n \frac{1}{[\lambda n]-n}\sum_{j=n+1}^{[\lambda n]}\left|V_j^{(0)}(a,1)\right|$$

$$(4.8) \qquad = (\lambda-1)\limsup_n \frac{1}{[\lambda n]-n}\sum_{j=n+1}^{[\lambda n]}\left|V_j^{(0)}(a,1)\right|$$

This completes the proof of the theorem.

Theorem 4.5 *For some integer $m \geq 2$ let $\{S_n(a)\}$ be (C,m)-summable. If*

$$V_n^{(0)}(V_n^{(m-2)}(a,1),1) = O(1), \qquad n \to \infty \tag{4.9}$$

then $\{S_n(a)\}$ is $(C,m-1)$-summable.

Proof The condition (4.9) implies that $\{S_n(a)\}$ is $(C,m-1)$-slowly oscillating. It is $(C,m-1)$-summable.

Corollary 4.5.1 Let $\{S_n(a)\}$ be $(C,2)$-summable and let $V_n^{(0)}(V_n^{(0)}(a,1),1) = O(1)$, $n \to \infty$. If $\{S_n(a)\}$ is slowly oscillating, then it converges.

Theorem 4.6 *For some integer $m \geq 1$ let $\{S_n(a)\}$ be (C,m)-summable. If $\{V_n^{(m-1)}(a,1)\}$ is very slowly oscillating then the sequence $\{S_n(a)\}$ is $(C,m-1)$-summable.*

Proof Let $\{V_n^{(m-1)}(am1)\}$ be very slowly oscillating. Then we have

$$V_n^{(m-1)}(a,1) - V_n^{(m)}(a,1) = o(1), \qquad n \to \infty\ .$$

Since $\{V_n^{(m-1)}(a,1)\}$ is very slowly oscillating, then it is slowly oscillating. We have the identities

$$V_n^{(m-1)}(a,1) = \sigma_n^{(m-1)}(S(a)) - \sigma_n^{(m)}(S(a)) \tag{4.10}$$

and

$$V_n^{(m)}(a,1) = \sigma_n^{(m)}(S(a)) - \sigma_n^{(m+1)}(S(a))\ .$$

Given $\{S_n(a)\}$ is (C,m)-summable. From the last identity we obtain that $\{V_n^{(m)}(a,1)\}$ is slowly oscillating. It follows from (4.10) that $\{\sigma_n^{(m-1)}(S(a))\}$ is slowly oscillating. Then, $\{\sigma_n^{(m-1)}(S(a))\}$ converges by the generalized Littlewood theorem. This completes the proof.

So far we have recovered the convergence of some processes out of its generalized Abel summability method together with some Tauberian condition(s). In that section we show that without mentioning the summability method in the theorems below we obtain the convergence of the series out of some condition related with the sequence . The following theorem shows that the slow oscillating of the sequence is a measurement how the original sequence converges. Concentrating on the slow oscillation as a condition which implies the convergence we obtain some important corollaries in the light of the first theorem in that chapter.

Theorem 4.7 *Let $\{S_n(a)\}$ be slowly oscillating. Then the sequence $\left\{\sum_{k=j}^{n} \frac{a_k}{k}\right\}$ converges.*

Proof Since the sequence $\{S_n(a)\}$ is slowly oscillating, there exists some $h \in H^S$, $s \geq 2$ such that $S_n(a) = n\hat{h}(n)$.

If we take the difference of both sides of the last identity we have

$$\begin{aligned} a_n &= n\hat{h}(n) - (n-1)\hat{h}(n-1) \\ &= n\left(\hat{h}(n) - \hat{h}(n-1)\right) + \hat{h}(n-1)\,. \end{aligned} \tag{4.11}$$

Divide both sides of (4.11) by n, and then sum up from $k = 1$ to n, we obtain that

$$\sum_{k=1}^{n} \frac{a_k}{k} = \sum_{k=1}^{n} \left(\hat{h}(k) - \hat{h}(k-1)\right) + \sum_{k=1}^{n} \frac{\hat{h}(k-1)}{k}$$

The first and second terms on the right of the last equality converge. This completes the proof of the theorem.

Corollaries 4.7.1 Let $\left\{\sum_{k=0}^{n} V_k^{(0)}(a,1)\right\}$ be slowly oscillating. Then the sequence $\{S_n(a)\}$ converges.

Proof Denote $W_n(a,1) = \sum_{k=0}^{n}(a,1)$. Then by theorem 4.7 the sequence $\{\sigma_n^{(1)}(S(a))\}$ converges. Since $\{W_n^{(1)}(a,1)\}$ is slowly oscillating, $V_n^{(0)}(a,1) = o(1)$, $n \to \infty$.

From the identity $S_n(a) - \sigma_n^{(1)}(S(a)) = V_n^{(0)}(a,1)$ it follows that the sequence $\{S_n(a)\}$ converges.

Corollary 4.7.2 For some $m \geq 1$ let $\left\{\sum_{k=0}^{n} V_k^{(m)}(a,1)\right\}$ be slowly oscillating and let the sequence $\{S_n(a)\}$ be slowly oscillating, then $\{S_n(a)\}$ converges.

Proof Set $W_n^{(m)}(a,1)=\sum_{k=0}^{n} V_k^{(m)}(a,1)$. The sequence $\{\sigma_n^{(m+1)}(S(a))\}$ converges. This implies that the sequence $\{S_n(a)\}$ is (A,o)-summable. Since it is slowly oscillating then it converges.

We have obtained the convergence of the sequence assuming the slow oscillation of some processes as in corollaries 4.7.1 and 4.7.2. If we assume the slow oscillation of the derivative of the partial sums of the sequence we have the convergence as showed below. For this we need to give some denotations .

Let $f(a) = f(a,x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < 1$. Denote the n^{th} partial sum of $f'(a)$ by $S'_n(a,x)$. Plugging in $x = 1$ in $S'_n(a,x)$ gives that $S'_n = \sum_{k=0}^{n} ka_k$.

Now we are ready to state a theorem which implies the convergence of the sequence $\{S_n(a)\}$ provided that $\{S'_n(a)\}$ is slowly oscillating.

Theorem 4.8 *If $\{S'_n(a)\}$ is slowly oscillating then the sequence $\{S_n(a)\}$ converges.*

Proof Applying summation by parts one obtains that

$$\begin{aligned} S_n(a) &= \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{ka_k}{k} \\ &= \frac{1}{n}\sum_{k=1}^{n} ka_k + \sum_{k=1}^{n-1} \frac{\sum_{j=1}^{k} ja_j}{k(k+1)} \\ &= \frac{1}{n}S'_n(a) + \sum_{k=1}^{n-1} \frac{S_k(a)}{k(k+1)} . \end{aligned}$$

Since $\{S'_n(a)\}$ is slowly oscillating , the first and the second term on the right converge. This completes the proof.

A positive sequence $\{M(n)\}$ is moderately divergent if for every $r > 1$

$$M(n) = o(n^{r-1}), \quad n \to \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{M(n)}{n^r} < \infty .$$

Denote by M the class of all moderately divergent sequences.

It is clear that every slowly oscillating sequence is moderately divergent.

Theorem 4.9 *For* $\gamma \in (0,1)$ *let* $\left\{\frac{1}{n^\gamma}\sum_{k=1}^{n} V_k^{(0)}(a,1)\right\} \in M$. *Then*

i) The sequence $\{S_n(a)\}$ *is* $(C,1)$*-summable,*

ii) $S_n(a) = O(n^\gamma M(n))$, $n \to \infty$.

Proof Applying summation by parts we obtain

$$\sigma_n^{(1)}(S(a)) = \sum_{k=1}^{n} \frac{V_k^{(0)}(a,1)}{k} = \frac{1}{n}\sum_{k=1}^{n} V_k^{(0)}(a,1) + \sum_{k=1}^{n-1} \frac{\sum_{j=1}^{k} V_j^{(0)}(a,1)}{k(k+1)}. \tag{4.12}$$

Since $M(n) = \frac{1}{n^\gamma}\sum_{k=1}^{n} V_k^{(0)}(a,1)$ by the definition it follows that

$$\frac{1}{n}\sum_{k=1}^{n} V_k^{(0)}(a,1) = o(1), \quad n \to \infty.$$

The second term in (4.12) is $\sum_{k=1}^{n-1} \frac{M(k)}{k^{2-\gamma}}$ and it converges as $n \to \infty$. Therefore $\{S_n(a)\}$ is $(C,1)$-summable.

ii) It follows from i)

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